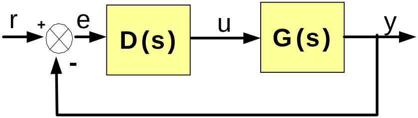
Control Systems
Lecture 3

Frequency response and neutral stability

Introduction



Objective:

- dynamically shape $D(j\omega)G(j\omega)$ to design the properties of the closed-loop system

  Map properties of DG to properties of the closed-loop $\frac{DG}{1+DG}$

  Tool: Shape the frequency response $D(j\omega)G(j\omega)$.

First address the question:

how can we manipulate the frequency response of a system?

(Bode plot analysis)

A. Objective (for the next few lectures)

Develop insight in how the properties of $D(j\omega)G(j\omega)$ relate to properties (stability, performance) of the closed loop system $\frac{DG}{1+DG}$.

If we know this relationship then we can design D to construct a suitable DG.

B. Preliminary for this

Understand how we can shape $D(j\omega)G(j\omega)$ by designing $D(j\omega)$.

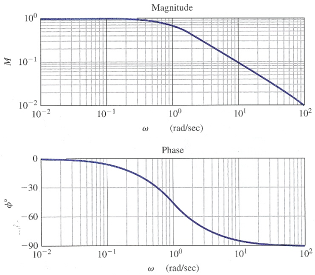
How can we multiply frequency responses? (Bode plot analysis)

Frequency response

The frequency response $G(j\omega)$ characterizes the dynamics of an LTI system, through its

$$\text{amplitude } |G(j\omega)| = \sqrt{\{\text{Re}[G(j\omega)]\}^2 + \{\text{Im}[G(j\omega)]\}^2},$$

$$\text{phase } \angle G(j\omega) = \arctan\left[\frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}\right]$$



Magnitude in dB:

$$A = 20 \cdot {}^{10}\log |G(j\omega)|$$

Composition of frequency responses, for distinct real-valued poles/zeros

From the general transfer function representation:

$$KG(s) = K\frac{(s - z_1)(s - z_2)\cdots}{(s - p_1)(s - p_2)\cdots}$$

we construct the alternative form:

$$KG(s) = K_0\frac{(s\tau_1 + 1)(s\tau_2 + 1)\cdots}{(s\tau_a + 1)(s\tau_b + 1)\cdots}$$

by simply scaling the numerator and denominator.
Note that $\tau_1 = -\frac{1}{z_1}$ etc., and $\tau_a = -\frac{1}{p_1}$ etc, real-valued
If there are poles and/or zeros in 0 then we expand to

$$KG(s) = K_0 s^n\frac{(s\tau_1 + 1)(s\tau_2 + 1)\cdots}{(s\tau_a + 1)(s\tau_b + 1)\cdots}$$

where n can be positive or negative.

For $s = j\omega$ the Bode form results:

$$KG(j\omega) = K_0 \cdot (j\omega)^n \cdot \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}$$

Amplitude response

$$\begin{aligned} \log|KG(j\omega)| &= \log K_0 + n\log|j\omega| + \\ &\quad + \log|j\omega\tau_1 + 1| + \log|j\omega\tau_2 + 1| + \\ &\quad - \log|j\omega\tau_a + 1| - \log|j\omega\tau_b + 1| \end{aligned}$$

Amplitude response

- Each pole and zero has its own -independent- contribution to $\log|KG(j\omega)|$
- All contributions are simply added (addition of log curves) to compose the total response

For $s = j\omega$ the Bode form results:

$$KG(j\omega) = K_0 \cdot (j\omega)^n \cdot \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}$$

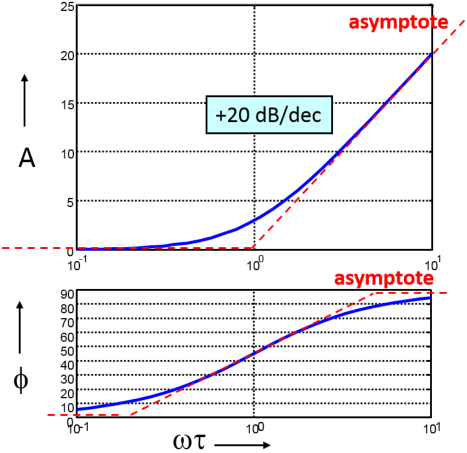
Phase response

$$\begin{aligned}\angle(KG(j\omega)) &= n\angle(j\omega) + \angle(j\omega\tau_1 + 1) + \angle(j\omega\tau_2 + 1) + \\ &\quad -\angle(j\omega\tau_a + 1) - \angle(j\omega\tau_b + 1)\end{aligned}$$

Phase response

- Each pole and zero has its own -independent- contribution to $\angle(KG(j\omega))$
- All contributions are simply added (addition of phase curves) to compose the total response

Response of $(j\omega\tau + 1)$



$$\text{for } \omega\tau \ll 1 \quad \rightarrow \quad j\omega\tau + 1 \approx 1$$
$$\text{for } \omega\tau \gg 1 \quad \rightarrow \quad j\omega\tau + 1 \approx j\omega\tau$$

Plot explanation: next slide

$$\text{for } \omega\tau \ll 1 \quad \rightarrow \quad \angle 1 = 0^o$$
$$\text{for } \omega\tau \gg 1 \quad \rightarrow \quad \angle j\omega\tau = 90^o$$
$$\text{for } \omega\tau = 1 \quad \rightarrow \quad \angle(j\omega\tau + 1) = 45^o$$

Break point is $\omega\tau = 1 \rightarrow \omega = 1/\tau$ rad/sec.

Note that for $\omega\tau \gg 1$

$$log|KG(j\omega)| \simeq log|j\omega\tau| = log\omega + log\tau$$

i.e. line with slope 1 in a log-log plot

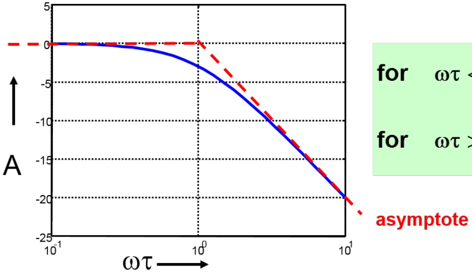
Since

$$|KG(j\omega)|_{db} = 20log|KG(j\omega)|$$

this becomes a line with slope 20 in the dB-plot $\Longrightarrow$

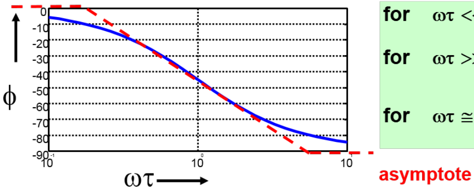
20 dB increase in amplitude plot for each decade in frequency

Response of $\dfrac{1}{(j\omega\tau + 1)}$



$$\text{for} \quad \omega\tau \ll 1 \quad \rightarrow \quad \frac{1}{j\omega\tau + 1} \cong 1$$

$$\text{for} \quad \omega\tau \gg 1 \quad \rightarrow \quad \frac{1}{j\omega\tau + 1} \cong \frac{1}{j\omega\tau}$$



$$\text{for} \quad \omega\tau \ll 1 \quad \rightarrow \quad \angle 1 = 0^\circ$$

$$\text{for} \quad \omega\tau \gg 1 \quad \rightarrow \quad \angle \frac{1}{j\omega\tau} = -90^\circ$$

$$\text{for} \quad \omega\tau \cong 1 \quad \rightarrow \quad \angle \frac{1}{(j\omega\tau + 1)} \cong -45^\circ$$

Low frequency behaviour

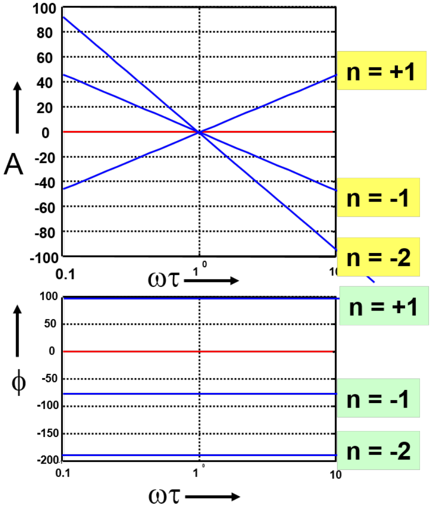
$$KG(j\omega) = K_0 \cdot (j\omega)^n \cdot \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}$$

For $\omega \to 0$:

$$KG(j\omega) \to K_0 \cdot (j\omega)^n$$

n determines the slope of the frequency response when $\omega \to 0$ (steady state gain)

Response of $n \log |j\omega|$



$$\log K_0 \left|(j\omega)^n\right| = \log K_0 + n \log |j\omega|$$

$$\phi = n \times 90^\circ$$

Example:

$$KG(s) = \frac{2000(s+0.5)}{s[(s+10)(s+50)]}$$

STEP 1

$$KG(j\omega) = \frac{2000(j\omega+0.5)}{j\omega[(j\omega+10)(j\omega+50)]} = \frac{2\left(\frac{j\omega}{0.5}+1\right)}{j\omega\left[\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)\right]}$$

STEP 2

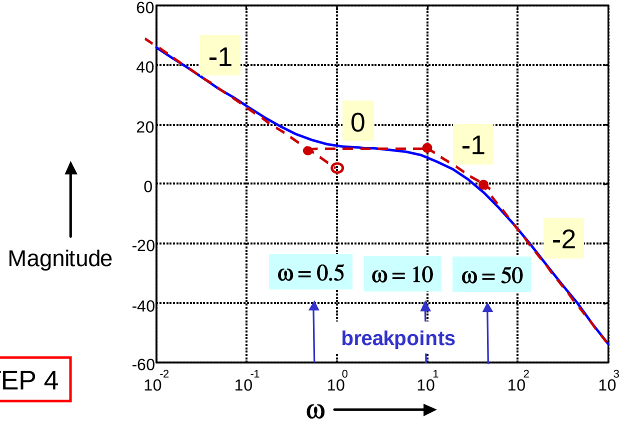
low frequency: $KG(j\omega) = \frac{2}{j\omega}$

for: $\omega = 0.1 \rightarrow |G| = \frac{2}{0.1} = 20 \Leftrightarrow$ **Magnitude is 26dB**

$$KG(s) = \frac{2000(s+0.5)}{s[(s+10)(s+50)]}$$

STEP 3 Remaining asymptotes

$$\omega = 1 \quad \rightarrow \quad |G| = 6\ \text{dB}$$

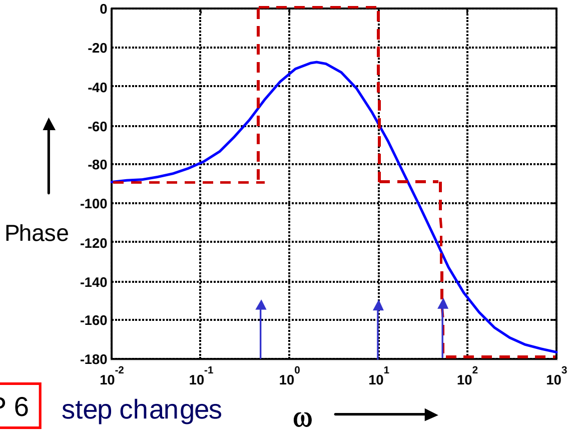


STEP 4

$$KG(s) = \frac{2000(s+0.5)}{s[(s+10)(s+50)]}$$

STEP 5

Phase starts at -90^0



STEP 6

step changes

Manipulation of frequency responses

The construction of Bode diagrams shows how we can manipulate frequency responses of systems, e.g. through multiplication with a controller: $G(j\omega) \rightarrow D(j\omega)G(j\omega)$

This is a mechanism of simply adding poles and zeros

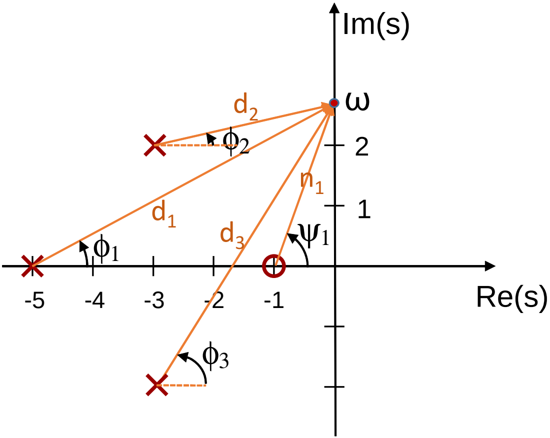
Construction of Bode Plots from pole/zero's

[Recall phase condition in root-locus plot (slides 2-14&15)]

$$G(s) = K \frac{\prod_k (s - z_k)}{\prod_i (s - p_i)}$$

leading to

$$\begin{aligned} |G(j\omega)| &= |K| \frac{\prod_k |j\omega - z_k|}{\prod_i |j\omega - p_i|} \\ \angle G(j\omega) &= \angle K + \sum_k \angle(j\omega - z_k) - \sum_i \angle(j\omega - p_i) \end{aligned}$$

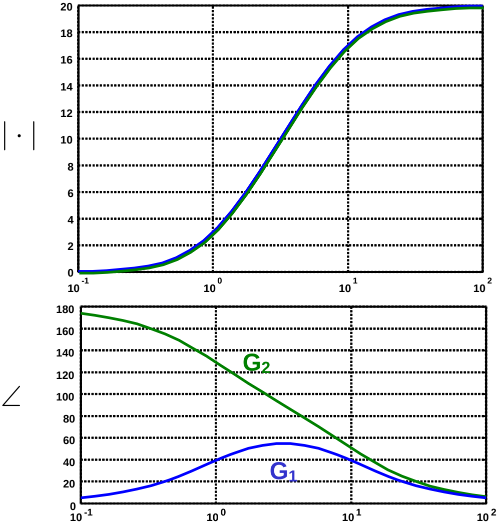


$$
\begin{aligned}
|G(j\omega)| &= |K| \frac{n_1}{d_1 \cdot d_2 \cdot d_3} \\
\angle G(j\omega) &= \angle K + \Psi_1 - \Phi_1 - \Phi_2 - \Phi_3
\end{aligned}
$$

Questions:

1. What happens with $|G(j\omega_0)|$ if G has a zero in $j\omega_0$?
2. What happens with $|G(j\omega_0)|$ if G has a pole in $j\omega_0$?

Non-minimum phase systems



$$G_1(s) = 10\frac{s+1}{s+10}$$

$$G_2(s) = 10\frac{s-1}{s+10}$$

Questions:

- Can we explain the phase change at $\omega = 0$ when a zero swaps from the LHP to the RHP?
- Can we explain why the magnitude Bode plot remains unchanged then for all ω?

Second order (two-pole) systems

Reasoning so far has been based on distinct real-valued poles/zeros

The characteristics of 2nd order (two-complex-pole) systems through

- step response properties: t_r, M_p, t_s
- pool locations: ω_n, ζ, σ

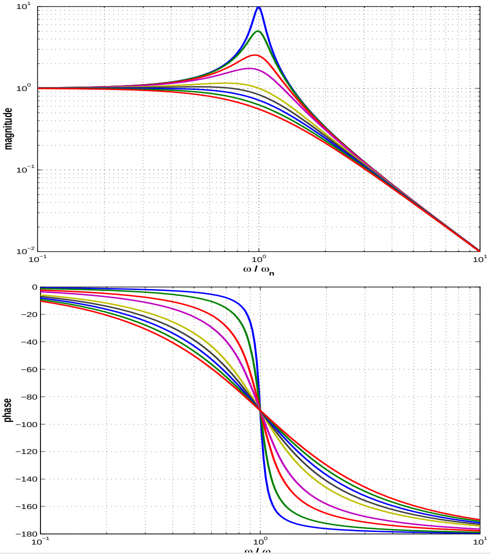
can also be phrased in properties of the frequency response:

$$G(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1}$$

"Asymptots" are characterized by evaluating

$$\omega/\omega_n >> 1 \quad \text{and} \quad \omega/\omega_n << 1$$

Frequency response for second order (two-pole) system



$\zeta = 0.05$

$\zeta = 0.1$

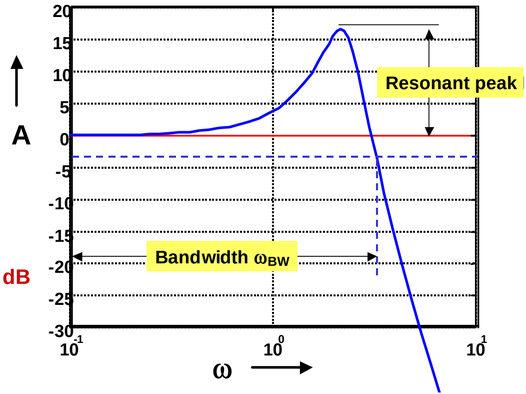
$\zeta = 0.6$

$\zeta = 0.9$

the less damping the higher the resonance peak

the frequency axis scales with ω_n

Resonance peak and bandwidth



For $\zeta < 0.5$: Resonance peak $M_r \simeq \dfrac{1}{2\zeta}$ (not in dB)

Summary Bode plots

- Frequency response properties characterized by stylistic curves with defined slopes
- Amplitude and phase properties of a system are analyzed (and predicted) relatively easily
- Multiplication of transfers is reflected by addition of log-curves in the ω-domain
- Several system properties can be analyzed with the frequency response

Closed-loop properties on the basis of open-loop Bode plots

How do properties of DG map to properties of the closed-loop system $\frac{DG}{1+DG}$?

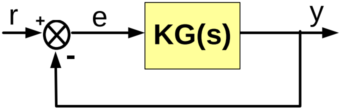
- Steady state response of the closed-loop → next
- Neutral stability (for simple systems) → next
- Nyquist stability (general case) → Lecture 4
- Bandwidth and damping → Lecture 5

Based on this mapping we can design controllers D to "shape" DG.

Steady state errors

The steady state errors $e(\infty)$ for

$$r(t) = \begin{cases} 1(t) & \text{step} \\ t & \text{ramp} \\ \frac{1}{2}t^2 & \text{parabola} \end{cases}$$



are determined by $\lim_{s\to 0} KG(s)$

This is based on the finite value theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+KG(s)} R(s)$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s}{1+KG(s)} R(s)$$

$$r(t) = \begin{cases} 1(t) & \text{step} \\ t & \text{ramp} \\ \frac{1}{2}t^2 & \text{parabola} \end{cases} \qquad R(s) = \begin{cases} \frac{1}{s} \\ \frac{1}{s^2} \\ \frac{1}{s^3} \end{cases}$$

For a step signal:

$$\lim_{t\to\infty} e(t) = \frac{1}{1+\lim_{s\to 0} KG(s)}$$

If $\lim_{\omega\to 0} |KG(j\omega)| = \infty$ then $\lim_{t\to\infty} e(t) = 0$.

This happens if the slope of the Bode plot for $\omega << 1$ is e.g. -1 or smaller.

Overview Steady state tracking errors

$$K_p := \lim_{s \to 0} KG(s)$$
$$K_v := \lim_{s \to 0} sKG(s)$$
$$K_a := \lim_{s \to 0} s^2 KG(s)$$

Steady state errors to selected inputs:

slope of $\|KG\|$ in $\omega = 0$	step	ramp	parabola	Syst. Type
0	$\frac{1}{1+K_P}$	∞	∞	0
-1	0	$\frac{1}{K_v}$	∞	1
-2	0	0	$\frac{1}{K_a}$	2

The higher the slope of $|KG(j\omega)|$ in $\omega = 0$, the better steady state properties for “faster” signals

For transfer $r \to y$ the system $KG(s)$ is defined to be of system type n if

$$\lim_{s \to 0} s^n KG(s) = c \neq 0,$$

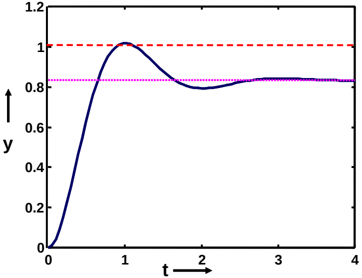
i.e. it is the degree of the factor

$$\frac{1}{s^n}$$

that appears in the transfer function $KG(s)$,
and is equal to the negative slope of the Bode plot

$$|KG(j\omega)| \quad \text{for } \omega << 1$$

Example step input



$$R(s) = \frac{1}{s}$$

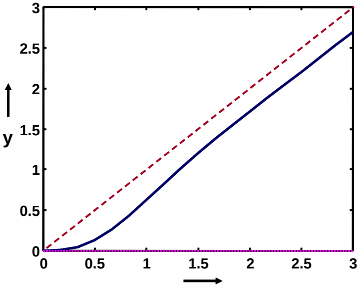
$$KG(s) = \frac{10}{(s+1)(s+2)}$$

$$K_p = \lim_{s \to 0} KG(s) = 5$$

$$E_{ss} = \frac{1}{1+K_p} = \frac{1}{6}$$

System type is 0; slope of $|KG(j\omega)|$ at $\omega = 0$ is 0.

Example ramp input



$$R(s) = \frac{1}{s^2}$$

$$KG(s) = \frac{10}{s(s+3)}$$

$$K_v = \lim_{s \to 0} sKG = 10/3$$

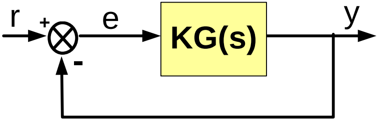
$$E_{ss} = \frac{1}{K_v} = \frac{3}{10}$$

System type is 1; slope of $|KG(j\omega)|$ at $\omega = 0$ is -1.

Summary

Steady state (control) errors are determined by the slope of the magnitude Bode plot for $\omega \to 0$.

Neutral stability



Closed-loop Stability

Which properties of KG ensure stability of $KG/(1+KG)$?

- Evaluation from Bode plot: Neutral stability (special case)

Later:

- Nyquist stability criterion based on the Nyquist plot (Bode amplitude and phase plot combined in 1 figure)
- Engineering approach through *Bode Gain-Phase relationship* in Bode plots

Neutral stability point

From Bode plot

Points on root locus (closed-loop poles) determined by $1 + KG(s) = 0$, i.e.

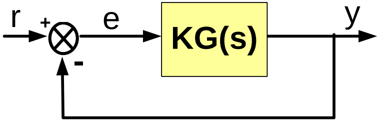
$$|KG(s)| = 1 \qquad \angle(G(s)) = 180^o$$

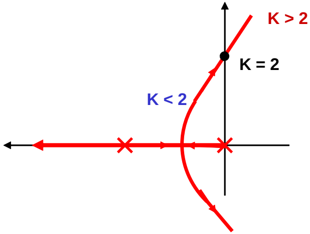
At the boundary of stability ($s = j\omega$), there exists a K^* such that:

$$|K^*G(j\omega)| = 1 \qquad \angle(G(j\omega)) = 180^o$$

For all systems where increasing gain (K) leads to instability:
Sufficient condition for stability: $K < K^*$, or

$$|KG(j\omega)| < 1 \quad \text{at } \omega \text{ where } \angle(G(j\omega)) = 180^o$$





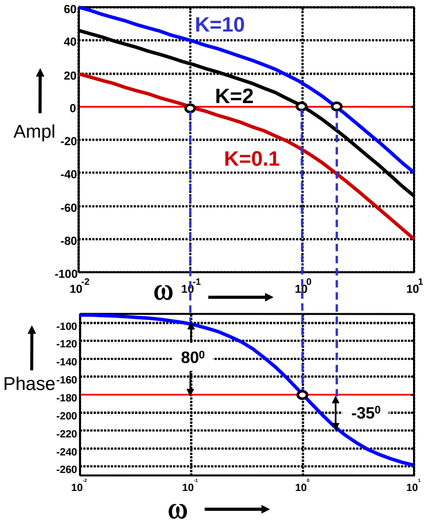
$$G(s) = \frac{1}{s(s+1)^2}$$

Root locus:

$$|KG(s)| = 1 \quad \text{and} \quad \angle G(s) = 180^o$$

For $K = 2$:

$$|KG(j\omega)| = 1 \quad \text{and} \quad \angle G(j\omega) = 180^o$$



Closed-loop system is stable if:

$$|KG(j\omega)| < 1$$

at ω where

$$\angle KG(j\omega) = 180^o$$

$$\Longrightarrow K < 2$$

This reasoning only applies to systems for which increasing gain leads to instability

Question:
Which systems fall outside of this category?

Summary

- Frequency responses can be understood with stylisic curves with well-defined slopes
- Multiplication of transfers is reflected by addition of log-curves in the ω-domain
- Mapping characteristics of KG to properties of the closed-loop system:
  - Steady state (control) errors are determined by slope at $\omega = 0$ in magnitude Bode plot of KG;
  - Neutral stability analysis (Bode plot) for "simple" systems: Loop gain $|KG(j\omega)| < 1$ at frequencies where $\angle KG(j\omega) = 180^o$

- Next:
  Designing controllers based on f-domain tuning.