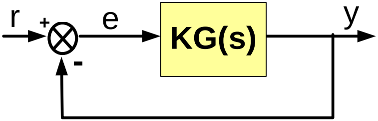
Control Systems
Lecture 5

Closed-loop Stability - Nyquist criterion

Introduction

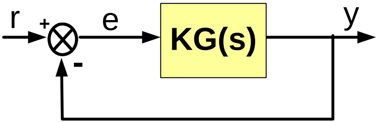


Closed-loop Stability

In Lecture 4:

- Shaping $KG(j\omega)$ so as to achieve attractive properties of the closed-loop system $KG/(1+KG)$;
- Stability considerations limited to neutral stability:
  - Considering systems for which increasing K leads to instability
  - Systems whose root locus start in the left half plane
  - Stable systems (including poles at the imaginary axis)

Introduction



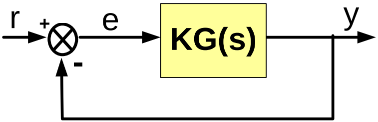
Closed-loop Stability

In Lecture 5:

- Analyzing stability in the general case (including unstable systems G)

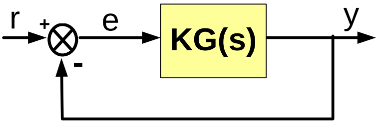
Tools and method: **Nyquist plot and Nyquist stability criterion**

Introduction



Note that in the above feedback loop, $KG(s)$ represents the product of plant and controller. The controller can be *any* dynamic controller, not necessarily a gain only. Its dynamics is then incorporated in $G(s)$.

Introduction



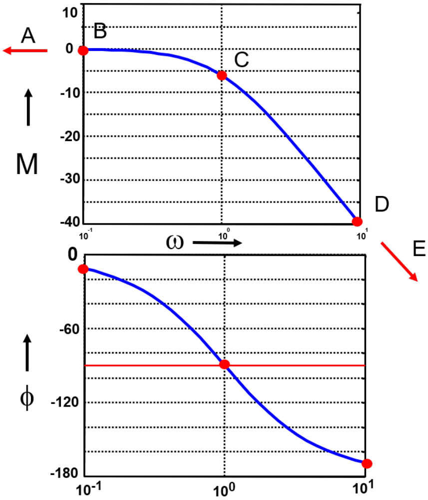
Our target remains:
Which properties of $KG(j\omega)$ ensure stability of $KG/(1+KG)$?

Nyquist plot

Nyquist plot

The Nyquist plot of $G(s)$ combines the Bode plots of $G(j\omega)$ and $G(-j\omega)$ in one (polar) figure in the complex plane.

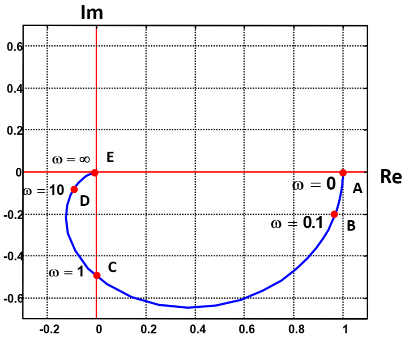
Example:



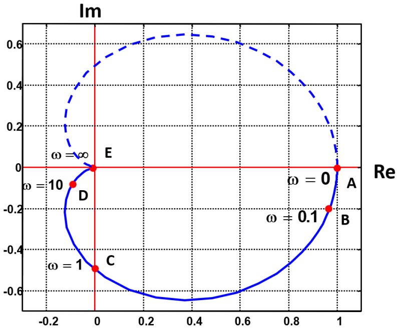
Bode plot

$$G(s) = \frac{1}{(s+1)^2}$$

Rather than separate amplitude and phase plots, now combined in the complex plane:



and complemented with the negative ω-part:



creates a closed contour in the complex plane: **(Nyquist plot)**

Nyquist stability criterion (in brief)

Let

N:= number of encirclement of Nyquist plot around the point -1

P:= number of RHP (unstable) poles of $KG(s)$

Z:= number of RHP (unstable) poles of $KG/(1+KG$.

Then

$$Z = N + P$$

For stable systems: $P = 0$, and closed-loop system is stable if the Nyquist plot does not encircle -1.

Nyquist stability criterium (see weblecture)

(Nyquist - 1932, Bell Labs)

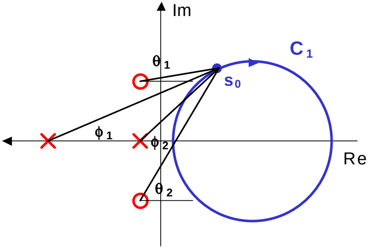
Thorough evaluation of the closed-loop stability of $KG(s)$ on the basis of its evaluation in the complex plain.

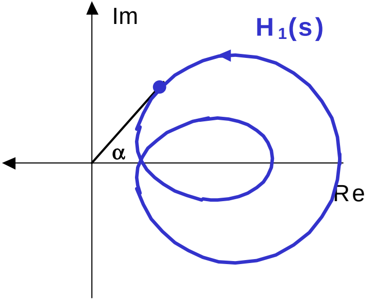
We know: stability property if all poles are in the LHP

How can a conclusion on closed-loop stability result from a function evaluation of the open-loop system $KG(s)$?

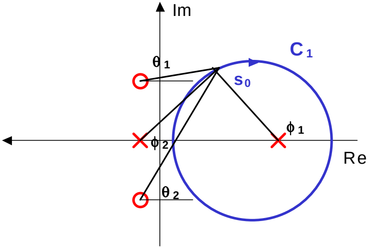
Why is this relevant?
Unlike determining the closed-loop poles for a given system (and checking closed-loop stability), the Nyquist plot indicates how a control design can influence stability

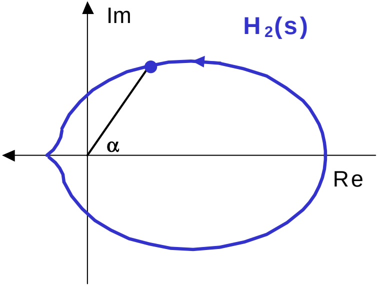
Evaluate any transfer $H_1(s)$ (represented by 2 poles and 2 zeros) over the closed contour C_1, and draw this contour map in the complex plain:





$$\alpha = \theta_1 + \theta_2 - \phi_1 - \phi_2$$





Argument principle:

Contour C_1 contains a singularity of $H(s)$ (pole/zero)

$$\Leftrightarrow$$

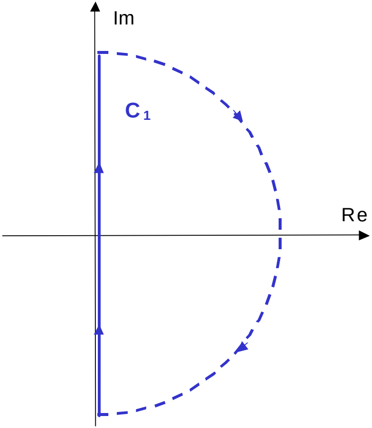
Contourmap of $H(s)$ encircles the origin

Clockwise encirclement of pole in $C_1 \Rightarrow$
$H(s)$ shows counter clockwise encirclement of 0

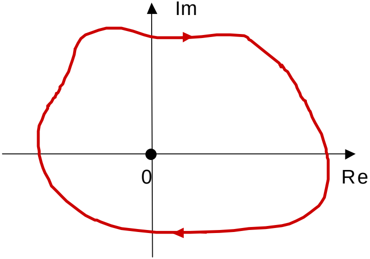
Clockwise encirclement of zero in $C_1 \Rightarrow$
$H(s)$ shows clockwise encirclement of 0

Use argument principle to study stability, i.e. existence of poles in the right half plane (RHP)

Choose contour C_1 = RHP:



Contourplot of $H(s)$ tells us something about RHP zeros and poles of $H(s)$



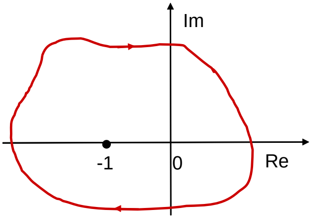
Applying to feedback control systems:

With the closed-loop transfer: $\frac{KG(s)}{1+KG(s)}$, closed-loop stability is represented by the solutions of:

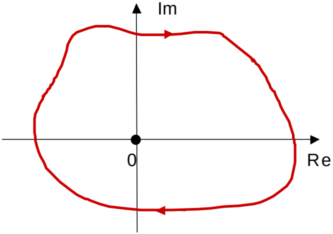
$$1 + KG(s) = 0$$

Contourplots of $KG(s)$ and $1+KG(s)$ are closely related:

$$[KG(s)]_{s\in C_1}$$



$$[1 + KG(s)]_{s\in C_1}$$



Line of reasoning

- If contour C_1 is defined by the right half plane (RHP)
- Then the plot of $KG(s)$ over C_1 can provide us with
- Information on the RHP zeros of $1 + KG(s)$
- Leading us to information on the RHP poles of the closed loop $KG/(1 + KG)$

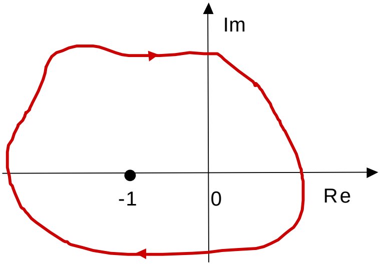
We will always count encirclements in clockwise direction

i.e.
an encirclement in a counterclockwise direction is counted as -1

Number of encirclements around -1 is counted by counting the number of times that the Nyquist curve crosses the real-axis line segment $(-\infty, -1)$;
in upwards direction $= +1$, in downwards direction $= -1$.

$\#$ encirclements of $1 + KG(s)$ around $0 =$
$\#$ zeros of $1 + KG(s)$ in C_1 -
$\#$ poles of $1 + KG(s)$ in C_1

$\#$ encirclements of $KG(s)$ around -1 (N) $=$
$\#$ poles of $KG/(1 + KG)$ in C_1 (Z) -
$\#$ poles of $KG(s)$ in C_1 (P)



$$Z = N + P$$

Number of unstable poles of $KG/(1+KG)$ (Z) $=$
Number of encirclements of $KG(s)$ around -1 (N) $+$
Number of (open-loop) unstable poles of KG (P)

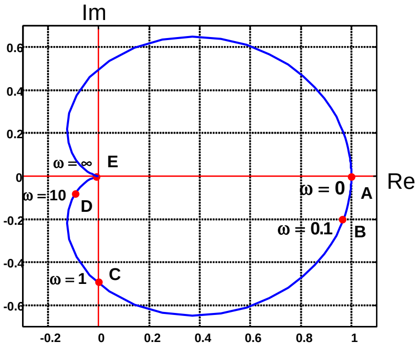
For open-loop stable plants and controllers (KG stable): $P = 0$ and
Closed-loop stability iff KG does not encircle -1

Contourmap of KG over C_1 is the Nyquist plot

Nyquist plot of G is evaluation of $G(s)$ over $s = C_1$ (RHP)

Example

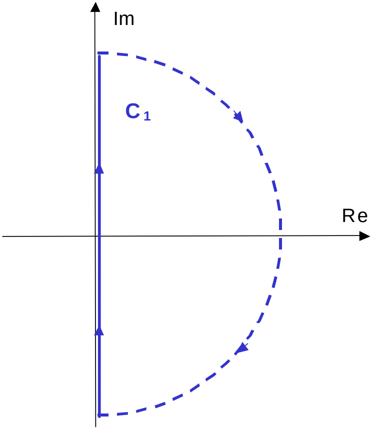
$$G(s) = \frac{1}{(s+1)^2}$$



The C_1 contour:

- $s = j\omega$ from $\omega = 0 \to \infty$
- at $s = \infty$ the curve around the RHP
- $s = j\omega$ from $\omega = -\infty \to 0$

Contour C_1:



- Since almost all physical systems have diminishing gain for $\omega \to \infty$, the function values on the dashed part of the contour are 0
- The relevant part of the contour becomes $s = j\omega$, $\omega \geq 0$ (same domain as for the Bode plot)
- Since $H(-j\omega) = H(j\omega)^*$ the Nyquist curve is composed of two parts which are symmetric w.r.t. the real axis.

Matlab commands:

```
numG = 1;
denG = [1 2 1];
sysG = tf(numG,denG);
nyquist(sysG)
axis([-1.2 1.2 -1 1])
```

Procedure for determining Nyquist stability

- Plot $KG(j\omega)$ for $\omega = 0 \to \infty$
- Complement the plot with its mirrored part w.r.t. the real axis
- Evaluate the number of clockwise encirclements of -1 and call this N
  (count the crossings of the line segment $(-\infty, -1)$)
- Determine the number of unstable poles (within the contour C_1) of $KG(s)$ and call this P
- The number of unstable closed-loop poles (within the contour C_1) is equal to

$$Z = N + P$$

Choice of contour C_1:

- C_1 is the contour within which we are going to evaluate the presence of closed-loop poles
- it is typically chosen as the open right half plane:
- C_1 should not run through singular points (poles or zeros) of KG
- If KG has a pole/zero on the imaginary axis: C_1 runs at an infinitesimal distance around this singularity (on the RHP side) - See book Figure 6.27 and next slide.
- If one would like to exclude (closed-loop) poles on the imaginary axis, then C_1 should be chosen to include the imaginary axis.

Summary Nyquist stability

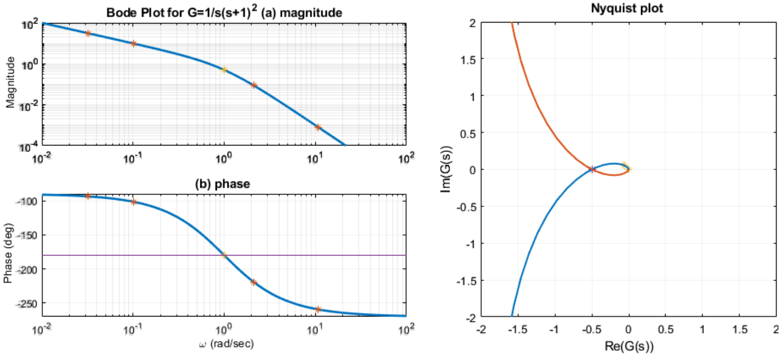
- Nyquist plot is the combination of Bode amplitude and phase plots in one figure
- Provides a basis for analysing closed-loop stability on the basis of open-loop frequency response
- Can handle all situations (including unstable and non minimum-phase plants)
- Therefore more general than the previous analysis of Bode plots only

If $KG(s)$ has a pole on the imaginary axis (e.g. in $s = 0$) then special care needs to be taken

- For $\omega = 0$ the Nyquist plot starts in ∞
- The combined Bodeplots for $j\omega$ and $-j\omega$ do not necessarily form a closed contour.
  (They meet in ∞)

Example:

$$KG(s) = \frac{K}{s(s+1)^2}$$

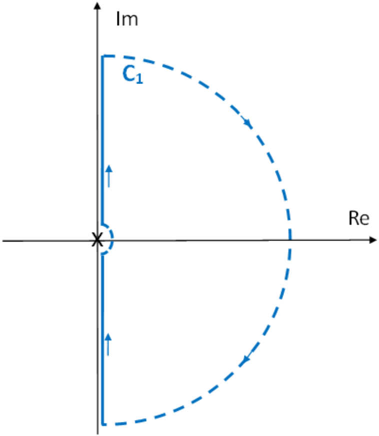


Question: how to create a closed contour in the Nyquist plot?

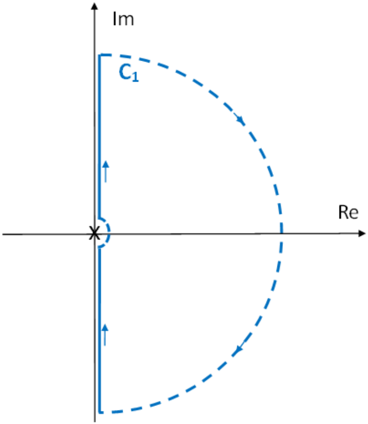
This is related to the choice of the contour C_1 in the complex plane, covering the RHP:

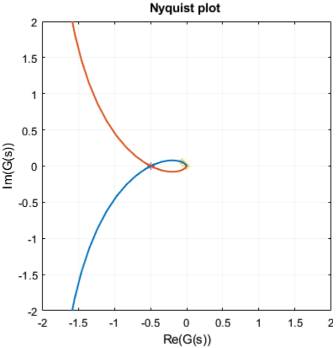
- C_1 is the contour within which we are going to evaluate the presence of closed-loop poles
- it is typically chosen as the open right half plane:
- C_1 should not run through singular points (poles or zeros) of KG
- If KG has a pole/zero on the imaginary axis: C_1 runs at an infinitesimal distance around this singularity (on the RHP side) - See book Figure 6.27 and next slide.
- If one would like to exclude (closed-loop) poles on the imaginary axis, then C_1 should be chosen to include the imaginary axis.

Choice of contour C_1 when $KG(s)$ has a pole in $s = 0$:

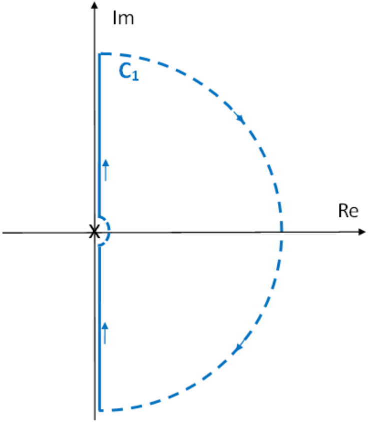


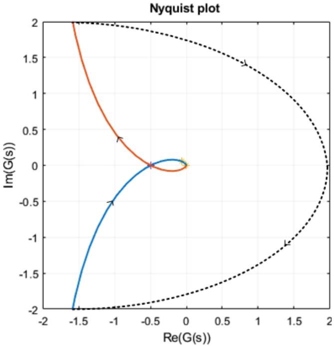
The Nyquist stability criterion then evaluates closed-loop RHP poles within C_1 (excluding $s = 0$).





Behaviour of $KG(s)$ around $s = 0$ determines behaviour in Nyquist plor around ∞.



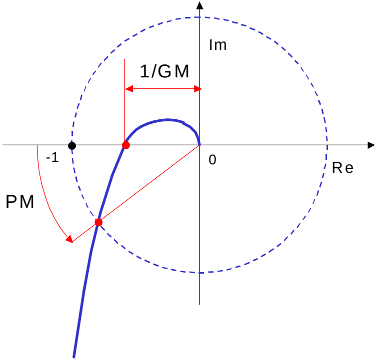


For $s = \alpha$ (real valued, $0 < \alpha << 1$), $KG(\alpha) = \dfrac{K}{\alpha(\alpha+1)^2}$, and $\angle KG(\alpha) = 0^o$, $\Longrightarrow$
The Nyquist plot is closed by crossing the 0^o line in infinity.

Stability margins

- Stability margins
  - Gain margin (GM) and phase margin (PM) to quantify the "safety distance" from instability
  - Relation between PM and damping (see lecture 4)

Stability margins defined on the basis of the Nyquist plot of $KG(s)$



GM: factor by which the gain can be increased

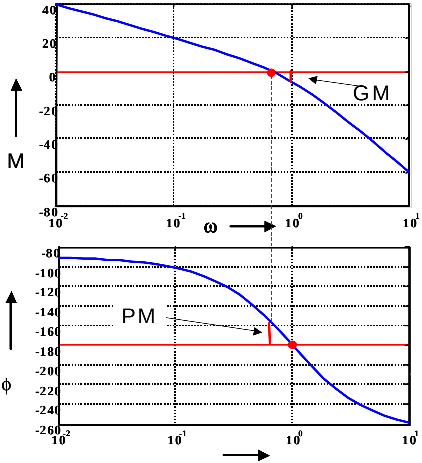
PM: amount by which the phase can be decreased before reaching instability

PM: phase margin:

phase-distance to 180^o at $\omega = \omega_c$, where $|KG(j\omega_c)| = 1$;

GM: gain margin

Gain and Phase Margin in Bode plot



GM and PM act as design objectives when designing controllers

Bode's Gain-Phase Relationship

To be used for a limited class of systems for guaranteeing stability:

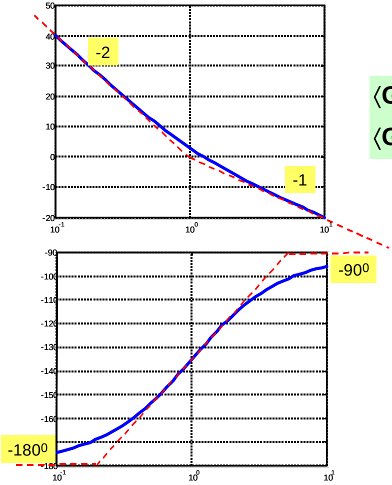
Bode's Gain-Phase Relationship

For every stable minimum-phase system:
$\angle G(j\omega)$ is uniquely determined by $|G(j\omega)|$.

If $|G(j\omega)|$ has slope -1: $\angle G(j\omega) \approx -90^o$
If $|G(j\omega)|$ has slope -2: $\angle G(j\omega) \approx -180^o$
etc.

For these special systems: If amplitude slope determines phase, then one could conclude stability on the basis of the amplitude characteristic only.

Bode's Gain-Phase relationship



$$\angle G(j\omega) \cong -90^\circ \quad \text{if} \quad n = -1$$
$$\angle G(j\omega) \cong -180^\circ \quad \text{if} \quad n = -2$$

at **crossover**
decade n = -1

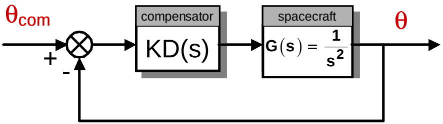
For "safe" stable control design $KD(s)$:
If around the Crossover frequency ω_c determined by

$$|KD(j\omega_c)G(j\omega_c)| = 1$$

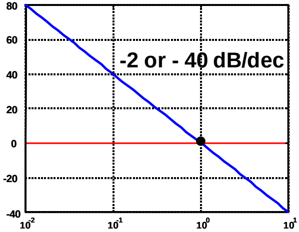
we can take care that the slope of $|KD(j\omega)G(j\omega)|$ is -1 then we have a guaranteed phase margin of 90^o.

Example

Spacecraft attitude control



Magnitude G(s)



phase shift

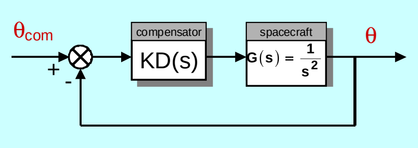
-180^0

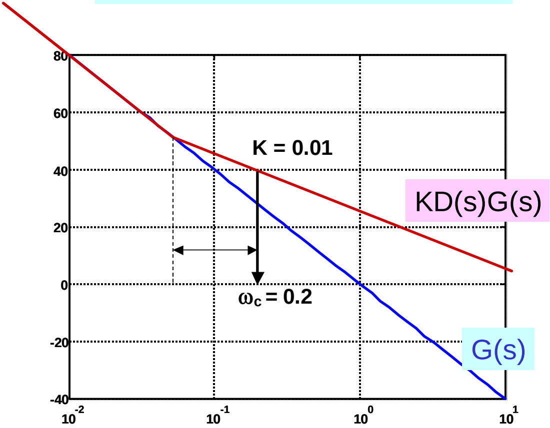
Objective

Well-damped control system with a bandwidth of around 0.2 rad/sec.

Start by setting

$$\omega_c \approx \omega_{BW} = 0.2 \text{ rad/sec}$$


θcom
+
-
compensator
KD(s)
spacecraft
G(s) = 1/s^2
θ


80
60
40
20
0
-20
-40
10^-2
10^-1
10^0
10^1
K = 0.01
ωc = 0.2
KD(s)G(s)
G(s)

Strategy

- Add a zero so that slope becomes -1 rather than -2 in the importent ω-range
- Structure for this:

$$KD(s) = K(T_D s + 1)$$

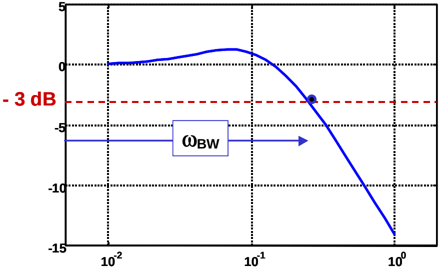
(PD action)

- Choose break point of controller at $\omega = 0.05$ rad/sec, in order to ensure full phase effect at $\omega = 0.2$, i.e.

$$KD(s) = K(20s + 1)$$

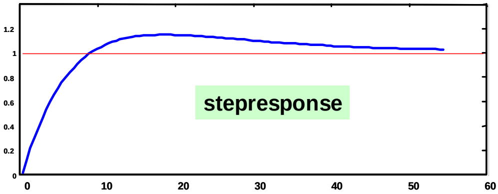
- Adapt K to guarantee a cross-over frequency of $\omega_c = 0.2$ rad/sec:

$$\left|\frac{K(20j\omega_c + 1)}{(j\omega_c)^2}\right| = \left|\frac{K(j4 + 1)}{0.04}\right| = 1 \Rightarrow K \approx 0.01$$



magnitude T(s)

$$T(s) = \frac{KD(s)G(s)}{1 + KD(s)G(s)}$$



Summary

- Nyquist plot combines both Bode plots
- Nyquist stability criterion for general situation
- Stability margins for "distance measures" towards instability
- Bode's gain-phase relation for stable minimum-phase plants