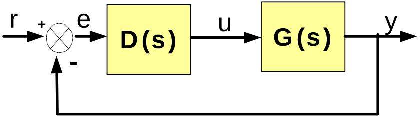
Control Systems
Lecture 2

Root locus

Introduction

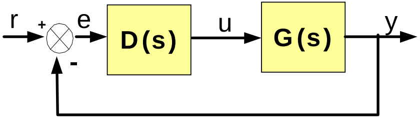
- Understanding - and then designing - the dynamic properties of closed-loop systems



Several analysis and synthesis tools for addressing

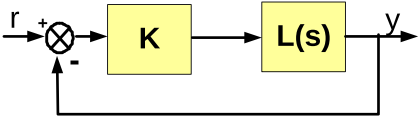
- Stability
- Dynamic responses (reference tracking)
- Disturbance rejection

Typical approach: use information on G to analyse/design D

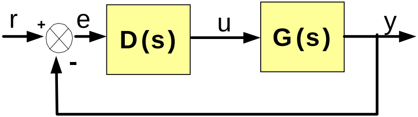


In this lecture focus on:

1. Root locus technique: analyse the stability properties of the closed loop for a varying $K \in \mathbb{R}$



Dynamic properties of $D(s)$ are assumed included in $L(s)$ and one scalar parameter K is isolated and its effect studied.



In next lecture(s):

2. Frequency response techniques: dynamically shape $D(j\omega)G(j\omega)$ to design the properties of the closed-loop system

   Map properties of DG to properties of the closed-loop $\frac{DG}{1+DG}$

   Tool: Shape the frequency response $D(j\omega)G(j\omega)$.

   First address the question (Lecture 3):

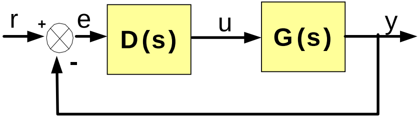
   how can we manipulate the frequency response of a system?

   (Bode plot analysis)

Root locus

The root-locus method (Evans, 1948)

Analyze the poles of the feedback system



as function of one scalar (varying) parameter K.

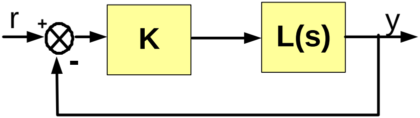
Write loop gain: $D(s)G(s) = KL(s)$ with $K \in \mathbb{R}$.

K is e.g. the controller gain k_P. By varying K the poles of the closed-loop system will move.

Closed-loop transfer: $\dfrac{KL(s)}{1+KL(s)}$.

Poles of the closed-loop system are the solutions to the characteristic equation

$$1 + KL(s) = 0$$



Root-locus: For a given $L(s)$, sketch in $\mathbb{C}$ the location/curve of the closed-loop poles as a function of K

Notation:

$$L(s) = \frac{b(s)}{a(s)}$$

with

$$\begin{aligned} b(s) &= s^m + b_1 s^{m-1} + \cdots + b_m \\ &= (s - z_1)(s - z_2)\cdots(s - z_m) \\ a(s) &= s^n + a_1 s^{n-1} + \cdots + a_n \\ &= (s - p_1)(s - p_2)\cdots(s - p_n) \end{aligned}$$

with $n \geq m$.
z_i: zeros of $L(s)$; p_i: poles of $L(s)$.

Note: Both $a(s)$ and $b(s)$ have $+1$ as the leading coefficient.
For $b(s)$ with leading term -1 see later.

Equivalent formulation of root conditions:

$$1 + KL(s) = 0$$

$$1 + K\frac{b(s)}{a(s)} = 0$$

$$a(s) + Kb(s) = 0$$

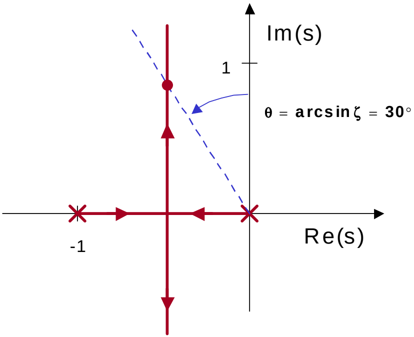
$$L(s) = -\frac{1}{K} \quad \text{(negative real)}$$

Example DC-motor

$$\frac{\Theta_M(s)}{V_a(s)} = \frac{Y(s)}{U(s)} = G(s) = \frac{A}{s(s+c)}$$

Use: $L(s) = \frac{1}{s(s+c)}$, $K = A$, $D = H = 1$, $c = 1$.
Closed-loop poles:

$$1 + K\frac{1}{s(s+1)} = 0 \;\Leftrightarrow\; s(s+1) + K = 0$$



Solutions for s:

$$r_{1,2} = \frac{-1 \pm \sqrt{1-4K}}{2}$$

$K = 0, \quad r_{1,2} = -1, \; 0$

$K = 0.25, \quad r_{1,2} = -0.5$

$K = 1, \quad r_{1,2} = -0.5 \pm j\frac{1}{2}\sqrt{3}$

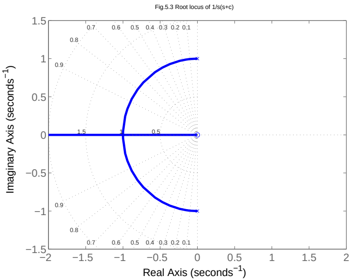
Matlab:

```
num=[0 0 1]; den=[1 1 0]; sys=tf(num,den);
rltool(sys)
```

Alteratives: `rlocus, sisotool`

Root locus as a function of system pole $-c$.
For $D = H = A = 1$,

$$1 + G(s) = 1 + \frac{1}{s(s+c)} = \frac{s(s+c)+1}{s(s+c)}$$



Characteristic equation:

$$s^2 + cs + 1 = 0$$

Bring this into form $KL(s)+1=0$:

$$c\frac{s}{s^2+1} + 1 = 0$$

With $c = K$, $L(s) = \frac{s}{s^2+1}$.
Poles of $L(s)$: $p_{1,2} = \pm j$, zero $z_1 = 0$.

Closed-loop poles: $r_{1,2} = -\frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}$

Phase-condition of root locus

Root-locus: $1 + KL(s) = 0$, $K \in \mathbb{R}$ and $K \geq 0$

$$\Leftrightarrow$$

$$L(s) = -\frac{1}{K} \quad \text{phase(L)} = \text{-}180^o$$

K can be controller gain k_P, but also

- k_I or k_D
- Any other variable system parameter (in D or G)

The root locus is composed of all points $s \in \mathbb{C}$ for which $\angle L(s) = -180^o$.

Construction of root-locus

$$1 + KL(s) = 0$$

$$\text{with } L(s) = \frac{b(s)}{a(s)}: \qquad a(s) + Kb(s) = 0$$

Step 1 - Start and end-points

$$\begin{array}{lll} K = 0 & a(s) = 0 & \Rightarrow \text{ poles of } L(s) \\ K = \infty & L(s) = \frac{-1}{K} = 0 & \Rightarrow \text{ zeros of } L(s) \end{array}$$

The root-locus starts in the poles and ends in the zeros of $L(s)$

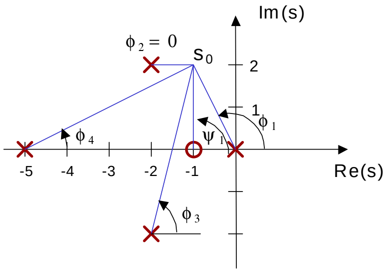
There are n finite poles and m finite zeros. If $n > m$: Step 3.

Step 2 - Real axis portion of root-locus

Example phase-condition

$$L(s) = \frac{s+1}{s(s+5)[(s+2)^2+4]} = \frac{s - z_1}{\prod_{i=1}^{4}(s - p_i)}$$

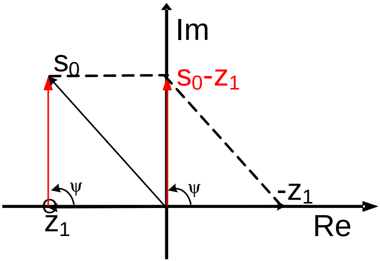
$$\angle(L(s_0)) = \angle(s_0 - z_1) - \sum_{i=1}^{4} \angle(s_0 - p_i)$$



$$z_1 = -1$$
$$p_1 = 0$$
$$p_4 = -5$$
$$p_{2,3} = -2 \pm 2j$$

$$\angle(L(s_0)) = \Psi_1 - \phi_1 - \phi_2 - \phi_3 - \phi_4 = -129.2$$

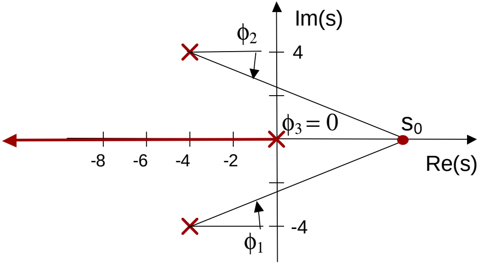
Analysis of the contribution of the zero:



Phase contribution Ψ of zero z_1= phase of $(s_0 - z_1)$
= angle that the vector that starts from z_1 and ends in s_0 makes with the real axis.

Similar holds for any of the poles (but ending up with a negative sign in $\angle L(s_0)$)

Step 2 - Real axis portion of root-locus



- For s_0 on real axis, only real-valued poles/zeros have a phase contribution
- Pole or zero left of s_0 has 0^o phase contribution
- Pole or zero right from s_0 has 180^o phase contribution

The part of the real axis that is to the left of an odd number of poles & zeros on the real axis, is on the root locus

Step 3 - Asymptotic behaviour for $K \to \infty$

$$L(s) = \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)}$$

n poles; m finite zeros; $n - m$ zeros in ∞
For $s \to \infty$, $L(s) \to 0$, and
$1/L(s)$ is dominated by the highest powers of s, i.e.:

$$1/L(s) \sim s^{n-m}.$$

A one-step more accurate approximation is obtained by taking

$$1/L(s) \sim (s - \alpha)^{n-m}$$

for a particular chosen value of α.

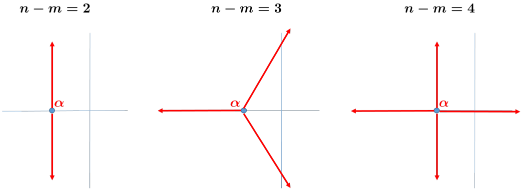
Locus asymptotes

For $K \to \infty$, there are $n - m$ roots in $s \to \infty$, and their behaviour is represented by the asymptotes of the root locus of

$$\frac{1}{(s-\alpha)^{n-m}} \quad \text{with } \alpha := \frac{\sum_i p_i - \sum_j z_j}{n-m}$$

which is

- $n - m$ straight lines that start in $s = \alpha$
- with angles symmetrically distributed around $\varphi = 180^o$.



Proof of root locus asymptotes:

Strategy:

- Write both $1/L(s)$ and $(s-\alpha)^{n-m}$ in a series expansion

$$s^{n-m} + x_1 s^{n-m-1} + \cdots$$

- and equate the first two terms in the expansion.
- This provides an "optimal" value for α.

Mathematical lemma

a. For a polynomial $\prod_{i=1}^{n}(s-p_i)$ the expansion $s^n + x_1 s^{n-1} + \cdots$ satisfies

$$x_1 = -\sum p_i$$

.

b. For polynomial $(s-\alpha)^{n-m}$ the expansion $s^{n-m} + x_1 s^{n-m-1} + \cdots$ satisfies

$$x_1 = -(n-m)\alpha$$

Proof: writing out the expressions.

Finding the optimal value of α

Equate the expansions of $1/L(s)$ and $(s-\alpha)^{n-m}$

1. Since $1/L(s) = a(s)/b(s)$ it follows that

$$a(s) = \underbrace{\left[s^{n-m} + x_1 s^{n-m-1} + \cdots\right]}_{1/L(s)} b(s)$$

$$s^n + a_1 s^{n-1} + \cdots = [s^{n-m} + x_1 s^{n-m-1} + \cdots][s^m + b_1 s^{m-1} + \cdots]$$

Then equating the terms for s^{n-1} (left and right) leads to

$$a_1 = b_1 + x_1$$

With part a of the Lemma: $a_1 = -\sum p_i$; $b_1 = -\sum z_i$, so

$$-\sum p_i = -\sum z_i + x_1.$$

So:

$$1/L(s) \sim s^{n-m} + x_1 s^{n-m-1} \text{ with } x_1 = \sum z_i - \sum p_i.$$

Finding the optimal value of α (cont'd)

Equate the expansions of $1/L(s)$ and $(s-\alpha)^{n-m}$

2. For large values of s we approximate:

$$(s-\alpha)^{n-m} \sim s^{n-m} + x_1 s^{n-m-1}, \text{ with } x_1 = -(n-m)\alpha$$

(according to Lemma b)

Equating the two expressions for x_1 now provides:

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}.$$

Behaviour of $L(s)$ for $s \to \infty$

For large values of s we can use the approximation

$$L(s) \sim \frac{1}{(s-\alpha)^{n-m}}$$

Root locus of the approximation

$$\frac{1}{(s-\alpha)^{n-m}}$$

For this approximation of $L(s)$, the root locus starts in $s = \alpha$

With $s_0 = re^{j\phi}$, for $r >> 1$: $\angle L(s_0) = -(n-m)\phi$ so in order to satisfy the angle condition of the root-locus:

$$(n-m)\phi = 180^o + \ell \cdot 360^o \qquad \ell = 0, \pm 1, \pm 2, \cdots$$

it follows that

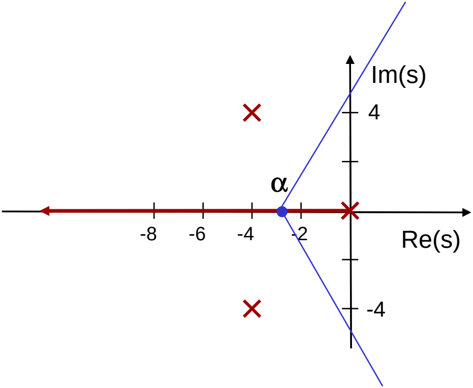
$$\phi = \frac{180^o + \ell \cdot 360^o}{n-m}$$

Asymptots are symmetrically distributed around $\phi = 180^o$

- $n-m=1$: $\phi = 180^o$
- $n-m=2$: $\phi = \pm 90^o$
- $n-m=3$: $\phi = 60^o,\ 180^o,\ 300^o$

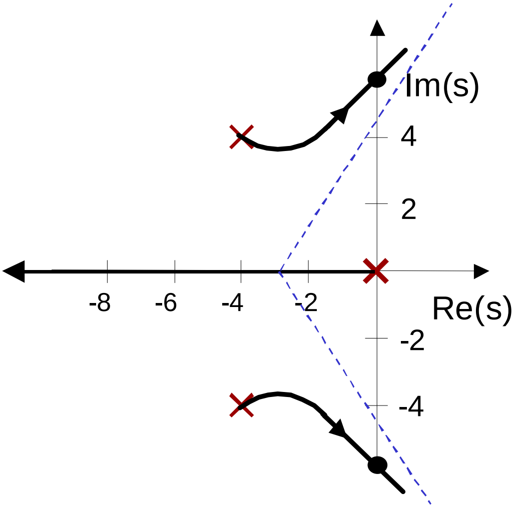
$$G(s) = \frac{1}{s\left[(s+4)^2+16\right]}$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$



$$\alpha = -\frac{8}{3}$$

3 finite poles, 3 zeros at infinity



Exact root locus to be calculated through Matlab.

Remarks on Root-locus

- Analysis tool rather than synthesis tool (former days)
- Gives insight into "role" of non-minimum-phase systems:

  Zeros in the right half plane limit the controller gain
  For increasing values of K, instability occurs

The negative root-locus (for negative values of K)

If $K < 0$ then

- All phase conditions become 0^o conditions (in stead of 180^o)
- Real-axis part of the root locus: to the left of an even number of poles/zeros
- Asymptotes appear symmetrically distributed around 0^o
- All other rules (including α) remain the same.

Sign changes of leading b-coefficient:

Positive root locus of $\frac{-s+2}{s(s+1)}$ is the negative root locus of $\frac{s-2}{s(s+1)}$

Summary root locus

- The root-locus displays the location of the closed-loop poles, as a function of a scalar gain K
- For $K = 0$ (the starting point) we get the open-loop poles
- The curves allow to analyse closed-loop stability as a function of K
- Simple rules drive the basic properties of the locus
  - Start in n poles of $L(s)$
  - End in m zeros of $L(s)$
  - If $n > m$ then $n - m$ asymptotes to ∞, distributed symmetrically around $\phi = 180^o$
- Automatic calculation in matlab through `rlocus`, `rltool` or `sisotool`