Network identifiability - Synthesis

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Network identifiability - Synthesis

Consider a network where all nodes are measured.

Question:

> Where to allocate external excitation signals in order to guarantee generic identifiability of the network model set?

Graphical approach to cover the network graph with pseudotrees[1]

[1] X. Cheng, S. Shi and P.M.J. Van den Hof, IEEE Trans. Automatic Control, Febr 2022.

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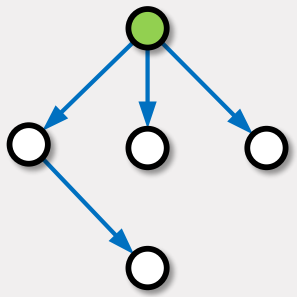
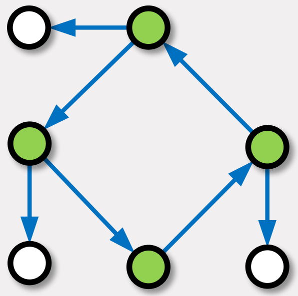
Definition Pseudotree:

A connected simple directed graph with number of vertices ≥ 2 is called a (directed) **pseudotree** if for all vertices i, the number of in-neighbors is ≤ 1.

Two typical examples:



rooted tree



cycle with outgoing tree

Observation:

An external signal added to any of the roots (green) reaches all vertices in the pseudotree

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Strategy:

- Cover the graph of a network with a set of **disjoint pseudotrees**
- Excite (one of the) root(s) of each pseudotree with an external excitation signal

(Edge-) disjoint pseudotrees

Two pseudotrees are (edge-) disjoint if

- They do not share any edges, and
- All outgoing edges of a vertex belong to the same pseudotree

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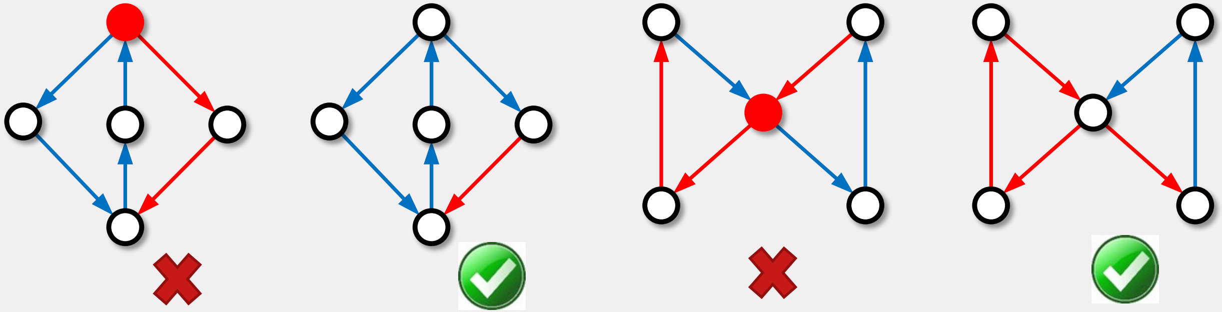
- Edges are **disjoint** and all out-neighbours of a node are in the same pseudo-tree



- Any network graph can be decomposed into a set of disjoint pseudo-trees

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Synthesis solution for network excitation

A network model set $\mathcal{M}$ is generically identifiable if

- its graph can be covered by K disjoint pseudotrees, and
- there are K independent external signals entering at a root of each pseudotree.

Sketch of Proof:

Let $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_K$ be disjoint pseudotrees that cover all the edges of the graph $\mathcal{G}$ and τ_k be an excited root node in pseudotree $\mathcal{T}_k$.
The definition of disjoint pseudotrees implies that

1. two disjoint pseudotrees cannot share common root nodes, i.e., $\tau_i \neq \tau_j$, for all $i \neq j$;

2. the in-neighbors of each node in $\mathcal{G}$ should be in distinct pseudotrees;

3. paths in different disjoint pseudotrees are vertex-disjoint, if they have no common starting or ending nodes.

The above three points guarantees that, for any node j in $\mathcal{G}$, there exist $|\mathcal{N}_j^-|$ vertex-disjoint paths from the set $\{\tau_1, \tau_2, ..., \tau_K\}$ to $\mathcal{N}_j^-$, where $\mathcal{N}_j^-$ is the set of in-neighbors of j. The result holds for all nodes in $\mathcal{G}$, thus generic identifiability of $\mathcal{M}$ follows.

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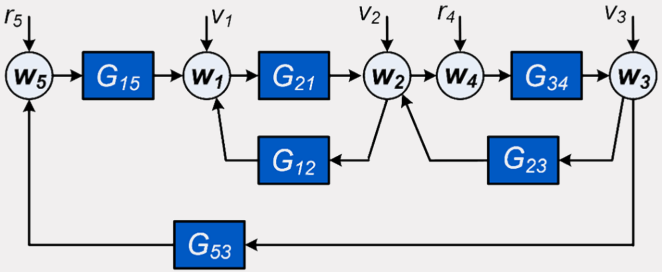
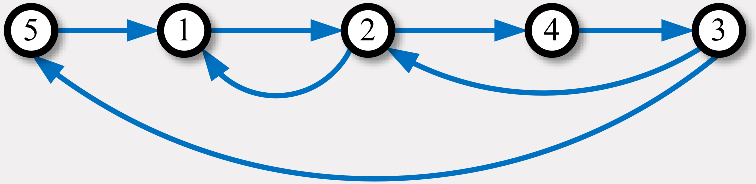
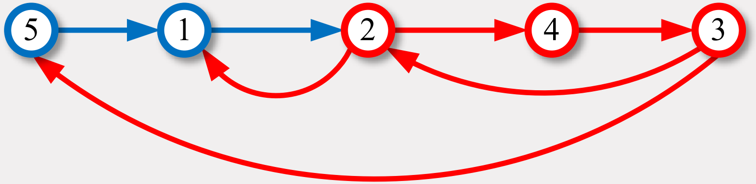
Example: 5 node network (revisited)



When discarding the present external signals, the graph becomes:

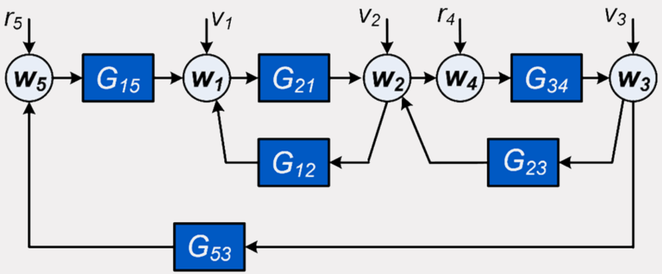


The graph can be covered by
Two disjoint pseudotrees:

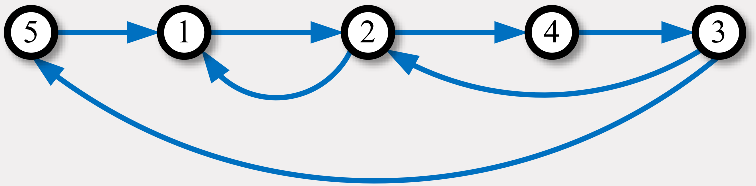


Note: this covering is non-unique!

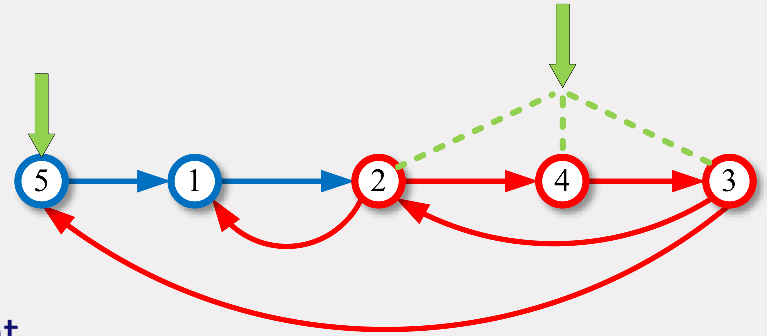
Example: 5 node network (revisited)



When discarding the present external signals, the graph becomes:

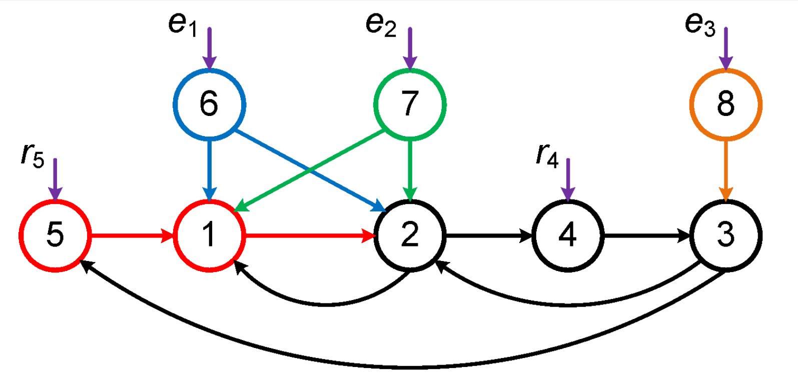


Two independent excitations guarantee network identifiability:



One of $v_2/r_4/v_3$ and r_5 would be sufficient

If parametrized noise models are included in the model set, then we use an extended graph, including the white noise disturbance inputs as nodes:



External signals $r_2/r_4/r_3$ and r_5 guarantee generic identifiability

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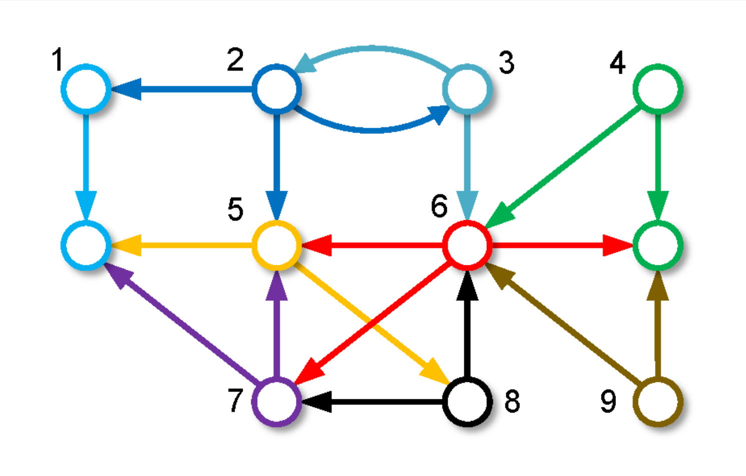
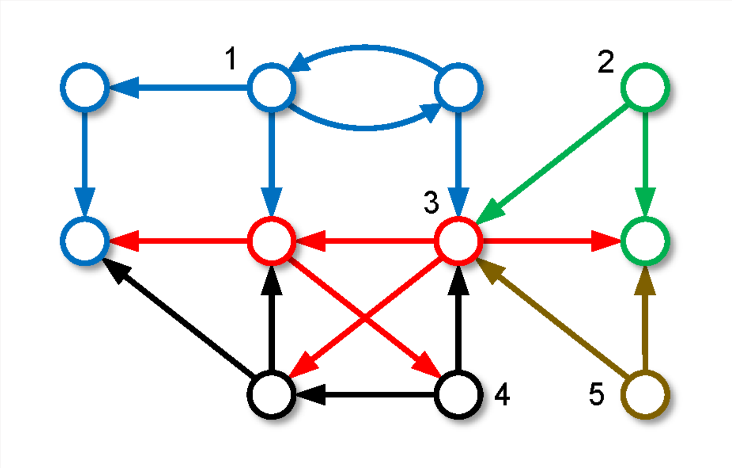
Where to allocate external excitations for network identifiability?



Pseudo-tree merging algorithm



Start from an elementary covering (all outgoing edges from a node in one pseudotree)

The merging can be done through an automated algorithm

Merging algorithm

Denote a set

$$\mathbb{M} = \{1, 0, \varnothing\}.$$

Let $\Pi = \{\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n\}$ be a disjoint pseudotree covering of a directed graph. The **characteristic matrix** of Π is denoted by $\mathscr{M} \in \mathbb{M}^{n\times n}$, whose (i, j)-th entry is defined as

$$\mathscr{M}_{ij} = \begin{cases} 1 & \text{if } \mathcal{T}_i \text{ is mergeable to } \mathcal{T}_j; \\ \varnothing & \text{if } V(\mathcal{T}_j) \cap V(\mathcal{T}_i) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

and define a commutative operator on $\mathbb{M}$ according to $1 \odot 1 = 1,\ 1 \odot 0 = 0,\ 1 \odot \varnothing = 1,$
$0 \odot 0 = 0,\ \varnothing \odot 0 = 0,\ \varnothing \odot \varnothing = \varnothing.$

defining a componentwise multiplication operation on rows of $\mathscr{M}$

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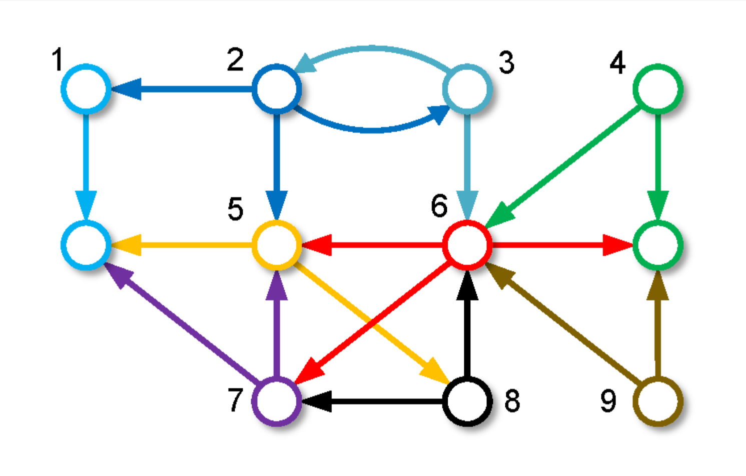
Start of the algorithm: elementary covering



$$\mathcal{M} = \begin{bmatrix} 0 & 1 & \varnothing & \varnothing & 0 & \varnothing & 0 & \varnothing & \varnothing \\ 0 & 0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\ \varnothing & 1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ \varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ 0 & 1 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\ \varnothing & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varnothing & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\ \varnothing & \varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \end{bmatrix}.$$

Merging algorithm

Merging of the i-th pseudotree into the j-th one now comes down to

- Replace row j by $\mathscr{M}_{i\star} \odot \mathscr{M}_{j\star}$
- Replace column j by $\mathscr{M}_{\star i} \odot \mathscr{M}_{\star j}$
- Remove the i-th row and column of $\mathscr{M}$

Ordering of the merging:

- Select the row with a (single) 1 entry and a maximum number $\varnothing$ entries, and merge this row;

At the end, the matrix $\mathscr{M}$ will have no more 1 entries.

Start of the algorithm: elementary covering

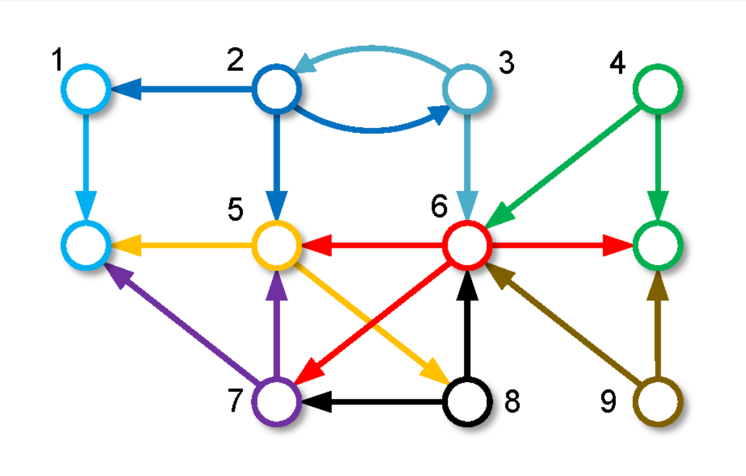
Given a graph $\mathcal{G}$ with the adjacency matrix $A(\mathcal{G})$. Denote

$$a_{ij} = ([A(\mathcal{G}) + I\mathrm{i}]_{\star i})^{\top}[A(\mathcal{G}) + I\mathrm{i}]_{\star j},$$

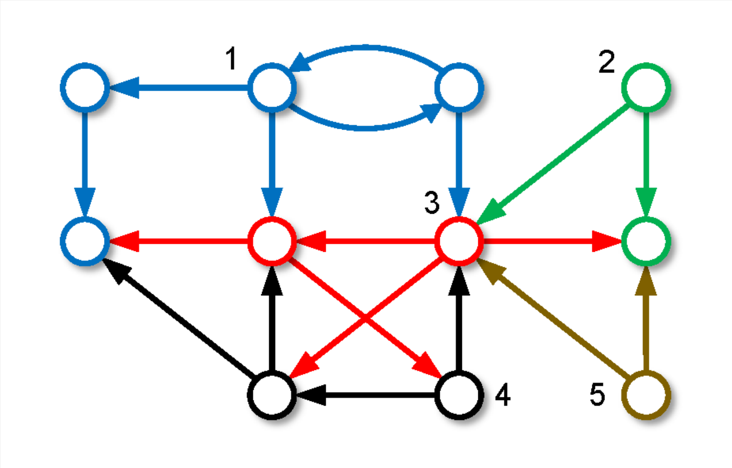
The characteristic matrix $\mathscr{M}$ is formulated as follows:
$\mathscr{M}_{ii} = 0$ for all i, while for $j \neq i$:

$$\mathscr{M}_{ij} = \begin{cases} 1, & \text{Re}(a_{ij}) = 0, \text{ and } \text{Im}(a_{ij}) \neq 0, \text{ and } [A(\mathcal{G})]_{ij} \neq 0. \\ 0, & \text{Re}(a_{ij}) \neq 0 \text{ or } \{\text{Re}(a_{ij}) = 0, \text{ and } \text{Im}(a_{ij}) \neq 0, \text{ and } [A(\mathcal{G})]_{ij} = 0\}. \\ \varnothing, & a_{ij} = 0, \end{cases}$$

Where to allocate external excitations for network identifiability?



Pseudo-tree merging algorithm

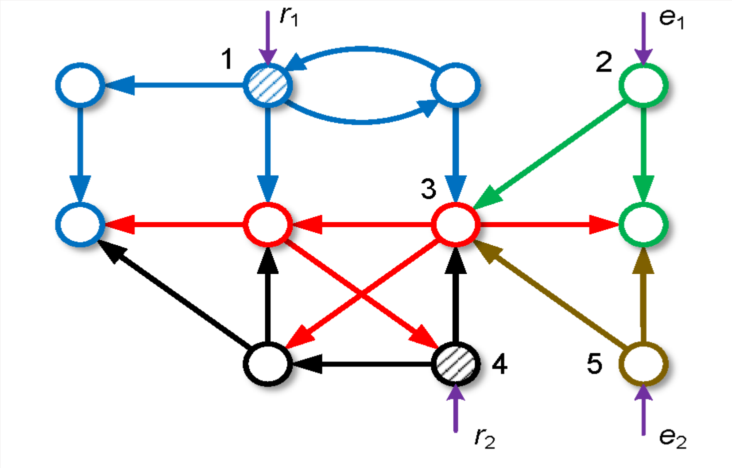


If white noises e_2 and e_5 are present, then it suffices to excite r_1, r_3 and r_4.

Where to allocate external excitations for network identifiability?

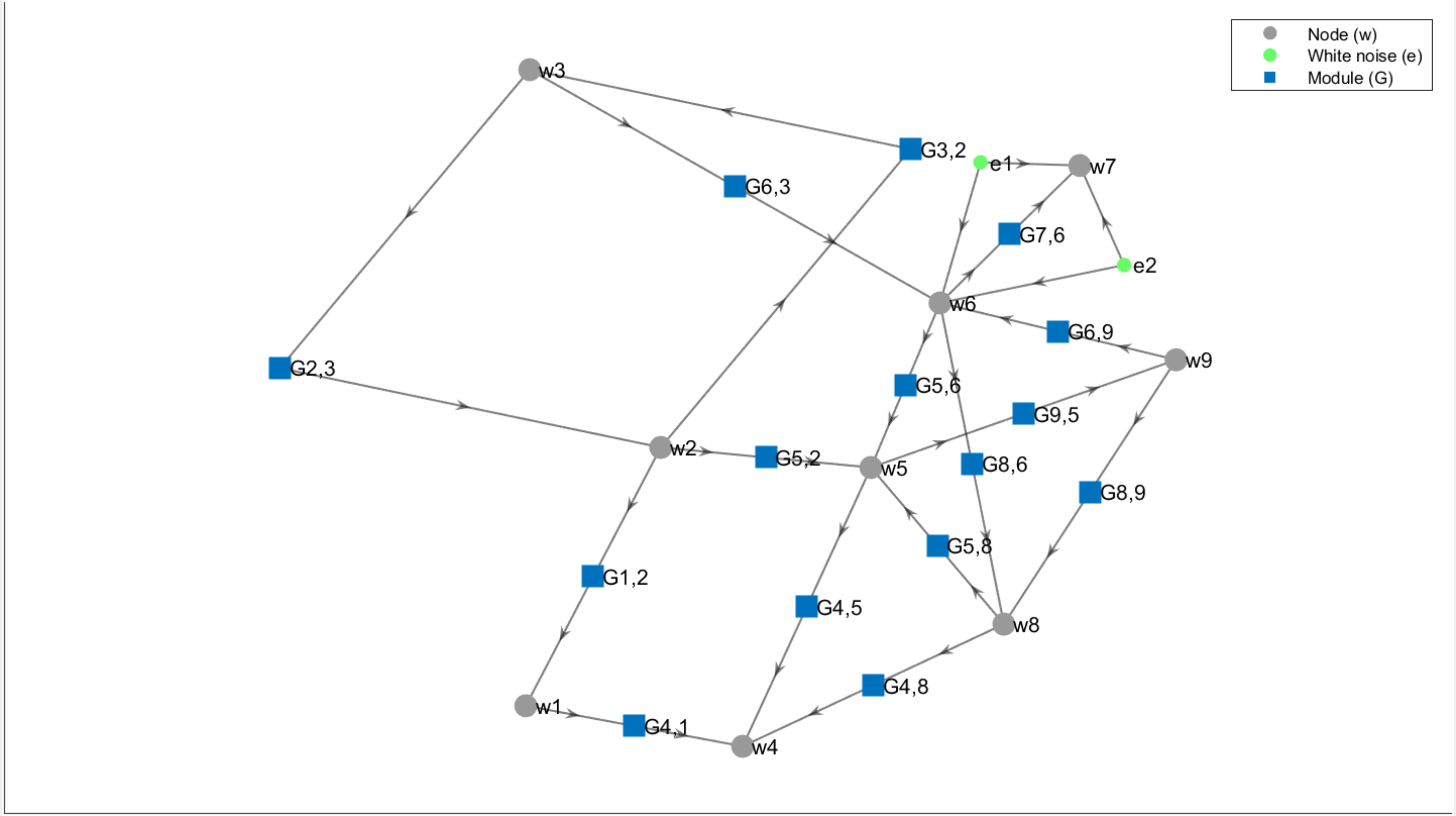
After selecting the roots of the pseudotrees:

Verify whether all root excitations are necessary for satisfying the path-based identifiability condition (# vertex disjoint paths)

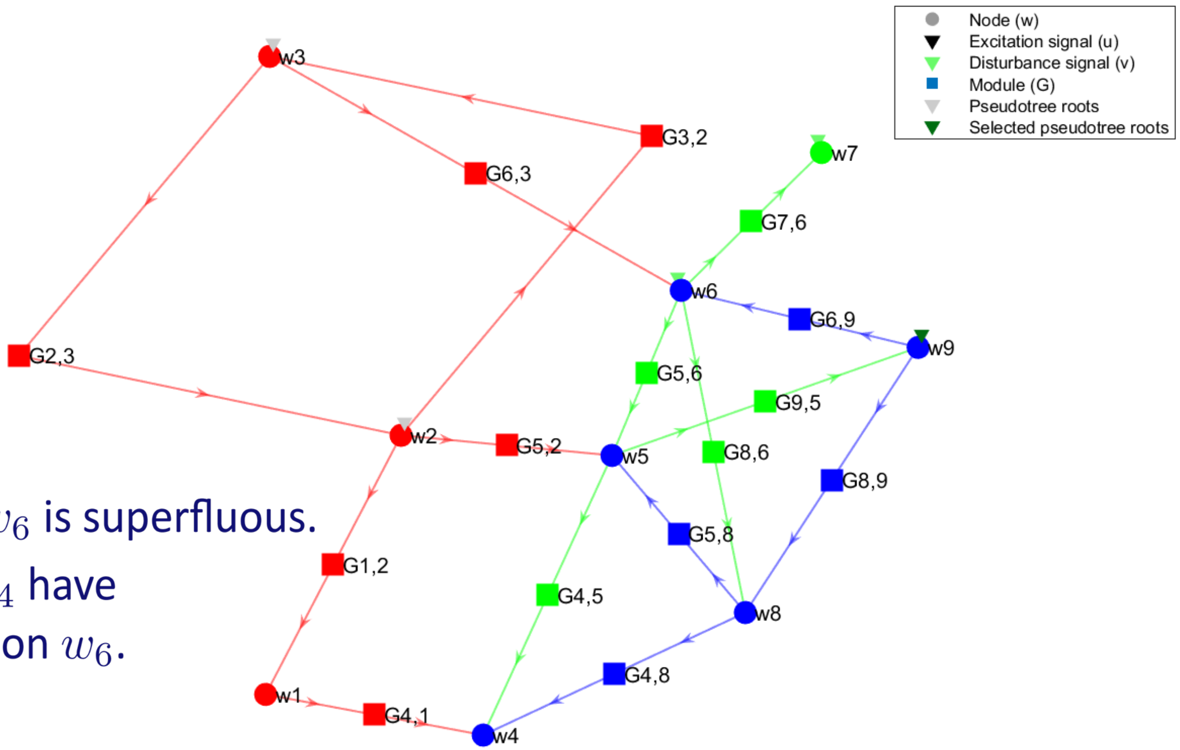


Since the path-based condition is satisfied for all nodes in pseudotree 3, even without the presence of r_3, this excitation can be removed.

Algorithm example



Algorithm example



Excitation at w_3 or w_2, w_6 and w_9.

Analysis shows that excitation at w_6 is superfluous.

Nodes with 3 in-neighbours w_5, w_4 have sufficient excitation by utilizing e_1 on w_6.

Summary identifiability synthesis algorithm

- Attractive graphical approach for verifying generic identifiability conditions.
- As well as for synthesizing the required experimental setup (allocating external signals), starting from existing disturbances.
- The results also apply to the situation of non-parametrized / fixed modules in $\mathcal{M}$; The fixed modules can then be excluded from the graph-covering.
- A less conservative way of including fixed modules is available by extending the concept of a pseudotree, to a graph with at most one parametrized link from an in-neighbor [1] ; this is implemented in the toolbox.

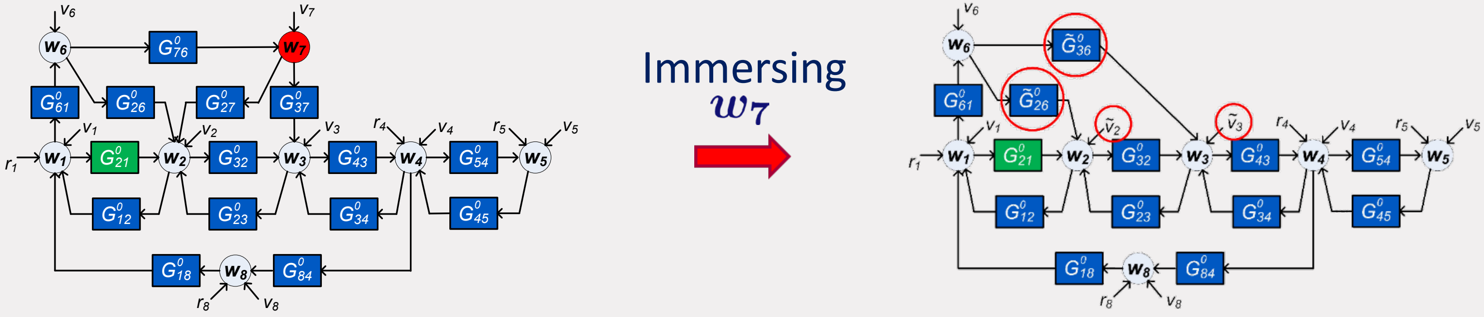
[1] Dreef et al., L-CSS, 2022.

Discussion identifiability

If node signals can not all be measured? (partial node measurement)

- Situation can be treated as separate problem[1],
  leading to statements that for identifiability each node should be measured or excited.

- Situation can partly be analysed by using the concept of **immersion**,
  i.e. removing a non-measured node from the network while keeping the other node signals invariant.[2]



[1] Bazanella et al., CDC 2019; Mapurunga et al., IFAC POL, Jan 2021; L-CSS, 2022; Cheng et al., IEEE-TAC, under review, 2022.

[2] Dankers et al., TAC 2016