Allocation of Excitation Signals for Generic Identifiability of Dynamic Networks

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***Abstract*— This paper studies the generic identifiability of dynamic networks, in which the edges connecting the vertex signals are described by proper transfer functions, and partial vertices are stimulated by white noises and designed external signals. We assume that the topology of the underlying graph is known, and all the vertex signals are measured. We show that the generic identifiability of a directed network is related to the existence of a set of disjoint directed pseudo-trees that cover all the edges of the underlying graph, based on which, an excitation allocation problem is studied, aiming to select the minimal number excitation signals to achieve the generic identifiability of the whole network. An algorithmic procedure thereby is devised for selecting locations of the external signals such that all the edges can be consistently estimated.**

I. Introduction

Dynamic networks can adequately describe a wide class of complex engineering systems, which appear in various applications, including multi-robot coordination and distributed control of power grids [1]. The conventional system identification mainly focuses on the systems with relatively simple dynamical structures, e.g., single-input-single-output (SISO), multiple-input-multiple-output (MIMO), open-loop or closed-loop systems [2], [3]. However, these classical data-driven tools seem to be limited when encountering dynamic networks with complex interconnection structures. The bridge connecting the classic identification framework and dynamic networks is initially built in [4], where the vertices in a network are interpreted as measured internal signals, and the directed edges represent transfer operators, referred as *modules*. Considering external noises and excitation signals, the identification of the modules in a network can be recast as a closed-loop system identification problem.

Based on this setup, three problems have been addressed. The first is to detect the topology of a network, see e.g., [5], [6], where techniques, such as Wiener filters or Bayesian approaches are taken to obtain sparse estimates. The second problem estimates a desired local module within a network, see e.g., [7]–[10], which focus on the question: under what conditions we are able to consistently identify the dynamics of a selected module in the network? In this paper, the problem in contrast, concerns the structural identifiability of a full dynamic network, see e.g., [11]–[14] and the references therein. Assuming that the topology of a network is known a priori, and a model set is considered in which all the models are associated with the given topology. Then the identifiability essentially reflects the ability to distinguish between these models on the basis of measurement data.

In the literature, there are two classes of network identifiability, namely, the *global identifiability* [12], [14] that requires *all* the models in the set to be distinguishable, and the *generic identifiability* [13], [15], which means that *almost all* models in the model set can be distinguished. In the study of identifiability, two problem settings of dynamic networks are considered. In e.g., [13], [14], all vertices are excited by external signals, while only a subset of vertices is measured, and in e.g., [12], [15], all internal variables are supposed to be measured, while only partial vertices are excited or influenced by noises. Thus, to achieve the identifiability of a dynamic network, we do not need to measure and excite all the vertices. This observation motivates the research in this paper, in which the generic identifiability is of particular interest. Given the topology of a network consisting of internal signals, we aim for a systematic scheme that finds the minimum number of excitation signals for the generic identifiability of a dynamic network. To the best of our knowledge, such a design problem has not been addressed in the context of generic identifiability for dynamic networks.

Inspired by [12], [13], we analyze the generic identifiability from a graph-theoretic point of view and provide a new characterization using the concept of *disjoint pseudo-tree covering*. It is shown that a directed dynamic network is generically identifiable if and only if there exist a set of disjoint pseudo-trees covering all the edges of the underlying graph, and the external signals are allocated at the roots of these pseudo-trees. Thereby, we propose a graph merging approach, which first partitions the network into several *minimal* disjoint pseudo-trees and then iteratively aggregates the pairs that are *mergeable*. The merging process is consistent with the operation on the characterization matrix of the disjoint pseudo-tree covering.

The rest of this paper is organized as follows: In Section II, we recap some basic notations in graph theory and introduce the network model. Section III then presents a graph-theoretic approach to the allocation of excitation signals, and finally, concluding remarks are made in Section IV.

Notation: Denote $\mathbb{R}$ as the set of real numbers, and $\mathbb{R}(q)$ is the rational function field over $\mathbb{R}$ with variable q. v_i denotes the i-th element of a vector v, and A_{ij} denotes the (i,j)-th entry of a matrix A. The cardinality of a set $\mathcal{V}$ is given by $|\mathcal{V}|$. Let $\mathcal{G}$ be a directed graph, and we denote $V(\mathcal{G})$ and $E(\mathcal{G})$ as the vertex set and edge set of $\mathcal{G}$, respectively. The

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union of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is denoted by $\mathcal{G} := G_1 \cup G_2$, where $V(\mathcal{G}) = V(\mathcal{G}_1) \cup V(\mathcal{G}_2)$ and $E(\mathcal{G}) = E(\mathcal{G}_1) \cup E(\mathcal{G}_2)$.

II. Preliminaries

A. Graph theory

We provide necessary terminologies and concepts from graph theory and refer to [1], [16] for more details. The topology of a network is characterized by a graph $\mathcal{G}$ that consists of a finite and nonempty vertex set $\mathcal{V} := \{1, 2, \cdots, L\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. For a directed graph, each element in $\mathcal{E}$ is an ordered pair of elements of $\mathcal{V}$, and if $(i, j) \in \mathcal{E}$, we say that the edge is incident from vertex i to vertex j. The vertex i is the *in-neighbor* of j, and j is the *out-neighbor* of i. Let $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$ be the sets that collect all the in-neighbors and out-neighbors of vertex i, respectively. Let S be the set that collects all sinks of $\mathcal{G}$, namely,

$$S(\mathcal{G}) := \{i \in V(\mathcal{G}) \mid |\mathcal{N}_i^+| = 0\}. \tag{1}$$

A graph $\mathcal{G}$ is called *simple*, if $\mathcal{G}$ does not contain self-loops (i.e., $\mathcal{E}$ does not contain edges of the form (i, i), $\forall\ i$), and there exists only one directed edge from one vertex to its each out-neighbor. In a simple graph, a directed path connecting vertices i_0 and i_n is a sequence of edges of the form (i_{k-1}, i_k), $k = 1, \cdots, n$. Two directed paths are *vertex disjoint* if they do not share any common vertex, including the start and the end vertices. A connected directed simple graph T is a *directed tree* (or an arborescence), if there exists a vertex r, known as the *root* vertex, from which there is exactly one directed path from r to every other vertices in T. A vertex i in a tree T is called a *leaf* if $|\mathcal{N}_i^+| = 0$, and the vertices that are neither the root nor leaves of T are called the *internal vertices* of T. In a rooted tree, a vertex i is a *child* of vertex j if there is a directed edge incident from j to i, in which case, vertex j is the (only) *parent* of i.

B. Dynamic network model

Consider a simple directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, L\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Then, following the basic setup of [4], [12], a dynamic network associated with $\mathcal{G}$ is defined as follows.

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t), \tag{2}$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j = w_j(t-1)$. $w(t) \in \mathbb{R}^L$, $r(t) \in \mathbb{R}^{K_r}$, and $e(t) \in \mathbb{R}^{K_e}$ are vectors of measured internal signals, external excitation signals, and external disturbances, respectively. Let $\mathcal{R}_r \subseteq \mathcal{V}$ and $\mathcal{R}_e \subseteq \mathcal{V}$ be the sets of vertices that are affected by the external excitation signals and disturbances, respectively. Denote $\mathcal{R} = \mathcal{R}_r \cup \mathcal{R}_e$, and $K := |\mathcal{R}|$. Each vertex in $\mathcal{R}$ is called a *stimulated vertex*. Throughout the paper, the transfer matrices $G(q)$, $R(q)$, and $H(q)$ satisfy the following properties [4], [12].

- All the entries of $G(q)$ are strictly proper transfer functions to be identified, and the entries of $R(q)$ and $H(q)$ are known proper transfer functions.
- Each row and each column of $R(q)$, $H(q)$ contain only one nonzero entry, i.e., each vertex in $\mathcal{R}$ is influenced by a unique external (excitation or disturbance) signal.
- The network $\mathcal{G}$ is *well-posed*, i.e., $(I - G(q))^{-1}$ is proper and stable. $R_{ik}(q)$ is stable, and $H_{ij}(q)$ is monic and minimum-phase, for all $i \in \mathcal{V}$, $k \in \mathcal{R}_r$, and $j \in \mathcal{R}_e$.

In this paper, we are interested in the condition under which all the transfer functions in $G(q)$ can be consistently identified from the external excitation signals $r(t)$ and the measurement data $w(t)$. Thereby, we assume that the topology is known a priori, and all the vertex signals $w(t)$ are measured. The disturbances are unmeasured white noises $e(t)$, whose locations are known. Then, we aim to allocate a minimal number of external excitation signals such that the full network is generically identifiable.

III. Main results

A. Generic identifiability: a graph-theoretical condition

The identifiability of the dynamic network in (2) is basically the ability to uniquely identify transfer functions in $G(q)$ from excitation signals $r(t)$ and the measurement date $w(t)$. Following [12], we denote a set of parameterized matrix-valued functions $\Sigma := \{\sigma(q, \theta) = (G(q, \theta), R(q), H(q)), \theta \in \Theta\}$ as the network model set for a network described in (2). Let

$$T(q, \theta) = (I - G(q, \theta))^{-1} \begin{bmatrix} R(q) & H(q) \end{bmatrix} \tag{3}$$

Then, the generic identifiability of Σ is defined as follows.

Definition 1 (Generic identifiability): The network model set Σ is generically identifiable if the implication

$$T(q, \theta_1) = T(q, \theta_0) \Rightarrow G_{ji}(q, \theta_1) = G_{ji}(q, \theta_0),\ \forall\ i, j \in \mathcal{V}$$

holds for almost all models $\sigma(\theta_1)$ and $\sigma(\theta_2)$ in Σ.

We refer to e.g., [13], [15] for more details. Moreover, a graph-theoretic condition has been studied for checking the generic identifiability of a dynamic network.

Lemma 1: [13], [15] A dynamic network is generically identifiable if and only if the maximum number of mutually vertex disjoint paths from $\mathcal{R}$ to $\mathcal{N}_i^-$ is equal to $|\mathcal{N}_i^-|$ for all $i \in \mathcal{V}$.

Note that the characterization of the generic identifiability in Lemma 1 relates to all the vertex disjoint paths from the stimulated vertices (i.e., the vertices in $\mathcal{R}$) to the in-neighbors of each vertex in a network. In contrast, this paper proposes a novel graph-theoretic condition for characterizing the generic identifiability for a given directed network. Before proceeding, the concept of directed pseudo-trees is introduced.

Definition 2 (Directed pseudo-trees): A connected directed graph $\mathcal{T}$ is called a (directed) **pseudo-tree** if $|\mathcal{N}_i^-| \leq 1$, for all $i \in V(\mathcal{T})$.

The above concept of pseudo-trees is an extension of its definition in the undirected case, in which they are also referred to as *unicyclic graphs*, see e.g., [17], [18]. Analogous to directed tree graphs, we call a vertex in the directed pseudo-tree $\mathcal{T}$ a *root* of $\mathcal{T}$, if there is exactly one directed path from this vertex to every other vertices in $\mathcal{T}$.

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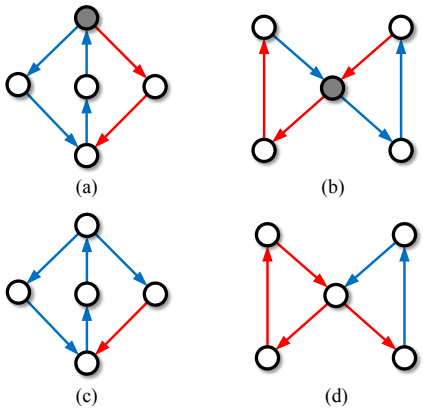

Fig. 1. Illustration of disjoint pseudo-trees, in which different pseudo-trees are induced by the edges with distinct colors. In (a) and (b), the pseudo-trees are not disjoint, since the out-neighbors of the gray vertices are assigned to different pseudo-trees. In contrast, the pseudo-trees in (c) and (d) are considered to be disjoint pairs.

Furthermore, a vertex in $\mathcal{T}$ is a *leaf* of $\mathcal{T}$, if it has no out-neighbors in $\mathcal{T}$. A vertex in $\mathcal{T}$ that is neither a root nor a leaf is an internal vertex of $\mathcal{T}$. We call a directed pseudo-tree *minimal* if it only contains one root and all the out-neighbors of the root as leaves.

From Definition 2, the class of directed pseudo-trees includes all the directed trees, but it also allows for a directed graph to contain a unique directed circle, and each vertex on the circle is a root of the pseudo-tree. Hereafter, we will drop the word 'directed' when we refer to a directed pseudo-tree, and denote $R(\mathcal{T})$ as a set that collects all the roots of the pseudo-tree $\mathcal{T}$. It is remarked that any pseudo-tree can be expressed as a union of a directed tree with a possible directed edge from a leaf or an internal vertex of the tree to its root. Related to the concept of vertex disjoint paths, disjoint directed pseudo-trees are defined.

Definition 3 (Disjoint pseudo-trees): Consider two pseudo-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ as subgraphs of a directed graph $\mathcal{G}$. $\mathcal{T}_1$ and $\mathcal{T}_2$ are called **disjoint** in $\mathcal{G}$ if the following two conditions hold.

1) $E(\mathcal{T}_1) \cap E(\mathcal{T}_2) = \emptyset$;
2) $\mathcal{N}_i^+ \subseteq V(\mathcal{T}_1)$ or $V(\mathcal{T}_2)$, for all $i \in V(\mathcal{T}_1) \cup V(\mathcal{T}_2)$.

The first condition means that disjoint pseudo-trees do not share common edges, and the second condition requires that all the edges incident from each vertex should be included in the same pseudo-tree. If both $\mathcal{T}_1$ and $\mathcal{T}_2$ are directed rooted trees, then $\mathcal{T}_1$ and $\mathcal{T}_2$ do not share the same root and any common internal vertex.

Example 1: A simple example of disjoint pseudo-trees is shown in Fig. 1. The pseudo-trees in both (a) and (b) are not disjoint, as the second condition in Definition 3 is not satisfied in both cases. With the same topologies, the two pseudo-trees depicted in (c) and (d) are disjoint.

It worth noting that the notion of disjoint pseudo-trees is closely related to that of vertex disjoint paths. If two pseudo trees are disjoint, then we can show that, for all $i \in V(\mathcal{T}_1) \cup V(\mathcal{T}_2)$, there exist $|\mathcal{N}_i^-|$ mutually vertex disjoint paths in the union $\mathcal{T}_1 \cup \mathcal{T}_2$ starting from $\{r_1, r_2\}$ to $\mathcal{N}_i^-$, with $r_1 \neq r_2$, $r_1 \in R(\mathcal{T}_1)$, and $r_2 \in R(\mathcal{T}_2)$.

Next, the concept of edge covering for a directed graph is introduced.

Definition 4 (Edge covering): Consider a directed graph $\mathcal{G}$, and let $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K$ be a collection of connected subgraphs of $\mathcal{G}$. The edges of $\mathcal{G}$ are covered by $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K$ if $E(\mathcal{T}_1) \cup E(\mathcal{T}_2) \cup \cdots \cup E(\mathcal{T}_K) = E(\mathcal{G})$. The set $\Pi := \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K\}$ is called a **covering** of $\mathcal{G}$. Moreover. Π is **disjoint pseudo-tree covering**, if all the elements in Π are pseudo-trees, which are disjoint to each other.

With the definition of the disjoint pseudo-tree covering, the following two lemmas are given.

Lemma 2: For any directed network $\mathcal{G}$, there always exists a set of disjoint pseudo-trees that cover all the edges of $\mathcal{G}$.

Proof: For any directed graph $\mathcal{G}$, a set of disjoint pseudo-trees can be constructed as follows. For each vertex $k \in V(\mathcal{G}) - S(\mathcal{G})$, we construct a directed star tree (i.e., a minimal pseudo-tree) with k as the root and the vertices in $\mathcal{N}_k^+$ as leaves. Then, $|V(\mathcal{G}) - S(\mathcal{G})|$ minimal pseudo-trees are formed, which are disjoint, since any two trees do not share a common root and any common internal vertex. ■

Lemma 3: If there exist K_1, with $K_1 < |V(\mathcal{G})|$ disjoint pseudo-trees covering all the edges of a graph $\mathcal{G}$, then there also exist K_2 disjoint pseudo-trees, for any $K_1 < K_2 \leq |V(\mathcal{G})|$, that cover all the edges of $\mathcal{G}$.

Proof: Note that if one of the K_1 disjoint pseudo-trees is a singleton vertex, then the statement is immediate. Thereby, in the rest of the proof, we exclude a singleton vertex being a pseudo-tree. Then, the maximal number of disjoint pseudo-trees that coexist in $\mathcal{G}$ does not exceed $|V(\mathcal{G}) - S(\mathcal{G})|$, where $S(\mathcal{G})$ is the set of the sinks in $\mathcal{G}$. It requires $K_1 < |V(\mathcal{G}) - S(\mathcal{G})|$, implying that in the K_1 disjoint pseudo-trees, there exists at least one pseudo-tree $\mathcal{T}_k$ such that $\mathcal{T}_k$ contains at least one internal vertex or contains multiple roots. We then show that in the both cases, $\mathcal{T}_k$ can be decomposed into two disjoint pseudo-trees.

Suppose $\mathcal{T}_k$ contains internal vertices. We can always find an internal vertex i with all its out-neighbors being the leaves of $\mathcal{T}_k$. Define a directed tree T_a with i as the root and $\mathcal{N}_i^+$ as the leaves. Thereby, $\mathcal{T}_k$ is decomposed into two a directed tree T_a and a pseudo-tree $\mathcal{T}_b$, where $R(T_b) := R(\mathcal{T}_k)$, $V(\mathcal{T}_b) \subseteq (\mathcal{T}_k)$, and $E(T_b) := E(\mathcal{T}_k) - E(T_a)$. Note that T_a and $\mathcal{T}_b$ are disjoint by Definition 3. Moreover, since T_a and $\mathcal{T}_b$ are subgraphs of $\mathcal{T}_k$, which is disjoint to the other trees, T_a and $\mathcal{T}_b$ are also disjoint to the other pseudo-trees. Next, suppose $\mathcal{T}_k$ does not contain any internal vertex, but more than one roots, i.e., $|R(\mathcal{T}_k)| \geq 2$. In this case, we define the directed tree T_a, which is rooted at one of $R(\mathcal{T}_k)$ and includes all the out-neighbors of this root as the leaves of T_a. Then, similar to the previous case, we can partition $\mathcal{T}_k$ into two disjoint pseudo-trees, which are disjoint to the other pseudo-trees in $\mathcal{G}$. Therefore, in the above cases, the graph $\mathcal{G}$ can be always decomposed into $K_1 + 1$ disjoint trees that still cover all the edges $E(\mathcal{G})$. The statement of this lemma follows by iteratively applying the above reasoning for all $K_2 \geq K_1 + 1$. ■

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Now, we are ready present the main conclusion in this section.

Theorem 1: Consider a directed network $\mathcal{G}$, and let $\mathcal{R} = \{\tau_1, \tau_2, \cdots, \tau_K\} \subseteq V(\mathcal{G})$ be the set of stimulated vertices. Then, the network $\mathcal{G}$ is generically identifiable if and only if there exists a disjoint pseudo-tree covering of $\mathcal{G}$, denoted by $\Pi = \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K\}$ such that $\tau_i \in R(\mathcal{T}_i)$, for all $i \in \{1, 2, \cdots, K\}$.

Proof: We first prove the 'if' statement. Assume that there exist a disjoint pseudo-tree covering, $\Pi = \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_K\}$, and in each pseudo-tree $\mathcal{T}_k$, one of its root vertices is stimulated by an external signal $r_k(t)$, i.e., $\tau_i \in R(\mathcal{T}_k)$. Since there does not exist any two vertices in a same pseudo-tree sharing a common child, for any vertex $i \in V(\mathcal{G})$, all the edges incident from vertices in $\mathcal{N}_i^-$ to i should belong to different pseudo-trees, and $|\mathcal{N}_i^-| \leq K$ holds. Thus, there exist at least $|\mathcal{N}_i^-|$ vertex disjoint paths from $\{\tau_1, \tau_2, \cdots, \tau_K\}$ to $\mathcal{N}_i^-$. As each τ_i, a root of a pseudo-tree $\mathcal{T}_i$, is chosen as an independent stimulation source, the generic identifiability of $\mathcal{G}$ then immediately follows from Lemma 1.

Next, the 'only if' statement is proven. Let $\mathcal{G}$ be generically identifiable, and let $\mathcal{R} \subseteq V(\mathcal{G})$ with the cardinality of K be the set that collects all the stimulated vertices in $\mathcal{G}$. Lemma 2 implies that there is a minimal number of pseudo-trees covering all the edges of $\mathcal{G}$. Let $\kappa(\mathcal{G})$ be this minimal number. We then show that $\mathcal{G}$ is generic identifiable only if $K \geq \kappa(\mathcal{G})$. The proof proceeds by contradiction. Assume that $K < \kappa(\mathcal{G})$, and we have $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_\kappa$ be disjoint pseudo-trees covering all the edges of $\mathcal{G}$, where $\kappa = \kappa(\mathcal{G})$. Then, there exists at least one pseudo-tree $\mathcal{T}_k$ whose vertices are not assigned with any external signals. Consequently, there exists a vertex $i \in V(\mathcal{T}_k)$ such that the mutually vertex disjoint paths from $\mathcal{R}$ to $|\mathcal{N}_i^-|$ will be less than $|\mathcal{N}_i^+|$, namely, the network $\mathcal{G}$ is not generically identifiable. That yields a contradiction. Therefore, if $\mathcal{G}$ is generic identifiable, then there exist at least K disjoint pseudo-trees that cover all the edges of $\mathcal{G}$.

That completes the proof. ■

The following result is obtained by directly applying Theorem 1.

Corollary 1: The minimal number of external signals that are needed for the generic identifiability of a directed graph $\mathcal{G}$ is equal to the minimal number of disjoint pseudo-trees that cover all the edges of $\mathcal{G}$.

We consider dynamic network $\mathcal{G}$ in (2). Suppose it is noise-free, i.e., $e(t) = 0$, and the minimal number of disjoint pseudo-trees that cover all the edges of $\mathcal{G}$ is K. Then, we only need K external excitation signals to guarantee the generic identifiability of $\mathcal{G}$.

B. Excitation allocation: a graph merging approach

In this section, we aim to solve an excitation allocation problem, which aims for a minimal number of external excitation signals which are used to consistently estimate the proper transfer functions associated with all the edges of a network.

Problem 1: Given a directed network $\mathcal{G}$ with known $\mathcal{R}_e \subseteq V(\mathcal{G})$, find $\mathcal{R}_r \subseteq V(\mathcal{G})$ as the set of excited vertices such that $|\mathcal{R}_r|$ is minimized, and $\mathcal{G}$ is generic identifiable.

The previous subsection shows the relation between the generic identifiability and the disjoint pseudo-tree covering. Note that the locations of noise signals fixed, and we are actually looking for the locations of excitation signals. Then, solving the above problem amounts to find the minimal number of disjoint pseudo-trees, which are not rooted at $\mathcal{R}_e$ in the network, such that all the edges are covered. At this point, Problem 1 is converted to a combinatorial optimization problem. In this section, we devise an algorithmic procedure to partition a graph into disjoint pseudo-trees with as less number as possible.

Lemma 2 indicates that for any directed graph $\mathcal{G}$, we can always find a disjoint pseudo-tree covering,

$$\Pi_0 = \{\mathcal{T}_1^0, \mathcal{T}_2^0, \cdots, \mathcal{T}_{|\Pi_0|}^0\}, \tag{4}$$

in which each element is a minimal pseudo-tree rooted at a vertex in $V(\mathcal{G}) - S(\mathcal{G})$ with $|\Pi_0| = |V(\mathcal{G}) - S(\mathcal{G})|$. We start with Π_0 as our initial disjoint pseudo-tree covering, and our strategy is to recursively merge the pseudo-trees until there are no mergeable pseudo-trees in a covering. Specifically, the mergeability of pseudo-trees are defined as follows.

Definition 5 (Mergeability): Consider two disjoint pseudo-trees $\mathcal{T}_1$ and $\mathcal{T}_2$. We say $\mathcal{T}_1$ is mergeable to $\mathcal{T}_2$, if

1) the union of $\mathcal{T}_1$ and $\mathcal{T}_2$, i.e., $(V(\mathcal{T}_1) \cup V(\mathcal{T}_2), E(\mathcal{T}_1) \cup E(\mathcal{T}_2))$ is also a pseudo-tree;
2) there is a directed path from vertex i to vertex j, for all $i \in R(\mathcal{T}_2)$ and $j \in V(\mathcal{T}_1)$.

The mergeability from a pseudo-tree $\mathcal{T}_1$ to $\mathcal{T}_2$ requires that $\mathcal{T}_1$ and $\mathcal{T}_2$ does not share any common leaf and internal vertex. As a result, merging $\mathcal{T}_1$ and $\mathcal{T}_2$ yields a new pseudo-tree $\mathcal{T}_3$, where $R(\mathcal{T}_3) = R(\mathcal{T}_2)$. Note that $\mathcal{T}_1$ being mergeable to $\mathcal{T}_2$ does not necessarily mean that $\mathcal{T}_2$ is also mergeable to $\mathcal{T}_1$.

Given a disjoint pseudo-tree covering of $\mathcal{G}$, in which some pseudo-trees are rooted at $\mathcal{R}_e$, i.e., the vertices that are directly affected by noises. If a pseudo-tree is rooted at a vertex in $\mathcal{R}_e$, then we prevent it from merging to the other pseudo-trees. We introduce a characteristic matrix $\mathcal{M}$, whose (i, j)-th entry is defined as follows.

$$\mathcal{M}(i,j) = \begin{cases} 1 & \text{if } \mathcal{T}_i \text{ is mergeable to } \mathcal{T}_j,\ R(\mathcal{T}_i) \cap \mathcal{R}_e = \emptyset; \\ \varnothing & \text{if } V(\mathcal{T}_i) \cap V(\mathcal{T}_j) = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Related to the characteristic matrix, the following notations and operations are defined. Let $\mathcal{M}(i,:)$ and $\mathcal{M}(:,j)$ be the i-th row and j-th column of a matrix $\mathcal{M} \in \mathbb{M}$, and denote $\mathbb{M} := \{1, \varnothing, 0\}$. Then we have $\mathcal{M} \in \mathbb{M}^{|\Pi| \times |\Pi|}$, where Π is a disjoint pseudo-tree covering. We define $c = a \odot b = b \odot a$, with $a, b, c \in \mathbb{M}$, as a commutative operation, which follows the rules:

$$\begin{aligned} &1 \odot 1 = 1,\ 1 \odot 0 = 0,\ 1 \odot \varnothing = 1, \\ &0 \odot 0 = 0,\ \varnothing \odot 0 = 0,\ \varnothing \odot \varnothing = \varnothing. \end{aligned} \tag{6}$$

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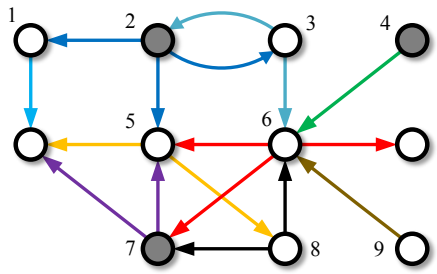

Fig. 2. A directed simple graph with 11 vertices, which is decomposed into 9 disjoint pseudo-trees, which are labeled with different colors. The gray vertices are directly affected by external noises.

Furthermore, let ρ, μ be two column (or row) vectors of the same dimensions in $\mathbb{M}$. Then, $\rho \odot \mu = \mu \odot \rho$ stands for an entrywise operation that returns a new column (or row) vector, whose i-th element is given by $\rho_i \odot \mu_i$.

Example 2: Consider a directed simple graph with 10 vertices, as shown in Fig. 2. Following Lemma 2, we find an initial disjoint pseudo-tree covering $\Pi_0 = \{\mathcal{T}_1^0, \mathcal{T}_2^0, \cdots, \mathcal{T}_9^0\}$, and each pseudo-tree has a single root vertex, which is not a sink of $\mathcal{G}$. By the definition in (5), we construct the following matrix for characterizing the mergeability of Π_0.

$$\mathcal{M}_0 = \begin{bmatrix} 0 & 1 & \varnothing & \varnothing & 0 & \varnothing & 0 & \varnothing & \varnothing \\ 0 & 0 & 0 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\ \varnothing & 1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ \varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ 0 & 1 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\ \varnothing & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \varnothing & \varnothing & 0 & 0 & 0 & 0 & \varnothing \\ \varnothing & \varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \end{bmatrix}. \quad (7)$$

The operation on the first two rows in $\mathcal{M}_0$ leads to

$$\mathcal{M}_0(1,:) \odot \mathcal{M}_0(2,:) = \begin{bmatrix} 0 & 0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \end{bmatrix}.$$

For the initial disjoint pseudo-tree covering Π_0, we obtain its characteristic matrix $\mathcal{M}_0$. Then, we aggregate mergeable pseudo-trees using a two-step approach.

Step 1: we merge $\mathcal{T}_i^0$ to $\mathcal{T}_j^0$, if $\mathcal{T}_j^0$ is the only pseudo-tree that $\mathcal{T}_i^0$ is mergeable to. If there are multiple pairs satisfying this condition, (e.g., in Fig. 2, both $\mathcal{T}_1$, $\mathcal{T}_3$ are only mergeable $\mathcal{T}_2$), we then select the one that has more non-overlapping pseudo-trees in Π_0. Two disjoint pseudo-trees are non-overlapping if they do not share any vertex in common. For example, in Fig. 2, as $\mathcal{T}_1$ has more non-overlapping pseudo-trees, it is preferable to merge $\mathcal{T}_1$ to $\mathcal{T}_2$ first.

From an algebraic point of view, this step is equivalent to seek a row of $\mathcal{M}_0$, in which there is only a single "1" entry, while the others are either "0" or "$\varnothing$". If there exist multiple rows containing a single "1" entry, then we choose the one with more "$\varnothing$" entries. Merging two disjoint pseudo-trees in a covering Π_0 can be represented by a reduction of the characteristic matrix $\mathcal{M}_0$. For a given disjoint pseudo-tree covering Π with $|\Pi| = n$ and a set $\mathbb{N} := \{1, 2, \cdots, n\}$, we define the following function

$$\mathscr{F} : \mathbb{M}^{n\times n} \times \mathbb{N} \times \mathbb{N} \to \mathbb{M}^{(n-1)\times(n-1)}, \quad (8)$$

and $\hat{\mathcal{M}} = \mathscr{F}(\mathcal{M}, i, j)$ is a reduction of $\mathcal{M}$ obtained by the following algebraic operations:

1) $\hat{\mathcal{M}} = \mathcal{M}$;
2) Row merging: $\hat{\mathcal{M}}(i,:) = \mathcal{M}(i,:) \odot \mathcal{M}(j,:)$;
3) Column merging: $\hat{\mathcal{M}}(:,i) = \mathcal{M}(:,i) \odot \mathcal{M}(:,i)$;
4) Remove j-th row and column of $\hat{\mathcal{M}}$.

It is also worth emphasizing that the order of the row and column operations can be switched, which will not affect the outcome $\hat{\mathcal{M}}$. Furthermore, reducing the characteristic matrix is consistent with the change in the disjoint pseudo-tree covering of the network. Specifically, if in Π_0, a pseudo-tree $\mathcal{T}_j^0$ is mergeable to $\mathcal{T}_i^0$, then $\mathscr{F}(\mathcal{M}_0, i, j)$ produces a $(|\Pi_0|-1)\times(|\Pi_0|-1)$ characteristic matrix $\mathcal{M}_1$ representing a new disjoint pseudo-tree covering Π_1, which contains one less element compared to Π_0. We can repeat the above reduction method until the newly generated characteristic matrix does not contain a row with a single "1" entry.

Example 3: Consider the network depicted in Fig. 2 and its initial disjoint pseudo-tree covering $\Pi_0 = \{\mathcal{T}_1^0, \mathcal{T}_2^0, \cdots, \mathcal{T}_9^0\}$, which is characterized by the matrix in (7). Note that $\mathcal{M}_0(12)$, $\mathcal{M}_0(23)$, $\mathcal{M}_0(32)$, $\mathcal{M}_0(78)$, and $\mathcal{M}_0(85)$ are the only "1" entries in their rows. As in the first row, there are more "$\varnothing$" elements, we therefore apply the reduction $\mathcal{M}_1 = \mathscr{F}(\mathcal{M}_0, 1, 2)$, which yields

$$\mathcal{M}_1 = \begin{bmatrix} 0 & 0 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\ 1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\ 0 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & \varnothing & \varnothing & 0 & 0 & 0 & 0 & \varnothing \\ \varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \end{bmatrix} \in \mathbb{M}^{8\times 8}. \quad (9)$$

The resulting disjoint pseudo-tree covering is then given $\Pi_1 = \{\mathcal{T}_1^1, \mathcal{T}_2^1, \cdots, \mathcal{T}_8^1\}$, where $\mathcal{T}_1^1 = \mathcal{T}_1^0 \cup \mathcal{T}_2^0$ and $\mathcal{T}_i^1 = \mathcal{T}_{i+1}^0$, for all $i = 2, 3, \cdots, 8$.

We continue to reduce the characteristic matrix using the same mechanism: $\mathcal{M}_2 = \mathscr{F}(\mathcal{M}_1, 1, 2)$ and $\mathcal{M}_3 = \mathscr{F}(\mathcal{M}_2, 3, 4)$, which finally yields

$$\mathcal{M}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varnothing & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & \varnothing & 0 & 0 & 0 & \varnothing \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}^{6\times 6}, \quad (10)$$

which leads to the disjoint pseudo-tree covering $\Pi_4 = \{\mathcal{T}_1^3, \mathcal{T}_2^3, \mathcal{T}_3^3, \mathcal{T}_4^3, \mathcal{T}_5^3, \mathcal{T}_6^3\}$, where $\mathcal{T}_1^3 = \mathcal{T}_1^0 \cup \mathcal{T}_2^0 \cup \mathcal{T}_3^0$, $\mathcal{T}_2^3 = \mathcal{T}_4^0$, $\mathcal{T}_3^3 = \mathcal{T}_5^0 \cup \mathcal{T}_6^0$, $\mathcal{T}_4^3 = \mathcal{T}_7^0$, $\mathcal{T}_5^3 = \mathcal{T}_8^0$, and $\mathcal{T}_6^3 = \mathcal{T}_9^0$. The corresponding disjoint pseudo-tree covering Π_3 is illustrated in Fig. 3.

Note that after Step 1, there may still exist disjoint pseudo-trees that are mergeable. For instant, in Fig. 3, $\mathcal{T}_3^3$ is

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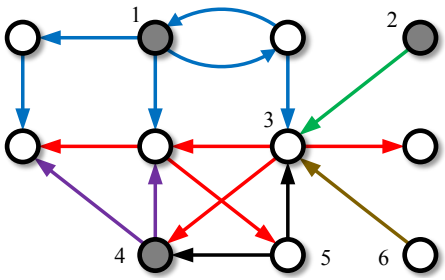

Fig. 3. After Step 1 of the proposed approach, the directed graph is decomposed into 6 disjoint pseudo-trees, labeled with different colors. The gray vertices are directly affected by external noises.

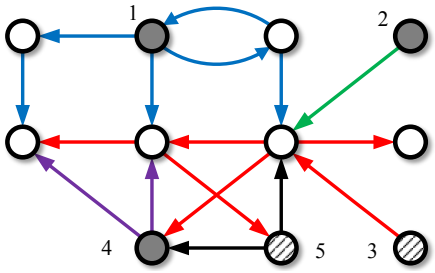

Fig. 4. The obtained disjoint pseudo-tree covering of the directed graph, where each pseudo-tree is labeled with a different color. One of the solution for allocating the excitation signals is to assign the shadowed vertices as the excited ones.

mergeable to $\mathcal{T}_2^3$ or $\mathcal{T}_5^3$. Therefore, a subsequent step of the characteristic matrix reduction is required.

Step 2: We find a row of the characteristic matrix with at least one "1" entry and then merge the two pseudo-trees corresponding to any "1" entry of this row. If there exists multiple rows containing "1" entries, we then select a row with more "$\varnothing$" entries. We repeat this step until there is no row containing "1" entries. Then, the pseudo-tree merging procedure is finalized. The number of the pseudo-trees in the final covering is the number of excitation signals that are designed to stimulate the network for the our identification purpose. The locations of these excitation signals can be assigned to distinct vertices that are the roots of the generated disjoint pseudo-trees.

Example 4: We continue the example in Fig. 3, where the disjoint pseudo-tree covering Π_3 is characterized by the matrix $\mathcal{M}_3$ in (10). Since the third row of $\mathcal{M}_3$ contains two "1" entries, we then take the following operation:

$$\mathcal{M}_4 = \mathscr{F}(\mathcal{M}_3, 6, 3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varnothing & 0 & 0 & \varnothing \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (11)$$

which yields the disjoint pseudo-tree covering $\Pi_4 = \{\mathcal{T}_1^4, \mathcal{T}_2^4, \mathcal{T}_3^4, \mathcal{T}_4^4, \mathcal{T}_5^4, \}$, as shown in Fig. 4, with $\mathcal{T}_1^4 = \mathcal{T}_1^0 \cup \mathcal{T}_2^0 \cup \mathcal{T}_3^0$, $\mathcal{T}_2^4 = \mathcal{T}_4^0$, $\mathcal{T}_3^4 = \mathcal{T}_5^0 \cup \mathcal{T}_6^0 \cup \mathcal{T}_9^0$, and $\mathcal{T}_4^4 = \mathcal{T}_7^0$ $\mathcal{T}_5^4 = \mathcal{T}_8^0$. (We may also do $\mathcal{M}_4 = \mathscr{F}(\mathcal{M}_4, 2, 3)$.) Note that there does not exist mergeable pairs in Π_4. Thus, two excitation signals are needed for the generic identifiability of the overall network. The roots of $\mathcal{T}_3^4$ and $\mathcal{T}_5^4$, as labeled by shadowed vertices in Fig. 4, can be chosen as excited vertices.

IV. Conclusion

In this paper, we have considered the allocation of external excitation signals for the generic identifiability of dynamical networks. A novel necessary and sufficient graph-theoretic condition of the generic identifiability has been provided, which relates to the disjoint pseudo-tree covering of the network. Based on the condition, we devise a efficient algorithm aiming to find the minimal number of excitation signals and their locations such that the network is generically identifiable.

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