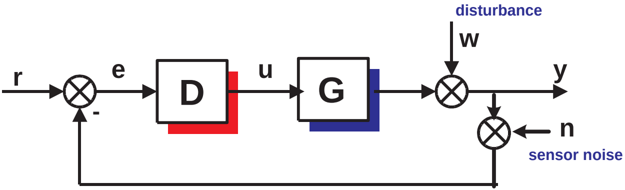
Control Systems
Lecture 6

Design trade-offs and the role of sensitivity functions

Design trade-offs

- From specific design tools (loop shaping, lead and lag, PID) for stability/damping and reference tracking, to more general design considerations and trade-offs
- Important role is being played by sensitivity function
- Sensitivity function is constrained by an important limitation: Bode's sensitivity integral (waterbed effect)
- Controllers should be robust with respect to model uncertainties

Design trade-offs: sensitivity functions



$$
\begin{aligned}
Y &= W + GD(R - N - Y) \\
(1+GD)Y &= W + GD(R - N) \\
Y &= \frac{1}{1+GD}W + \frac{GD}{1+GD}(R - N)
\end{aligned}
$$

Sensitivity function $\dfrac{1}{1+GD}$ is transfer from outer loop to inner loop signal

$$
\begin{aligned}
R - Y &= R - (1+GD)^{-1}GD(R-N) - (1+GD)^{-1}W \\
&= \underbrace{\frac{1}{1+GD}}_{S}(R-W) + \underbrace{\frac{GD}{1+GD}}_{T}N
\end{aligned}
$$

Sensitivity S: small in relevant area of R and W

Complementary sensitivity T: small in relevant area of N

Fundamental limitation in linear control:

$$S(j\omega) + T(j\omega) = 1 \qquad \forall\omega$$

Note: sum of complex numbers.

One cannot simultaneously reduce the effect of disturbances and sensor noise on the output, if they occur at similar frequencies

For "physical" plants:

$$|D(j\omega)G(j\omega)| \to 0 \quad \text{for } \omega \to \infty$$

Therefore:

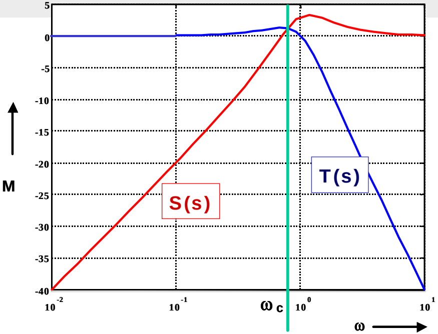
$$\begin{aligned} |S(j\omega)| &\to 1 \quad \text{for } \omega \to \infty \\ |T(j\omega)| &\to 0 \quad \text{for } \omega \to \infty \end{aligned}$$

If $|D(j\omega)G(j\omega)| \gg 1$ for $\omega \to 0$ then

$$\begin{aligned} |S(j\omega)| &\to 0 \quad \text{for } \omega \to 0 \\ |T(j\omega)| &\to 1 \quad \text{for } \omega \to 0 \end{aligned}$$

Since $|D(j\omega_c)G(j\omega_c)| = 1$, $|S|$ and $|T|$ cross at ω_c

$$|S(j\omega_c)| = |T(j\omega_c)|$$



Enlarging ω_c (higher bandwidth) leads to

- Improvement of tracking behaviour (r)
  (pass-band of T is wider)
- Improvement of disturbance rejection (w)
  (S is small over a wider band)
- More sensitive to sensor noise (n)
  (T increases for high ω)

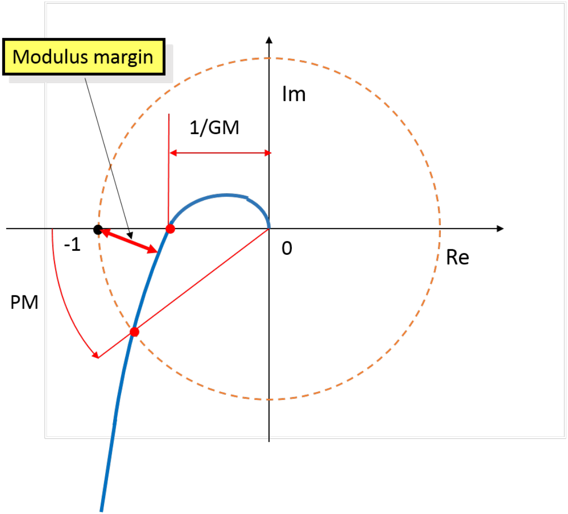
Interpretation of sensitivity function S

- Shows over which frequency range disturbances w can be attenuated
  (frequency area where $|S(j\omega)|$ is small)
- In areas where $|S|$ peaks over 1: disturbances are amplified!
- Peak value of $|S(j\omega)|$ can be interpreted in terms of Nyquist curve:

$$\frac{1}{\max_\omega |S(j\omega)|} = \min_\omega |1 + D(j\omega)G(j\omega)|$$

  is minimum distance of Nyquist curve of DG to the point -1.
  ["modulus margin" or in FP: "vector margin"]

  The higher the peak of $|S|$ the closer the Nyquist curve gets to -1.


Modulus margin
Im
1/GM
-1
0
Re
PM

Disturbance rejection is an additional performance specification of feedback controllers, that comes on top of the 4 specs from Lecture 4: steady state tracking, stability, damping and bandwidth.

Disturbance rejection is typically covered by the design of the sensitivity function $S(j\omega) = \frac{1}{1+D(j\omega)G(j\omega)}$

Design of sensitivity functions

A general design rule is to push down the sensitivity function over the widest possible frequency range, (as long as sensor noise and high-frequency disturbances do not have a detrimental effect)

Bode Sensitivity Integral

Can sensitivity functions be given any shape?

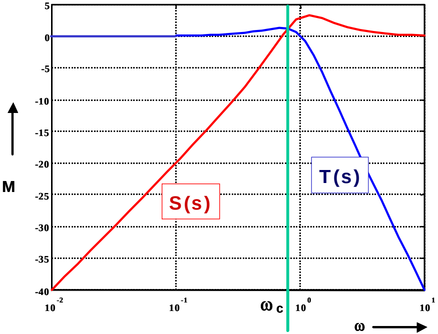
Bode's sensitivity integral:

$$\int_0^\infty \ln(|S(j\omega)|)d\omega = \pi \sum_{i=1}^{n_p} \text{Re}\{p_i\}$$

where:
p_i, $i = 1, \cdots n_p$ are the right half plane poles of DG_0,
and it is assumed that DG_0 rolls off at high frequencies with a slope faster than -1.

Consequence: The integral of $\ln |S(j\omega)|$ is basically constant, and independent of the controller.

For a sketch of derivation of Bode's sensitivity integral see Åström and Murray, pp. 338-340.



If DG_0 has no RHP poles: $\int_0^\infty \ln(|S(j\omega)|)d\omega = 0$

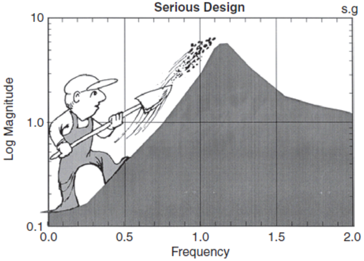
If $|S(j\omega)| \ll 1$ in a particular frequency region, this is compensated for by an area where $|S(j\omega)| > 1$ (disturbance amplification). This is also called the waterbed effect

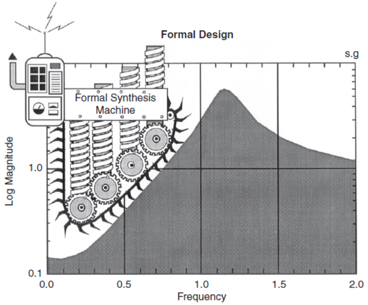
We can not attenuate disturbances at all frequencies simultaneously

And no linear controller can change this!

Bode sensitivity integral is a

"law on the preservation of dirt"





The trade-off is not related to any specific design methodology.

Gunter Stein, Bode Lecture (1989).

For the historical lecture see the (old-fashioned) video:

http://www.ieeecss-oll.org/lectures/1989/respect-unstable

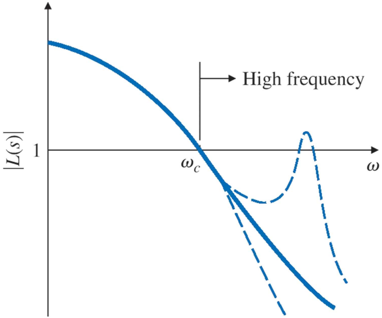
In case of unstable plants:

$$\int_0^\infty \ln(|S(j\omega)|)d\omega = \pi \sum_{i=1}^{n_p} \text{Re}\{p_i\}$$

The effect is even worse if there are unstable (RHP) poles (integral is positive, and the "bad" side of $|S|$ "wins")).

Stability Robustness

What if the plant's frequency response is not exactly known?



We think that we designed the right cross-over frequency, but the real plant's frequency response pops up beyond 1 at a higher frequency? $\rightarrow$ Serious risk of instability

Model uncertainty

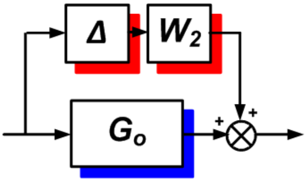
- Very often our models are not 100% accurate
- Models are merely an approximation of reality
- In mass-produced equipment dynamics might vary (DVD)

Robustness of controllers (i.e. ability to deal with model uncertainty) is an important step towards reliable high-performance engineering solutions.

Model uncertainty

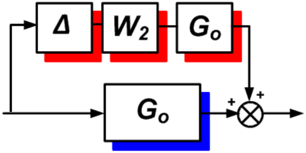
What is the effect on stability if we design a controller on the basis of $G_0(s)$ while the real plant is given by $G(s)$

Additive uncertainty



$$G = G_0 + W_2\Delta$$

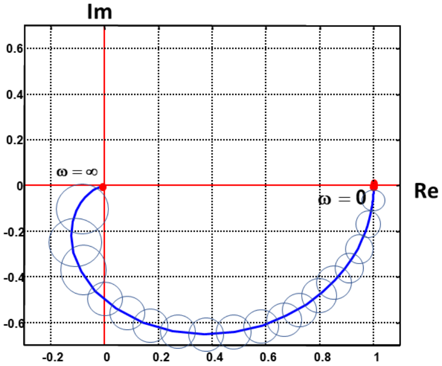
Multiplicative uncertainty



$$G = G_0 + G_0 \cdot W_2\Delta$$

$\Delta(j\omega)$ is a f-response function that is bounded,
e.g. $|\Delta(j\omega)| \leq 1$, and $W_2(\omega)$ a positive real weighting function.

Every point on the Nyquist curve becomes "affected" by a circular uncertainty.



Frequency-dependent diameter of circle is determined by W_2 (additive) or W_2 and G_0 (multiplicative)

Analysis stability robustness for multiplicative uncertainty

What is the effect on stability if we design a controller on the basis of $G_0(s)$ while the real plant is given by $G(s)$ with

$$\frac{G(j\omega) - G_0(j\omega)}{G_0(j\omega)} = W_2(\omega)\Delta(j\omega) \quad \text{with } |\Delta(j\omega)| \leq 1$$

$W_2(\omega)$ a positive, real-valued frequency function.

Equivalent formulation:

$$G(j\omega) = G_0(j\omega)[1 + W_2(\omega)\Delta(j\omega)]$$

If model is accurate for small ω: $W_2(\omega)$ is small there
If model is uncertain for large ω: $W_2(\omega)$ is larger there

(cont'd) stability robustness:

Suppose G_0 is stabilized by D and $G_0 D$ stable (for simplicity).
Then D stabilizes all G if the critical point:

$$|DG| = 1, \quad \angle DG = 180^o$$

is not reached for *any* G at *any* ω: $1 + DG \neq 0 \quad \forall \omega, \Delta$.
Apply this to all G in

$$G(j\omega) = G_0(j\omega)[1 + W_2(\omega)\Delta(j\omega)]$$

for any point point in the ball $\Delta(\omega)$.
Then

$$1 + DG_0(1 + W_2\Delta) \neq 0 \quad \forall \omega, \Delta \quad \Leftrightarrow$$
$$(1 + DG_0)[1 + \tfrac{DG_0}{1+DG_0} W_2 \Delta] \neq 0 \quad \forall \omega, \Delta$$

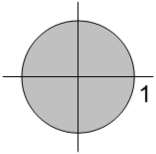
This can be guaranteed for all Δ (see explanation next slide)

Reasoning for handling Δ:

$$1 + \frac{DG_0}{1 + DG_0} W_2 \Delta \neq 0 \quad \forall \omega, \Delta$$

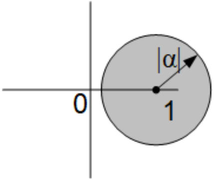
For a single ω this reads as $1 + \alpha\delta \neq 0$

with $\alpha \in \mathbb{C}$, and δ any complex number in the grey set of the figure.



Then it is clear that $1 + \alpha\delta \neq 0$ **for all** δ, if

$$|\alpha| < 1$$



Result:

Stability robustness is guaranteed if

$$|TW_2| < 1 \quad \forall\omega$$

or equivalently

$$|T(j\omega)| < W_2^{-1}(\omega) \quad \forall\omega.$$

For higher frequencies, W_2 typically increases, and puts a limitation on T through

$$|T(j\omega)| < W_2^{-1}(\omega)$$

For higher frequencies, also $|DG_0| \ll 1$ and therefore

$$|T(j\omega)| \approx |DG_0(j\omega)|$$

The design constraint then reduces to

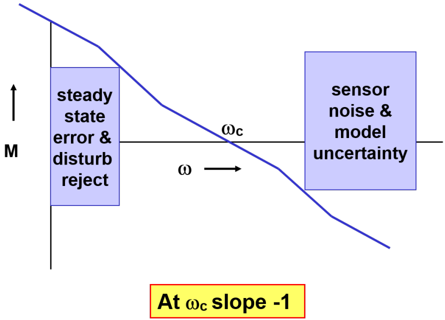
$$|DG_0(j\omega)| < W_2^{-1}(\omega)$$

High-frequent model uncertainty that is bounded by $W_2(\omega)$ in a multiplicative uncertainty, leads to a design constraint for robust stability:

$$|DG_0(j\omega)| < W_2^{-1}(\omega)$$

In other words: the loop gain roll-off should be sufficiently fast

Design specifications for DG_0:



- High gain for small ω
  (steady state errors, disturbance rejection, tracking)
- Slope of -1 around ω_c (stability)
- Fast roll-off for higher ω (plant uncertainty, sensor noise)

Summary

- Control design often comes down to trading off different/conflicting objectives
- Sensitivity functions are very instructive in interpreting control performance behaviour (disturbance rejection, bandwidth)
- $S + T = 1$ implying that when present at the same frequencies, sensor noise and disturbances can not be suppressed simultaneously
- In linear control, sensitivity functions are constrained by the Bode sensitivity integral
- Robustness considerations require loop transfers to roll-off strongly beyond the cross-over frequency