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On dynamic network modeling of stationary multivariate processes

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Abstract: We study the modeling of a stationary multivariate stochastic process as the output of a dynamic network driven by white noise. When this noise corresponds to the innovation, i.e. the unpredictable part of the process, we show that the network satisfies certain stability conditions. Restricting the network model to having diagonal noise structure, we show that the innovation-driven representation is unique and internally stable. We provide a one-to-one correspondence between this representation and the spectral factor associated with the innovation model. For two-node networks, we show that a representation with diagonal noise model can be obtained from a generic one through an explicit map.

1. INTRODUCTION

A dynamic network model can be seen as the natural extension to a general multivariate scenario of input-output feedback models *a la* Caines-Chan-Gevers-Anderson (Caines and Chan, 1976; Gevers and Anderson, 1981, 1982). These models encode the (dynamical) dependences among components of a multivariate stochastic process in terms of a graph (*network*) whose links are dynamical systems (*dynamic*).

The study of such models has gained considerable attention in the recent years. The main reason is that dynamic network models can help gaining insights on the underlying mechanisms of complex systems, ranging from biological systems (e.g. gene regulatory networks, Hecker et al. (2009); Nimmegeers et al. (2017), and whole brain network models, Friston et al. (2003); Prando et al. (2017)), econometrics (see e.g. Materassi and Innocenti (2010) and references therein), and engineering (see e.g. Hill and Chen (2006); Kotevska et al. (2017)) just to mention a few.

In the past decade, several techniques for inference in dynamic networks have been proposed. These methods may be grouped into two main categories. In the first category are methods for topology detection (Materassi and Salapaka, 2012; Sanandaji et al., 2011; Chiuso and Pillonetto, 2012), where the main goal is to find direct relations among the network output signals. The second category comprises methods for estimating the direct relation – usually modeled as a linear time-invariant system – between two or more outputs (see, e.g., Dankers et al. (2015, 2016); Van den Hof et al. (2013); Everitt et al. (2017)).

More recently, increasing attention has been devoted to the study of the identifiability of dynamic networks (Hayden et al., 2016; Weerts et al., 2015, 2018; Gevers and Bazanella, 2015; Bazanella et al., 2017). Here, the focus is on understanding the uniqueness of a dynamic network representation of a given multivariate process (possibly driven by external controllable inputs). Results show that, for parametric descriptions of dynamic networks, uniqueness is guaranteed under a persistency of excitation condition from the external signals (Weerts et al., 2018). When there is no external excitation, the noise model characterizing the network must be spatially uncorrelated to guarantee identifiability (Weerts et al., 2018).

This paper also deals with identifiability of dynamic networks. Focusing on networks without external references, we explore different ways to model a p-dimensional stationary stochastic process as the output of a dynamic network driven by white noise. In particular, our interest is on network representations where the driving noise corresponds to the innovation process (Lindquist and Picci, 2015, Ch. 4), due to its close connection to the best linear predictor of the process and Granger causality graphs. The main contributions of this paper are as follows: (i) we show that networks driven by the innovation satisfy, under the assumption that the noise model is diagonal, certain internal stability properties; (ii) restricting to diagonal noise models, we show that there always exists only one dynamic network model driven by the innovation process, and this representation can be constructed using the elements of the spectral factor of the output process associated with the innovation. Our results extend those of Gevers and Anderson (1982, 1981) for closed-loop systems to general dynamic networks; we also demonstrate that this model is always internally stable, i.e. the sensitivity transfer function

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associated with the network is always stable; (iii) finally, for a two-node network we also give a closed form expression of the unique internally stable model with diagonal noise process from a generic one.

2. DYNAMIC NETWORK REPRESENTATION OF STATIONARY PROCESSES

We consider a zero-mean p-dimensional stationary stochastic process $w(t)$. In particular, we focus on its power spectrum, denoted by $\Phi(z) \in \mathbb{C}^{p\times p}$. We assume that $\Phi(e^{j\omega}) > 0$ for every $\omega \in [0, 2\pi)$.

We want to represent the dynamics of $w(t)$ as the output of a structured multivariate system fed by a white noise process of the same dimension of $w(t)$. In particular, we are interested in the representations stated in the following definitions.

Definition 1. (Innovation Model) (Lindquist and Picci, 2015, Ch. 4) The Innovation Model (IM) of $w(t)$ is a representation of the type

$$w(t) = \Gamma(z)\varepsilon(t)\,, \tag{1}$$

where $\varepsilon(t)$ is white noise with positive definite covariance matrix Λ, and $\Gamma(z) \in \mathbb{C}^{p\times p}$ is such that:

- $\Phi(z) = \Gamma(z)\Lambda\Gamma^*(z)$;
- $\Gamma(z)$ and $\Gamma^{-1}(z)$ are stable;
- $\Gamma(\infty) = I_p$.

Under the conditions stated in the definition of IM, $\varepsilon(t)$ is called the innovation of $w(t)$ and is such that

$$w(t) = \hat{w}(t|t-1) + \varepsilon(t)\,,$$

where $\hat{w}(t|t-1)$ is the minimum variance predictor of $w(t)$ given its past values up to $t-1$. This predictor can be obtained as

$$\hat{w}(t|t-1) = (I - \Gamma^{-1}(z))w(t)\,. \tag{2}$$

While this is a standard representation of a stochastic process, it does not give particular insights on the relations among the components of the process. Therefore, we will focus on structured representation for $w(t)$. A first network representation is given in the following definition.

Definition 2. (Dynamic Network Model). A Dynamic Network Model (DNM) is a representation of the type

$$w(t) = \tilde{G}(z)w(t) + \tilde{H}(z)\tilde{e}(t)\,, \tag{3}$$

where $\tilde{G}(z) \in \mathbb{C}^{p\times p}$ is a rational transfer matrix such that

- $\tilde{G}_{ii}(z) = 0$, for any $i = 1,\ldots,p$,
- $\tilde{G}_{ij}(z)$, $i,j = 1,\ldots,p$, $i \neq j$ is a strictly causal transfer function,

$\tilde{H}(z) \in \mathbb{C}^{p\times p}$ is a rational transfer matrix such that $\tilde{H}(\infty) = I_p$, and $\tilde{e}(t)$ is white noise with positive definite covariance matrix.

We now consider innovation-driven models.

Definition 3. (Innovation-driven DNM).
An Innovation–driven Dynamic Network Model (IDNM) is a DNM with $\tilde{e}(t) = \varepsilon(t)$.

A special case of interest, as we shall see below, is when the matrix $\tilde{H}(z)$ is diagonal; we define formally this class of Dynamic Network Models as follows:

Definition 4. (Diagonal Dynamic Network Model). A Diagonal Dynamic Network Model (DDNM) is a representation of the type

$$w(t) = G(z)w(t) + H(z)e(t)\,, \tag{4}$$

where $G(z) \in \mathbb{C}^{p\times p}$ is a rational transfer matrix such that

- $G_{ii}(z) = 0$, for any $i = 1,\ldots,p$,
- $G_{ij}(z)$ $i,j = 1,\ldots,p$, $i \neq j$ is a strictly causal (i.e., $G_{ij}(\infty) = 0$) transfer function,

$H(z) \in \mathbb{C}^{p\times p}$ is a rational transfer matrix such that

- $H_{ii}(z)$ is monic (i.e., $H_{ii}(\infty) = 1$ for any $i = 1,\ldots,p$,
- $H_{ij}(z) = 0$ $i,j = 1,\ldots,p$, $i \neq j$,

and $e(t)$ is white noise with positive definite covariance matrix.

Despite the restriction to diagonal noise models, there are infinite representations of the DDNM type. For instance, given the DDNM (4), by multiplying each $H_{ii}(z)$ by an all-pass filter $Q_{ii}(z)$ one can obtain another DDNM representation where the driving noise is $Q_{ii}^{-1}(z)e(t)$. We therefore concentrate on a particular DDNM representation, given by the following definition.

Definition 5. (Innovation-driven DDNM). An Innovation-driven Diagonal Dynamic Network Model (IDDNM) is a representation of the type (4) with $e(t) = \varepsilon(t)$.

For completeness, we also introduce the following fundamental property of dynamic networks.

Definition 6. (Internally stable dynamic network). A dynamic network such that $(I - G(z))^{-1}$ is stable is referred to as *internally stable*.

3. PROPERTIES OF THE DYNAMIC NETWORK MODELS

In this section, we establish connections between the network representations introduced in the previous section, showing their stability properties.

Proposition 1. Every DNM such that $(I - \tilde{G}(z))^{-1}\tilde{H}(z)$ is stable, defines a stationary process $w(t)$ with spectral density

$$\Phi(z) = (I - \tilde{G}(z))^{-1}\tilde{H}(z)\Lambda\tilde{H}^\top(1/z)(I - \tilde{G}(1/z))^{-\top}.$$

Proof: The proof is obvious from the fact that given a DNM $w(t) = \tilde{G}(z)w(t) + \tilde{H}(z)\tilde{e}(t)$, the output $w(t)$ can be written as

$$w(t) = (I - \tilde{G}(z))^{-1}\tilde{H}(z)\tilde{e}(t)$$

i.e. the output of the stable linear model with transfer function $(I - \tilde{G}(z))^{-1}\tilde{H}(z)$, fed with white noise input $e(t)$. □

3.1 Properties of Innovation driven DNM

Given a stationary processes $w(t)$, we first discuss the properties that an innovation-driven DNM satisfies.

Proposition 2. Any DNM, which satisfies the conditions:

(i) $\tilde{H}^{-1}(z)$, $\tilde{H}^{-1}(z)\tilde{G}(z)$ and $(I - \tilde{G}(z))^{-1}\tilde{H}(z)$ are stable

is an innovation model, i.e. $\tilde{e}(t) = \varepsilon(t)$.

Proof: We have that $w(t) = (I - G(z))^{-1}H(z)\tilde{e}(t) = \Psi(z)\tilde{e}(t)$. From (i) it follows that $\Psi(z) = (I-G(z))^{-1}H(z)$ as well as $\Psi^{-1}(z) = H^{-1}(z)(I - G(z)) = H^{-1}(z) - H^{-1}(z)G(z)$ are both stable. In addition $\Psi(\infty) = I$ holds true. Therefore the model $w(t) = \Psi(z)\tilde{e}(t)$ is the innovation model. , i.e. $\Psi(z) = \Gamma(z)$ and $\tilde{e}(t) = \varepsilon(t)$. □

Proposition 2 states sufficient conditions for a DNM to be an IDNM. However, these conditions are not necessary as the following example shows:

Example 1. Consider a 2-node DNM where

$$\tilde{G}_{12}(z) = \frac{-2}{z-0.6}\,,\ \tilde{G}_{21}(z) = \frac{1}{z+2}\,,$$

and

$$\tilde{H}(z) = \begin{bmatrix} \frac{z^3+0.1z^2-4.5z-0.352}{z^3-0.6z^2-0.01z+0.006} & \frac{-3z^2-4.9z-0.32}{z^3+2z^2-0.01z-0.02} \\ \frac{2.1z-0.98}{z^2-0.4z-0.12} & \frac{z^3+0.9z^2-2z+0.58}{z^3+1.6z^2-0.92z-0.24} \end{bmatrix}.$$

We note that this network is internally stable, since $(I - \tilde{G}(z))^{-1}$ is stable, while $\tilde{H}^{-1}(z)$ is unstable, having a pole at $z = -2$. If we compute the spectral factor of $w(t)$ we get

$$\Psi(z) = \begin{bmatrix} \frac{z+0.8}{z+0.1} & \frac{-2}{z-0.1} \\ \frac{0.1}{z-0.6} & \frac{z-0.5}{z+0.2} \end{bmatrix}, \tag{5}$$

which is stable and with stable inverse. Since also $\Psi(\infty) = I$ holds, we conclude that the model is an IDNM, showing that not all the conditions of Proposition 2 are necessary.

In case $H(z)$ is diagonal, instead, necessary and sufficient conditions for a DDNM to be in innovation form can be given:

Proposition 3. Given a DDNM, the following two conditions are equivalent:

(i) $H^{-1}(z)$, $H^{-1}(z)G(z)$ and $(I - G(z))^{-1}H(z)$ are stable;
(ii) $e(t) = \varepsilon(t)$, i.e. the model is an IDDNM.

Proof: 1) $\Rightarrow$ 2) follows from the previous Proposition since a DDNM is a special case of a DNM.

Let us now prove that 2) $\Rightarrow$ 1): if the model $w(t) = \Gamma(z)\varepsilon(t)$ with $\Gamma(z) = (I - G(z))^{-1}H(z)$ is the innovation model, then $(I - G(z))^{-1}H(z)$ is stable. In addition $\Gamma^{-1}(z) = H^{-1}(z)(I - G(z)) = H^{-1}(z) - H^{-1}(z)G(z)$ is stable. Since $G(z)$ has zero diagonal entries and $H(z)$ is diagonal, $[\Gamma(z)]_{ii} = [H^{-1}(z)]_{ii}$ which implies that $H^{-1}(z)$ (which is diagonal by assumption) is stable. Therefore $(I - G(z))^{-1} = \Gamma(z)H^{-1}(z)$ is stable (since both $\Gamma(z)$ and $H^{-1}(z)$ are so; the same conclusion holds for $H^{-1}(z)G(z) = H^{-1}(z) - \Gamma^{-1}(z)$, which concludes the proof. □

The following result is a direct consequence of the previous one.

Corollary 1. An IDDNM is internally stable.

Proof: Since in an IDDNM $H^{-1}(z)$ is stable and diagonal, $(I - G(z))^{-1}$ must be stable, for otherwise $\Gamma(z) = (I - G(z))^{-1}H(z)$ would be unstable, due to the fact that eventual unstable poles of $(I - G(z))^{-1}$ could not be canceled by poles of $H^{-1}(z)$. □

It is useful to observe that, while for a generic DNM (i.e. without the assumption that $H(z)$ is diagonal) the conditions (i) in Proposition 2 are only sufficient for a DNM to be an innovation model (see Proposition 2 and Example 1), in the diagonal case the same conditions are also necessary. This fact has been exploited in Corollary 1 to show that a diagonal innovation network model is always internally stable. This is not true in general as the following example shows:

Example 2. Consider a 2-node DNM where

$$\tilde{G}_{12}(z) = \frac{-2}{z-0.6}\,,\ \tilde{G}_{21}(z) = \frac{1}{z+0.2}\,,$$

and

$$\tilde{H}(z) = \begin{bmatrix} \frac{z^2-0.6z+1}{z^2-1.1z+0.3} & \frac{-0.5z+0.1}{z^2-0.3z-0.1} \\ \frac{2.5z-0.3}{z^2-1.1z+0.3} & \frac{z^2+0.2z-0.5}{z^2-0.3z-0.1} \end{bmatrix}.$$

We note that both $\tilde{G}(z)$ and $\tilde{H}(z)$ are stable, but the network is not internally stable, since

$$(I - \tilde{G}(z))^{-1} = \begin{bmatrix} \frac{z^2-0.4z-0.12}{z^2-0.4z+1.88} & \frac{z-0.6}{z^2-0.4z+1.88} \\ \frac{-2z-0.4}{z^2-0.4z+1.88} & \frac{z^2-0.4z-0.12}{z^2-0.4z+1.88} \end{bmatrix}$$

has unstable poles at $z = 0.2 \pm j1.36$. Nevertheless, the resulting spectral factor is

$$\Gamma(z) = \begin{bmatrix} \frac{z}{z-0.5} & \frac{0.5}{z-0.5} \\ \frac{0.5}{z-0.5} & \frac{z}{z-0.5} \end{bmatrix},$$

because the unstable poles cancel with unstable poles of $\tilde{H}^{-1}(z)$ at the same location. Therefore, the network is an IDNM, since $\Gamma(z)$ is stable and with stable inverse.

3.2 Canonical IDDNM

We now show that every stationary process $w(t)$ admits a canonical IDDNM, which is unique under certain conditions given in the following proposition

Proposition 4. Any stationary process $w(t)$ can be modeled as the output of an IDDNM as introduced in Definition 6, in which $H(z)$ and $G(z)$ satisfy the conditions of Proposition 3. In addition, there is a canonical link between the pair $(G(z), H(z))$ and $\Gamma(z)$ in (1) given by equations:

$$G_{ij}(z) = -\left(\left[\Gamma^{-1}(z)\right]_{ii}\right)^{-1}\left[\Gamma^{-1}(z)\right]_{ij} \tag{6}$$

$$H_{ii}(z) = \left[\Gamma^{-1}(z)\right]_{ii} \tag{7}$$

showing that such model is unique.

Proof: The stationary process $w(t)$ can represented using the canonical (stable, minimum phase and normalised at infinity) spectral factor $\Gamma(z)$ as follows:

$$\begin{aligned} w(t) &= \Gamma(z)\varepsilon(t) = (\Gamma(z) - I)\varepsilon(t) + \varepsilon(t) \\ &= (\Gamma(z) - I)\Gamma^{-1}(z)w(t) + \varepsilon(t)\,, \end{aligned}$$

so that

$$\hat{w}(t|t-1) := (\Gamma(z) - I)\Gamma^{-1}(z)w(t) = L(z)w(t)$$

is the one step-ahead predictor of $w(t)$ and the last equation defines $L(z) = I - \Gamma^{-1}(z)$. Let us also define $L_D(z)$ as the square transfer matrix which coincide with $L(z)$ on the diagonal and is zero out of the diagonal, thus leading to the decomposition

$$L(z) = L_D(z) + \tilde{L}_D(z)$$

where $\tilde{L}_D(z) := L(z) - L_D(z)$ equals $L(z)$ for the off-diagonal terms and is zero on the main diagonal. Since $L(z) = I - \Gamma^{-1}(z)$ is stable, so are $L_D(z)$ and $\tilde{L}_D(z)$. It thus follows that we can write the output process $w(t)$ as follows:

$$w(t) = (L(z) - L_D(z) + L_D(z))w(t) + \varepsilon(t)$$

and thus

$$w(t) = (I - L_D(z))^{-1}\tilde{L}_D(z)w(t) + (I - L_D(z))^{-1}\varepsilon(t)$$

Defining

$$G(z) = (I - L_D(z))^{-1}\tilde{L}_D(z) \qquad H(z) = (I - L_D(z))^{-1}. \tag{8}$$

Since $H(z)$ as defined in (8) is diagonal we have shown that $w(t)$ can be represented as the output of a DDNM

$$w(t) = G(z)w(t) + H(z)\varepsilon(t)$$

Recalling that $L(z) = L_D(z) + \tilde{L}_D(z) = I - \Gamma^{-1}(z)$ we have that

$$H(z) = (I - L_D(z))^{-1} = \left[\mathrm{diag}\left\{\left[\Gamma^{-1}(z)\right]_{ii}\right\}\right]^{-1}$$

so that

$$H_{ii}(z) = \left(\left[\Gamma^{-1}(z)\right]_{ii}\right)^{-1} \tag{9}$$

while

$$\begin{aligned} G(z) &= (I - L_D(z))^{-1}\tilde{L}_D(z) \\ &= \left[\mathrm{diag}\left\{\left[\Gamma^{-1}(z)\right]_{ii}\right\}\right]^{-1}\tilde{L}_D(z)\,. \end{aligned}$$

Since $(I - L_D(z))$ is diagonal, the element in position i, j of $G(z)$ is given by

$$\begin{aligned} G_{ij}(z) &= \left[(I - L_D(z))^{-1}\right]_{ii}\left[\tilde{L}_D(z)\right]_{ij} \\ &= \left(\left[\Gamma^{-1}(z)\right]_{ii}\right)^{-1}\left[\tilde{L}_D(z)\right]_{ij}\,. \end{aligned}$$

Using now the fact that, for $i \neq j$

$$\left[\tilde{L}_D(z)\right]_{ij} = L_{ij}(z) = \left[I - \Gamma^{-1}(z)\right]_{ij} = -\left[\Gamma^{-1}(z)\right]_{ij}\,,$$

we conclude that

$$G_{ij}(z) = -\left(\left[\Gamma^{-1}(z)\right]_{ii}\right)^{-1}\left[\Gamma^{-1}(z)\right]_{ij}\,. \tag{10}$$

Conversely, given an IDDNM

$$w(t) = G(z)w(t) + H(z)\varepsilon(t)$$

we can write

$$H^{-1}(z)w(t) = H^{-1}(z)G(z)w(t) + \varepsilon(t)$$

so that

$$w(t) = (I - H^{-1}(z) + H^{-1}(z)G(z))w(t) + \varepsilon(t)$$

It thus follows that $(I - H^{-1}(z) + H^{-1}(z)G(z))w(t)$ is the one step ahead predictor of $w(t)$ and therefore

$$(I - H^{-1}(z) + H^{-1}(z)G(z))w(t) = I - \Gamma^{-1}(z)$$

since $G_{ii}(z) = 0$ (and thus $[H^{-1(z)}G(z)]_{ii} = 0$) it follows that $[H^{-1}(z)]_{ii} = [\Gamma^{-1}(z)]_{ii}$ and

$$[H^{-1}(z)G(z)]_{ij} = -[\Gamma^{-1}(z)]_{ij}$$

implying that

$$G_{ij}(z) = -[H(z)]_{ii}[\Gamma^{-1}(z)]_{ij} = -[\Gamma^{-1}(z)]_{ii}^{-1}[\Gamma^{-1}(z)]_{ij}.$$

This observation concludes the proof showing that the IDDNM is unique. $\square$

As an illustration we specialize this decomposition to the two-nodes network case.

Example 3. Consider the case of a two-node network (i.e. $i \in \{1, 2\}$); we have that

$$\Gamma(z) = \begin{bmatrix} \Gamma_{11}(z) & \Gamma_{12}(z) \\ \Gamma_{21}(z) & \Gamma_{22}(z) \end{bmatrix}$$

$$\Gamma^{-1}(z) = \frac{1}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\begin{bmatrix} \Gamma_{22}(z) & -\Gamma_{12}(z) \\ -\Gamma_{21}(z) & \Gamma_{11}(z) \end{bmatrix}$$

Therefore, equation (9) can be rewritten as

$$\begin{aligned} H_{11}(z) &= \left(\left[\Gamma^{-1}(z)\right]_{11}\right)^{-1} \\ &= \left(\frac{\Gamma_{22}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right)^{-1} \\ &= \Gamma_{11}(z) - \Gamma_{12}(z)\Gamma_{22}^{-1}(z)\Gamma_{21}(z) \\ H_{22}(z) &= \left(\left[\Gamma^{-1}(z)\right]_{22}\right)^{-1} \\ &= \left(\frac{\Gamma_{11}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right)^{-1} \\ &= \Gamma_{22}(z) - \Gamma_{21}(z)\Gamma_{11}^{-1}(z)\Gamma_{12}(z) \end{aligned} \tag{11}$$

Similarly, as far as the $\tilde{G}_{ij}(z)$'s are concerned, we have

$$\begin{aligned} G_{12}(z) &= -\left(\left[\Gamma^{-1}(z)\right]_{11}\right)^{-1}\left[\Gamma^{-1}(z)\right]_{12} \\ &= -\left(\frac{\Gamma_{22}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right)^{-1} \\ &\quad\times\left(\frac{-\Gamma_{12}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right) \\ &= \Gamma_{22}^{-1}(z)\Gamma_{12}(z) \\ G_{21}(z) &= -\left(\left[\Gamma^{-1}(z)\right]_{22}\right)^{-1}\left[\Gamma^{-1}(z)\right]_{21} \\ &= -\left(\frac{\Gamma_{11}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right)^{-1} \\ &\quad\times\left(\frac{-\Gamma_{21}(z)}{\Gamma_{11}(z)\Gamma_{22}(z) - \Gamma_{12}(z)\Gamma_{21}(z)}\right) \\ &= \Gamma_{11}^{-1}(z)\Gamma_{21}(z) \end{aligned} \tag{12}$$

Equations (11) and (12) are the classical expressions found in the literature (see (Gevers and Anderson, 1981)) which link the minimum phase spectral factor, normalized at infinity, and the (unique) internally stable feedback representation (with the terminology of this proposition the "diagonal network innovation model").

4. TWO NODE NETWORKS

In this section, we focus on networks having two nodes, and establish an explicit map that transforms a DNM representation into DDNM representation, keeping the same driving noise (whether or not it is the innovation process). The result is particularly important since, according to the network identifiability results of Weerts et al. (2018), having a diagonal noise model leads to identifiability of the network. In other words, when the noise model is diagonal, the network structure (and dynamics) is uniquely determined from the spectrum of the noise-contribution on the node signals, and the transfer from reference signals to node signals. This result has an interpretation in a relation setting, where we consider whether any dynamic network with non-diagonal noise model, can be equivalently written in a network with diagonal noise model, which would result to be identifiable.

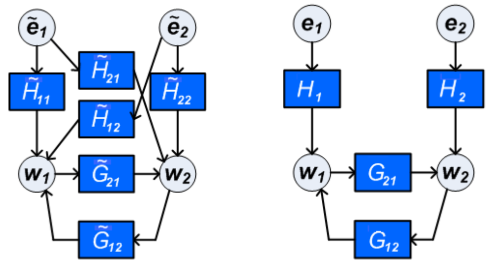

Fig. 1. Two-node case: DNM (left) and DDNM (right).

Proposition 5. Consider a two-node DNM. There exists an equivalent DDNM driven by the same noise process, i.e. $e(t) = \tilde{e}(t)$, such that (the argument z is omitted)

$$G_{21} = \frac{\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21}}{\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12}} \qquad G_{12} = \frac{\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12}}{\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21}}$$

$$H_1 = \frac{(\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})}{(1 - \tilde{G}_{12}\tilde{G}_{21})} - \frac{(\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21})(\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12})}{(1 - \tilde{G}_{12}\tilde{G}_{21})(\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})}$$

$$H_2 = \frac{(\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})}{(1 - \tilde{G}_{12}\tilde{G}_{21})} - \frac{(\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12})(\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21})}{(1 - \tilde{G}_{12}\tilde{G}_{21})(\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})}.$$

Proof: The system equations of the DNM are written as:

$$w_2 = (\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21})\tilde{S}\tilde{e}_1 + (\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})\tilde{S}\tilde{e}_2 \quad (13)$$

$$w_1 = (\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})\tilde{S}\tilde{e}_1 + (\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12})\tilde{S}\tilde{e}_2 \quad (14)$$

with $\tilde{S} := (1 - \tilde{G}_{12}\tilde{G}_{21})^{-1}$, while the DDNM follows

$$w_2 = H_1 G_{21} S e_1 + H_2 S e_2 \quad (15)$$

$$w_1 = H_1 S e_1 + H_2 G_{12} S e_2 \quad (16)$$

with $S := (1 - G_{12}\tilde{G}_{21})^{-1}$.

We now derive a DNM representation enforcing $\tilde{e} = e$. In order to guarantee that the w_i's are the same in the two representations, the transfer G_{21} then needs to be given by the quotient of the transfers $e_1 \rightarrow w_2$ and $e_1 \rightarrow w_1$. In other words, by using (13) and (14) we obtain

$$G_{21} = \frac{\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21}}{\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12}}.$$

A similar type of reasoning will providfe an expression for G_{12} by taking the quotient of the transfers $e_2 \rightarrow w_1$ and $e_2 \rightarrow w_2$, leading to

$$G_{12} = \frac{\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12}}{\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21}}.$$

In order to find the expressions for H_1 and H_2 we first need to evaluate S. Note that

$$S = \frac{1}{1 - G_{12}G_{21}} = \frac{1}{1 - \frac{\tilde{H}_{12}+\tilde{H}_{22}\tilde{G}_{12}}{\tilde{H}_{22}+\tilde{H}_{12}\tilde{G}_{21}} \frac{\tilde{H}_{21}+\tilde{H}_{11}\tilde{G}_{21}}{\tilde{H}_{11}+\tilde{H}_{21}\tilde{G}_{12}}}$$

$$= \frac{(\tilde{H}_{22}+\tilde{H}_{12}\tilde{G}_{21})(\tilde{H}_{11}+\tilde{H}_{21}\tilde{G}_{12})}{(\tilde{H}_{22}+\tilde{H}_{12}\tilde{G}_{21})(\tilde{H}_{11}+\tilde{H}_{21}\tilde{G}_{12}) - (\tilde{H}_{12}+\tilde{H}_{22}\tilde{G}_{12})(\tilde{H}_{21}+\tilde{H}_{11}\tilde{G}_{21})}.$$

We can now determine H_1 on the basis of $H_1 S = (\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})S$, leading to

$$H_1 = \frac{(\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})\tilde{S}}{S},$$

which can be shown to be equal to the expression reported in the statement of the proposition. Using the properties of $\tilde{H}$ that $\tilde{H}_{11}$ and $\tilde{H}_{22}$ are monic, and $\tilde{H}_{12}$ and $\tilde{H}_{21}$ are strictly proper, it follows that

$$H_1^{\infty} := \lim_{z\to\infty} H_1(z) = \frac{1 - G_{21}^{\infty}G_{12}^{\infty}}{1 - \tilde{G}_{21}^{\infty}\tilde{G}_{12}^{\infty}} = 1$$

so that H_1 is monic too, implying that the requirements of Definition 4 are satisfied.

For H_2 we utilize the equality $H_2 S = (\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})\tilde{S}$, leading to

$$H_2 = \frac{(\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})\tilde{S}}{S}$$

and with the same reasoning as above it follows that $H_2^{\infty} = 1$. □

Remark 1. We note that $\tilde{H}_{21}(z) = 0$ implies that $G_{21}(z) = \tilde{G}_{21}(z)$. This means that, if one applies the direct method to identify $\tilde{G}_{21}(z)$ (assuming that the noise sources on $w_1(t)$ and $w_2(t)$ are uncorrelated), the resulting estimate would be consistent. This does not hold if $\tilde{H}_{21}(z) \neq 0$. This result is in line with the findings of Van den Hof et al. (2017), where a method called the *joint direct method* is introduced to deal with dynamic networks with correlated noise sources.

We also observe that, even if $\tilde{G}_{21}(z) = 0$, we have that $G_{21}(z) \neq 0$. The latter term appears in the DDNM to account for correlation among the original noise sources.

Example 4. Consider a DNM consisting of a two nodes, with

$$\tilde{H}(z) = \begin{bmatrix} \tilde{H}_{11}(z) & \tilde{H}_{12}(z) \\ \tilde{H}_{21}(z) & \tilde{H}_{22}(z) \end{bmatrix} = \begin{bmatrix} \frac{z+0.8}{z+0.7} & \frac{0.1}{z-0.9} \\ \frac{-2}{z+0.3} & \frac{z-0.2}{z+0.8} \end{bmatrix}$$

and

$$\tilde{G}_{12}(z) = \frac{-0.58z + 0.186}{z^2 + 1.2z + 0.32}, \; \tilde{G}_{21}(z) = \frac{z + 0.4}{z^2 - 1.2z + 0.27}.$$

Note that $\tilde{H}(z)$ is stable, minimum phase, and monic, while $\tilde{G}(z)$ and $(I - \tilde{G}(z))^{-1}$ are both stable, thus fulfilling the assumptions of Proposition 2 and Proposition 5. Therefore, this is an IDNM Using Proposition 5, we transform the network into an IDDNM representation, obtaining

$$H_1(z) = \frac{z^6 - 1.2z^5 - 0.32z^4 + 0.94z^3 - 0.28z^2 - 0.12z + 0.04}{z^6 - 1.3z^5 - 0.22z^4 + 0.1z^3 + 0.08z^2 - 0.002z + 0.02},$$

$$H_2(z) = \frac{z^5 + 0.4z^4 - 0.59z^3 - 0.004z^2 + 0.25z + 0.06}{z^5 + 1.4z^4 + 0.97z^3 - 1.66z^2 - 1.16z + 0.17},$$

which can be shown to be minimum phase (and monic) but not stable, since $H_1(z)$ has the unstable poles at $z = 1.0531 \pm j0.5436$, while $H_2(z)$ has the unstable poles at $z = -0.8621 \pm j1.1790$. We also get

$$G_{12}(z) = \frac{-0.48z^5 + 1.6z^4 - 1.54z^3 + 0.55z^2 - 0.12z + 0.02}{z^6 - 1.1z^5 - 0.57z^4 + 1.11z^3 + 0.21z^2 - 0.03z + 0.03},$$

$$G_{21}(z) = \frac{-z^5 + 1.3z^4 + 4.5z^3 + 2.7z^2 + 0.24z - 0.09}{z^6 + 1.1z^5 + 0.55z^4 - 1.94z^3 - 0.66z^2 + 0.51z - 0.05},$$

which have the same unstable poles of $H_1(z)$ and $H_2(z)$, respectively. Hence, $H^{-1}(z)G(z)$ is stable (as also shown by Proposition 3). Finally, it can be also shown that the sensitivity function

$$S(z) = \frac{z^3 - 0.4z^2 - 0.69z + 0.216}{z^3 - 0.4z^2 - 0.11z + 0.03}$$

is stable, a fact that agrees with Corollary 1.

5. CONCLUSIONS

In this paper we have discussed the representation of a multivariate stationary stochastic process as the output of a dynamic network model driven by noise. Our results extend those of Gevers and Anderson (1982, 1981) from the single feedback loop case to the dynamic network case. We have provided stability conditions on the network that depend on whether the noise model is general (full) or diagonal. Furthermore, focusing on dynamic networks driven by the innovation process and with diagonal noise model, we have shown that these networks are internally stable and provided a constructive way to obtain the network components from the elements of the spectral factor associated with innovation process. For two-node networks, we have shown how to explicitly obtain a diagonal noise representation starting from a full noise one.

We are currently studying how the models analyzed in this paper relate to those proposed by Materassi and Salapaka (2012), where non-strictly causal elements are allowed, and by Hayden et al. (2016), where specific state-space network models are studied.

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