Control Systems
Lecture 10b

Nonlinear systems and linearization

Nonlinear systems

Many physical systems will exhibit nonlinear dynamics

In state space form this is reflected by replacing the (linear) matrix operations by a general operator:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$$

where $\mathbf{f}$ can now include nonlinear terms (in $\mathbf{x}$ and/or in u).

However: for small signal variations around an equilibrium we can find a

Linearized model

that is valid around the particular equilibrium (working point).

We first determine an equilibrium $(\mathbf{x}_0, u_0)$ for which holds that

$$\dot{\mathbf{x}}_0 = 0 = \mathbf{f}(\mathbf{x}_0, u_0).$$

Then we choose a small-signal variation around this equilibrium:

$$\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x} \quad \text{and} \quad u = u_0 + \delta u$$

to be substituted in model $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$.

This delivers

$$\begin{aligned} \dot{\mathbf{x}}_0 + \delta\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}_0 + \delta\mathbf{x}, u_0 + \delta u) \\ &\cong \mathbf{f}(\mathbf{x}_0, u_0) + \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta u \end{aligned}$$

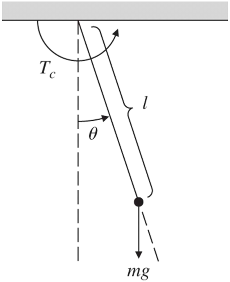
by using a first order Taylor expansion, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_0, u_0} \quad \text{and} \quad \mathbf{B} = \left[\frac{\partial \mathbf{f}}{\partial u}\right]_{\mathbf{x}_0, u_0}.$$

This leads to the (linearized) state space model:

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta u$$

Example 9.1 - Pendulum (2.5)



See example 2.5:

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = \frac{T_c}{m\ell^2}$$

With $\omega_0^2 = g/\ell$ and $u = T_c/(m\ell^2)$ this is rewritten as

$$\ddot{\theta} + \omega_0^2 \sin(\theta) = u$$

With $\mathbf{x} = [x_1 \;\; x_2]^T = [\theta \;\; \dot{\theta}]^T$ a state space model is given by:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -\omega_0^2 \sin(x_1) + u \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}, u) \\ f_2(\mathbf{x}, u) \end{bmatrix} = \mathbf{f}(\mathbf{x}, u)$$

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -\omega_0^2 \sin(x_1) + u \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}, u) \\ f_2(\mathbf{x}, u) \end{bmatrix} = \mathbf{f}(\mathbf{x}, u)$$

Equilibrium point for a nominal value of $u_0 = 0$ is

$$\begin{aligned} \dot{x}_1 &= \dot{\theta} = 0 \\ \dot{x}_2 &= \ddot{\theta} = -\omega_0^2 \sin(\theta) = 0 \end{aligned}$$

which corresponds to equilibrium conditions $\theta_0 = 0, \pi$.
$\mathbf{x}_0 = [\theta_0 \;\; 0]^T$, $u_0 = 0$, leading to

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 \cos(\theta_0) & 0 \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

The linearized model

- is valid only for small signal variations around the equilibrium point
- can be used to stabilize/control the system around this point through feedback