Paper submitted to 59th IEEE Conf. Decision and Control, Jeju Island, Republic of Korea, 8-11 December 2020.

# Identifiability of linear dynamic networks through switching modules

H.J. Dreef M.C.F. Donkers P.M.J. Van den Hof

***Abstract*— Identifiability of linear dynamic networks typically depends on the presence and location of external (excitation or disturbance) signals, in relation to the topology of the parametrized network model set. For closed-loop identification, it is known that switching (non-parameterized) controllers can also provide excitation, thereby rendering the model set identifiable. In this paper, we derive verifiable conditions for network identifiability of the non-switching modules in presence of (non-parameterized) switching modules. These conditions generalize the classical result in closed-loop identification towards network identification. Furthermore, verifiable path-based conditions for identifiability in a generic sense are developed on the graph of the network model set.**

## I. Introduction

Technological progress towards large-scale interconnections of dynamical systems requires monitoring, control and optimization techniques to operate safely and efficiently. An attractive approach to achieve this is by using model-based techniques, for which a representation in terms of interconnected modules is advantageous. The vast availability of data, as a result of the ubiquitous presence of sensors in modern systems, motivates research of identification in linear dynamic networks. The objective and prior knowledge of the structure or dynamics generate different identification problems. These include topology detection, see e.g., [1], [2], identification of a single module, see e.g., [3]–[5], or identification of the full network dynamics, see e.g., [6], [7].

Before performing the actual identification, it is important to know whether a unique model can be retrieved from the identification setup, which is captured by the notion of identifiability. The derived conditions for identifiability depend on the network interconnection structure, the selection of measured nodes and the locations where external excitation signals enter the network. These conditions are typically verified on the rank of a particular transfer matrix of the network, which, for the full network problem, is repeated for every node in the network, see e.g., [8] for the situation that all nodes are measured and some nodes are excited. Another concept called generic identifiability is independent of numerical values of transfer matrices and relies solely on the network interconnection structure. This notion allows the rank conditions to be translated to graph-based conditions, see e.g., [9], [10] for the dual problem in [8] where all nodes are excited and some are measured.

This project has received funding from the European Research Council (ERC), Advanced Research Grant SYSDYNET, under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694504).

The authors are with the Dept. of Electrical Engineering, Eindhoven University of Technology, Netherlands. Email: {h.j.dreef,m.c.f.donkers,p.m.j.vandenhof}@tue.nl.

Besides the development in interconnected systems, the combination of digital and physical components poses additional challenges in the near future. Discrete decisions or switches can also appear in dynamic networks through different operation modes or (dis)connection of links between nodes or (sub)networks. Although identification of these hybrid systems has been studied extensively, see e.g., [11]–[13], they have, to the authors' knowledge, not been studied in a feedback situation, let alone in dynamic networks.

While identification of switching systems in dynamic networks remains challenging, it has been shown that a switching controller connected in feedback to a linear time-invariant plant provides additional excitation that renders the plant identifiable, even without an external reference signal, see e.g., [14]. Given the crucial role of the location and presence of excitation signals on identifiability of dynamic networks, switching modules can be used for identification of the remaining non-switching, or *mode-invariant*, modules. This leads to the following research question: "under which conditions is additional excitation provided by switching modules such that identifiability of the remaining non-switching modules is ensured?"

In this paper, verifiable conditions for identifiability of non-switching modules in dynamic networks in presence of switching modules are derived. The analysis is based on rank conditions on a specific part of transfer matrices from external signals to internal node signals that were derived for non-switching networks, see e.g., [8]. The switching modules introduce different transfer matrices for each network mode, which leads to additional information on the mode-invariant parts that can be extracted, as has been done for the closed-loop setup with switching controllers, see e.g., [14]. The presented results can be seen as a generalization of [8] and [14]. Furthermore, the developed rank conditions are connected to path-based conditions using the generic identifiability setting in a similar way as in [9], [10].

*Nomenclature:* Denote $\mathbb{R}$ as the set of real numbers, and $\mathbb{R}(z)$ is the rational function field over $\mathbb{R}$ with variable $z$. $v_i$ denotes the $i$-th element of a vector $v$, and $A_{ij}$ denotes the $(i, j)$-th entry of a matrix $A$. The cardinality of a set $\mathcal{V}$ is given by $|\mathcal{V}|$.

## II. Problem statement

This section formulates a dynamic network model and an associated model set in which switching modules can be captured. The model set is based on the separation of parametrized mode-invariant modules and switching modules.

Following this setup, identifiability is defined on transfer matrices for all the network modes.

### A. Module switching dynamic network setup

Following the dynamic network setup proposed in [3], [8], a dynamic network consists of $L$ scalar *internal variables*, *nodes* or *vertices* $w_j, j \in \{1,\dots,L\}$, and $K$ *external variables* $r_k, k \in \{1,\dots,K\}$ that can be manipulated by the user. Each internal variable is described as:

$$w_j(t) = \sum_{i=1}^{L} G_{ji}(q)w_i(t) + \sum_{k=1}^{K} R_{jk}(q)r_k(t) + v_j(t) \quad (1)$$

where $q^{-1}$ is the delay operator, i.e., $q^{-1}w_j(t) = w_j(t-1)$ and $t \in \mathbb{N}$ denotes time. The *modules* $G_{ji}$ contain the dynamic relationship between the nodes, excluding *self-loops*, i.e., $G_{jj} = 0$ for all $j$. Unmeasured *process noise* variables $v_j$ are included, where the vector process $v = [v_1 \dots v_L]^\top$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a $p$-dimensional (zero-mean) white noise process $e := [e_1 \dots e_p]^\top$, with $p \leq L$ and covariance matrix $\Lambda > 0$ such that

$$v(t) = H(q)e(t).$$

The combination of all the $L$ nodes can be written in terms of the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1L} \\ G_{L1} & \cdots & G_{LL-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} v_e \\ v_e \\ \vdots \\ e_p \end{bmatrix}, \quad (2)$$

where dependence on $q$ is omitted for compactness of notation. The compact form of (2) is given by

$$w = Gw + Rr + He. \quad (3)$$

A *network model* that includes these concepts and the considered properties in this work is defined below.

**Definition 1 (Network model)** *A network model of $L$ nodes, and $K$ external excitation signals, with a noise process of rank $p \leq L$ is defined by the quadruple:*

$$M = (G, R, H, \Lambda)$$

*with*

- *$G \in \mathbb{R}^{L\times L}(z)$, diagonal entries 0, all modules proper and stable;*[1]
- *$R \in \mathbb{R}^{L\times K}(z)$, proper;*
- *$H \in \mathbb{R}^{L\times p}(z)$, stable, with a left stable inverse and satisfying $H(q) = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$ with $H_a$ square, proper, monic, stable and minimum-phase;*
- *$\Lambda \in \mathbb{R}^{p\times p}$, $\Lambda > 0$;*
- *The network is well-posed*[2]*, see e.g., [5], with $(I-G)^{-1}$ proper and stable.*

In this definition, the flexibility of a rank-reduced noise process ($p < L$), i.e., a non-square noise model $H$, is allowed, see e.g., [8].

In order to include the possibility of switching modules in the defined network model, a *multimode network model* is defined, which contains $m \in \mathbb{N}$ different *network modes*, where the network mode is denoted by $\ell$.

**Definition 2 (Multimode network model)** *A multimode dynamic network with $m$ modes is defined as a finite set of network models $M_\ell$, i.e.,*

$$\mathbb{M} := \{M_\ell\}_{\ell\in\{1,\dots,m\}}, \quad (4)$$

*with $M_\ell = (G_\ell, R_\ell, H_\ell, \Lambda_\ell)$.*

This model does not specify restrictions on the switching behavior between the network modes. We therefore assume that switching does not destabilize the network. This can be achieved by assuming that for every mode the inter-switching time adheres to minimum dwell-time conditions for switched systems, see e.g., [**?**], for details. This means that the time between every switch is sufficiently long that potential transient effects have sufficiently decayed.

The multimode network model requires further specification, since the dynamics between some nodes remain the same for every network mode. These modules are referred to as *mode-invariant*, while the remaining links are switching. Since these links may contain different dynamics or modules for some network modes, it is more convenient to introduce some graph-based concepts that specify the structure and distinguish between mode-invariant and switching links. In this setting the nodes are referred to as *vertices* and links as *edges*, for which the distinction is defined below.

**Definition 3 (Multimode network graph)** *A multimode network graph is a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which captures the structure of multimode dynamic network $\mathbb{M}$, with vertex set $\mathcal{V} := \{1,\dots,L\}$ and the set of edges by*

$$\mathcal{E} := \{(i,j) \in \mathcal{V}\times\mathcal{V} \mid G_{ji,\ell} \neq 0 \text{ for some } \ell \in \{1,\dots,m\}\}, \quad (5)$$

*where the mode-invariant edges are defined by*

$$\mathcal{E}_{inv} := \{(i,j) \in \mathcal{E} \mid G_{ji,\ell_1} = G_{ji,\ell_2}\ \forall\, \ell_1,\ell_2 \in \{1,\dots,m\}\}, \quad (6)$$

*and the set of switching edges as $\mathcal{E}_s = \mathcal{E}\backslash\mathcal{E}_{inv}$.*

This multimode network graph is helpful in explicitly specifying the mode-invariant and switching parts in the multimode dynamic network. For this purpose, the *module switching network model* is defined below.

[1]The assumption of having all modules stable is made in order to guarantee that $T_{we}$ is a stable spectral factor of the noise process that affects the node variables.

[2]This implies that all principal minors of $(I - G(\infty))^{-1}$ are nonzero.

**Definition 4 (Module switching network model)** *A module switching dynamic network with $m$ modes, is a multimode dynamic network $\mathbb{M}$, with $M_\ell = (G_\ell, R_\ell, H_\ell, \Lambda_\ell) \in \mathbb{M}$ that satisfy*

$$\begin{aligned} R_\ell &= R, && \forall \ell \in \{1,\dots,m\} \\ H_\ell &= H, && \forall \ell \in \{1,\dots,m\} \\ \Lambda_\ell &= \Lambda, && \forall \ell \in \{1,\dots,m\} \\ G_\ell &= G^{inv} + G^s_\ell, && \end{aligned} \tag{7}$$

*where for each edge either the element in $G^{inv}$ or $G^s_\ell$ is 0, i.e., for $(i,j) \notin \mathcal{E}_{inv}$, it holds that $G^{inv}_{ji} = 0$ and for $(i,j) \notin \mathcal{E}_s$, it holds that $G^s_{ji,\ell} = 0 \ \forall\ \ell \in \{1,\dots,m\}$.*

### B. Full network identifiability problem

The identifiability problem that is considered in this paper is that of the full network, where known switching modules are present. Therefore, the problem is restricted to parametrize only mode-invariant modules, for which a set of *parametrized edges* $\mathcal{E}_p \subseteq \mathcal{E}_{inv}$ is defined. This allows for using prior knowledge of dynamics of mode-invariant modules. The restriction in parametrization leads to the following definition of the model set that is considered for the identifiability problem.

**Definition 5 (Module switching network model set)** *A module switching network model set $\mathcal{M}$ for a module switching network $\mathbb{M}$ with $m$ modes, is defined as a set of networks, according to*

$$\mathcal{M} := \{\mathbb{M}(\theta), \theta \in \Theta\} \tag{8}$$

*where $\mathbb{M}(\theta) = \{M_\ell(\theta)\}_{\ell \in \{1,\dots,m\}}$ and*

$$M_\ell(\theta) = (G^{inv}(\theta) + G^s_\ell, R(\theta), H(\theta), \Lambda(\theta)) \tag{9}$$

*with $G^{inv}_{ji}(\theta), (i,j) \in \mathcal{E}_p \subseteq \mathcal{E}_{inv}$ and $\Theta$ a finite dimensional parameter space, $\Theta \subset \mathbb{R}^{n_\theta}$.*

The objective for this model set is to assess whether the structural properties in terms of location and presence of excitation signals and topology of the network can distinguish different candidate models. In other words, the objective is to assess whether a unique model exists, without taking into account whether the data is informative enough. To do so, the concepts of identifiability for non-switching networks in [8] are generalized to include excitation through known switching modules. For this, the transfer matrices from external signals $r$ and $e$ to internal nodes $w$ for all the network modes $\ell$ are used, which for each mode are given by

$$T_\ell(\theta) := (I - G_\ell(\theta))^{-1} U(\theta) \tag{10}$$

with

$$U(\theta) := \begin{bmatrix} R(\theta) & H(\theta) \end{bmatrix}. \tag{11}$$

These transfer matrices contain all the structural information used by most identification methods under mild conditions, see e.g., [8]. That is, either absence of direct feedthrough terms or of algebraic loops, which occur when all modules in a loop contain direct feedthrough terms. By this result, the problem of assessing identifiability of the full module switching network will be done using the following definition of identifiability.

**Definition 6 (Module switching network identifiability)** *The module switching network model set $\mathcal{M}$ is identifiable at $\mathbb{M}_0 := \mathbb{M}(\theta_0)$ if for all models $\mathbb{M}(\theta_1) \in \mathcal{M}$,*

$$T_\ell(\theta_1) = T_\ell(\theta_0)\ \forall \ell \in \{1,\dots,m\} \Rightarrow \mathbb{M}(\theta_1) = \mathbb{M}(\theta_0). \tag{12}$$

*We call $\mathcal{M}$ identifiable if* (12) *holds for all $\mathbb{M}_0 \in \mathcal{M}$.*

This definition differs from the ones in literature by allowing information contained in transfer matrices for all network modes together to draw conclusions on the existence of a unique model. All the transfer matrices act as additional equations to solve for the unknown elements, which is the main aspect that will be exploited in this paper.

## III. Rank conditions

In this section, verifiable conditions for identifiability of the defined model set are derived on the basis of transfer matrices, as a natural approach arising from the used definition. These conditions generalize the developed rank conditions in [8], where switching modules are not considered, and the identifiability results on a closed-loop setup with a switching controller in [14]. The relation of the developed conditions to these methods will also be studied.

### A. Identifiability of module switching networks

Let us start from a similar reasoning on the transfer matrices (10) as in [8]. Suppose that row $j$ of $(I - G_\ell(\theta))$ has $\alpha_j$ parametrized transfer functions, and, similarly, each row $j$ of $U(\theta)$ has $\beta_j$ parametrized transfer functions. The objective is to isolate these parametrized transfer functions for each row and assess whether they can be uniquely retrieved from the transfer matrices. This is done through permutation of both matrices, for which permutation matrices $P_j \in \mathbb{N}^{L \times L}$ and $Q_j \in \mathbb{N}^{K+p \times K+p}$ are introduced. These gather all parametrized entries in row $j$ of $(I - G_\ell(\theta))$ on the left hand side, and all parametrized entries in the considered row of $U(\theta)$ on the right hand side, i.e.,

$$(I - G_\ell(\theta))_{j\star} P_j = \begin{bmatrix} (I - G(\theta))^{(1)}_{j\star} & (I - G_\ell)^{(2)}_{j\star} \end{bmatrix} \tag{13}$$

$$\begin{aligned} U(\theta)_{j\star} Q_j &= \begin{bmatrix} U^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star} \end{bmatrix} \\ &= \begin{bmatrix} U^{(1)}_{j\star} & 0 \end{bmatrix} + U(\theta)^{(2)}_{j\star} V_j \end{aligned} \tag{14}$$

with $V_j = \begin{bmatrix} 0 & I_{\beta_j} \end{bmatrix}$ of appropriate dimensions. These permutation matrices allow to formulate a particular sub-matrix of the transfer matrix (10). This sub-matrix consists of the mapping for network mode $\ell$ from all the external excitation signals to the input nodes of the parametrized modules that connect to node $j$, which is given by

$$\breve{T}_{j,\ell}(\theta) = \begin{bmatrix} I_{\alpha_j} & 0 \end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j. \tag{15}$$

This matrix may contain different dynamics for some network modes, which is information that is helpful in uniquely retrieving the parametrized modules. Therefore, this matrix is central in formulation of the following result for identifiability of module switching network model sets.

**Theorem 7** *The module switching network model set* $\mathcal{M}$

*I) is network identifiable at* $\mathbb{M}(\theta_0)$ *if for each row* $j$ *the matrix* $\begin{bmatrix}\breve{T}_{j,1}(\theta_0) & \dots & \breve{T}_{j,m}(\theta_0)\\ V_j & \dots & V_j\end{bmatrix}$ *has full row rank for* $\theta_0 \in \Theta$*;*

*II) is network identifiable if for each row* $j$ *the matrix* $\begin{bmatrix}\breve{T}_{j,1}(\theta) & \dots & \breve{T}_{j,m}(\theta)\\ V_j & \dots & V_j\end{bmatrix}$ *has full row rank for all* $\theta \in \Theta$*.*

*III) is generically identifiable if for each row* $j$ *the matrix* $\begin{bmatrix}\breve{T}_{j,1}(\theta) & \dots & \breve{T}_{j,m}(\theta)\\ V_j & \dots & V_j\end{bmatrix}$ *has full row rank for almost all* $\theta \in \Theta$*.*

*Proof:* The proof is given in the Appendix. ■

The above theorem includes a concept called generic identifiability, which in comparison to network identifiability is aimed at excluding a particular set of models for which the rank of these matrices is lost, see e.g., [9], [10]. This puts more emphasis on the network structure rather than on pathological cases in which particular numerical values for some model that might lead to loss of rank in these conditions. Theorem 7 effectively generalizes two results. Namely, the presented derivation to reach the matrix (15) corresponds to the results in [8] for one network mode, whereas "stacking" all the transfer matrices next to each other corresponds to the results in [14] restricted to a closed-loop structure. The relation between these and the presented method is studied further below.

### B. Non-switching networks

In order to show that the presented method generalizes the results in [8], they are compared to the conditions in Theorem 7 for a non-switching network, i.e., $m = 1$. In application of Theorem 7, the matrix $V_j$ is by definition full row rank at the last $\beta_j$ columns, meaning that only full row rank of the first $K + p - \beta_j$ columns of $\breve{T}_j$ implies full row rank of the entire matrix. These $\beta_j$ excluded columns correspond to the external signals that enter node $j$ through a parametrized module in $U_{j\star}$. The interpretation being that the information of that signal has already been used to identify this module. This realization has been used in [8] to derive conditions on a smaller dimensional matrix, while the conditions are equivalent.

### C. Switching controllers in closed-loop

The role of switching controllers on identifiability has been studied in a closed-loop setup, which can be represented by a particular dynamic network. The relation between Theorem 7 and those in [14] will be showed using the following network, corresponding to a closed-loop setup,

$$\begin{bmatrix}w_1\\ w_2\end{bmatrix} = \begin{bmatrix}0 & K_\ell\\ G(\theta) & 0\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} + \begin{bmatrix}L_\ell & 0\\ 0 & H(\theta)\end{bmatrix}\begin{bmatrix}r\\ e\end{bmatrix}. \quad (16)$$

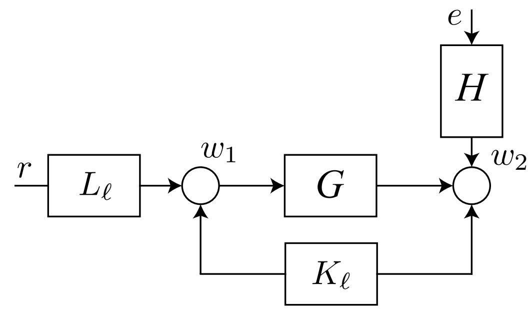


Fig. 1. A multimode closed-loop setup where the controller $K_\ell$ and the prefilter $L_\ell$ switch between different modes.

This network gives a transfer matrix given by (10)

$$T_\ell(\theta) = \begin{bmatrix}(I - K_\ell G(\theta))^{-1}L_\ell & K_\ell(I - G(\theta)K_\ell)^{-1}H(\theta)\\ (I - G(\theta)K_\ell)^{-1}GL_\ell & (I - G(\theta)K_\ell)^{-1}H(\theta)\end{bmatrix}. \quad (17)$$

The analysis presented in this paper is performed for every node $j$, while there is no parametrized module connected to node 1, i.e., there are no parametrized entries in row 1 of (16). Therefore, it suffices to analyze the situation for node 2 only, which has the convenient structure that the permutation matrices are given by $P_2 = Q_2 = I_2$, since the parametrized entries are already correctly ordered in (16). Then, application of the conditions in Theorem 7 leads to verifying that the following matrix has to be full row rank

$$\begin{bmatrix}\breve{T}_{j,1} \cdots \breve{T}_{j,m}\\ V_j \cdots V_j\end{bmatrix} = \begin{bmatrix}\hat{S}_1 L_1 & K_1\hat{S}_1\hat{H} & \cdots & \hat{S}_m L_m & K_m\hat{S}_m\hat{H}\\ 0 & I & \cdots & 0 & I\end{bmatrix} \quad (18)$$

with $\hat{S}_\ell = (I - \hat{G}K_\ell)^{-1}$ where the elements that depend on $\theta$ are denoted by $\hat{(\cdot)}$ for brevity. Now, the following decomposition can be made

$$\begin{bmatrix}\hat{S}_1 L_1 & K_1\hat{S}_1\hat{H} \cdots & \hat{S}_m L_m & K_m\hat{S}_m\hat{H}\\ 0 & I \quad \cdots & 0 & I\end{bmatrix} = \Xi_1\Xi_2\Xi_3, \quad (19)$$

where

$$\Xi_1 = \begin{bmatrix}0 & I\\ \hat{H}^{-1} & -\hat{H}^{-1}\hat{G}\end{bmatrix}, \Xi_2 = \begin{bmatrix}I & \cdots & I & 0 & \cdots & 0\\ K_1 & \cdots & K_m & L_1 & \cdots & L_m\end{bmatrix},$$

$$\Xi_3 = \begin{bmatrix}\mathrm{diag}(\hat{P}_1 L_1, \dots, \hat{P}_m L_m) & \mathrm{diag}(\hat{S}_1\hat{H}, \dots, \hat{S}_m\hat{H})\\ I & 0\end{bmatrix}$$

with $\hat{P}_\ell = (I - \hat{G}K_\ell)^{-1}\hat{G}$ and where $\mathrm{diag}(\hat{P}_1 L_1, \dots, \hat{P}_m L_m)$ is the block diagonal matrix with $\hat{P}_1 L_1, \dots, \hat{P}_m L_m$ on the diagonal entries, and $\mathrm{diag}(\hat{S}_1\hat{H}, \dots, \hat{S}_m\hat{H})$ is defined similarly. Since, the matrices $\Xi_1$ and $\Xi_3$ are always full rank, due to the well-posedness of the network and the monicity of the noise model $H$, the rank condition in (18) depends on the row rank of $\Xi_2$. This rank condition corresponds exactly to conditions of [14], which shows that the conditions of Theorem 7 generalizes this result.

**Remark 8** *The defined module switching network model does not allow external signals to enter through switching modules, whereas the closed-loop setup in [14] does consider those. The reason is not a limitation of the condition in Theorem 7, but rather for ease of exposition. Furthermore, have multiple modes for these modules no effect on the identifiability result, as is illustrated in* (19)*, where the row rank remains the same for different modes of* $L_\ell$*.*

## IV. Path-based conditions

The developed rank conditions of Theorem 7 have as disadvantage that it is practically inconvenient and, in particular for a large number of network modes, not scalable to evaluate a priori unknown transfer matrices. Therefore, the analysis is preferably performed on a graphical basis, for which efficient algorithms exist. In this section, path-based conditions for identifiability of module switching network models are presented. This is done by relating the rank conditions on transfer matrices, as in [8], for generic identifiability to conditions in terms of vertex-disjoint paths, similar to results in [9], [10]. The conditions for generic identifiability in Theorem 7, therefore act as a bridge to connect the rank conditions to path-based ones. These conditions are formulated in terms of a graph that is independent of the network mode, meaning that general graph theory can be applied and prevents increased complexity caused by dealing with multiple network modes.

To this end, let us recall graph $\mathcal{G}$ as defined in Section II, for which an extended graph can be formulated that includes the correlation structure in $H$ and $R$ from (2), as in [15]. This is defined below.

**Definition 9 (Extended graph [15])** *Consider a directed dynamic network* (2)*. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be its underlying graph. An extended graph $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is defined as*

$$\hat{\mathcal{V}} := \mathcal{V} \cup \{L+1, L+2, \dots, L+K+p\} \tag{20}$$

$$\hat{\mathcal{E}} := \mathcal{E} \cup \{(i,j) \mid i \in \hat{\mathcal{V}}, j \in \mathcal{V}, U_{j,i-L}(\theta) \neq 0\}. \tag{21}$$

Here, the additional vertices $\hat{\mathcal{V}}$ consist of all external signals $e(t)$ and $r(t)$, whereas the structure of the mapping from these signals into the network are captured in the additional set of edges $\hat{\mathcal{E}}$ with respect to $\mathcal{E}$. This extended graph captures the structure of graph $\mathcal{G}$ and correlation structure of external signals simultaneously. This allows to formulate the set of in-neighbors of node $j$ as

$$\mathcal{W}_j := \{i \in \hat{\mathcal{V}} \mid (i,j) \in \mathcal{E}_p \text{ or } U_{j,i-L} \text{ is parametrized}\}, \tag{22}$$

which are inputs to parametrized elements in the module switching network. Let $\mathcal{U}$ be the set of all external signals $r(t)$ and $e(t)$ entering the network.

Let us consider a directed graph without taking the effect of switching modules into account, i.e., switching edges are assumed to be known, but have no other role in the graph. Then, generic identifiability can be verified using the following lemma, which was inspired by results in, e.g., [10]. This result is formulated using the concept of *vertex-disjoint paths* from a set $\mathcal{A}$ to $\mathcal{B}$, denoted by $b_{\mathcal{A}\to\mathcal{B}}$, which corresponds to the maximum number of paths from the set of nodes $\mathcal{A}$ to the set of nodes $\mathcal{B}$ that do not share any nodes or vertices.

**Lemma 10** *[15] A network model set $\mathcal{M}$ for $m = 1$ is generically identifiable if and only if in its extended graph $\hat{\mathcal{G}}$*

$$b_{\mathcal{U}\to\mathcal{W}_j} = |\mathcal{W}_j| \tag{23}$$

*holds for all $j \in \mathcal{V}$.*

This path-based result has also been connected to the rank conditions in [8], see e.g., [10], [16]. Application of this relation in the presented framework in this paper for the situation $m = 1$ leads to

$$b_{\mathcal{U}\to\mathcal{W}_j} = \operatorname{rank} T_{\mathcal{W}_j\mathcal{U},1} \tag{24}$$

with

$$T_{\mathcal{W}_j\mathcal{U},\ell} = \begin{bmatrix} \breve{T}_{j,\ell} \\ V_j \end{bmatrix} \tag{25}$$

for $\ell \in \{1, \dots, m\}$. This close relation will be used to establish path-based conditions for general module switching networks.

In the path-based conditions conditions developed below, the partitioning of the edge set of Definition 3 is used. The particular set of interest is the set of switching edges $\mathcal{E}_s$, which is used to specify the output nodes of these edges as *stimulated*. These stimulated nodes can be defined by the following set

$$\mathcal{S} := \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}_s \wedge b_{\mathcal{U}\to\{w_i\}} \neq 0\}, \tag{26}$$

where the second condition ensures that at least one path from an external signal to the input node of the switching edge exists. Without an excitation signal reaching the input, the switching module would not appear in any transfer matrix in Theorem 7 and this situation is therefore excluded. The set of stimulated nodes by switching modules plays a key role, since the excitation that these switching modules produces is closely related to the excitation by external signals. This close relation is formulated in the following Theorem.

**Theorem 11** *The module switching dynamic network model set $\mathcal{M}$ is generically identifiable if in its extended graph $\hat{\mathcal{G}}$, it holds that*

$$b_{(\mathcal{U}\cup\mathcal{S})\to\mathcal{W}_j} = |\mathcal{W}_j| \tag{27}$$

*holds for every node $j \in \mathcal{V}$.*

The result of Theorem 11 shows that switching modules provide excitation in a similar way as external signals, with the main difference that switching modules require input from an external excitation signal. This formulation allows for easier interpretation on the influence of switching modules on identifiability of dynamic networks than conditions in Theorem 7, while in addition efficient existing algorithms like the Ford-Fulkerson algorithm can be used for evaluation, see e.g., [17].

## V. Example

In this section, the presented theory will be illustrated using a simple example. We will consider the network depicted in Figure 2, which includes a known switching module with two different operating modes. The network can be described as follows

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ G_{21,\ell} & 0 & 0 \\ G_{31}(\theta) & G_{32}(\theta) & 0 \end{bmatrix}}_{G_\ell(\theta)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{U} r, \tag{28}$$

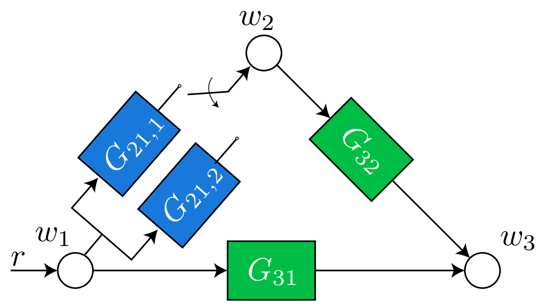


Fig. 2. A module switching network example with one switching module (blue) that ensures identifiability of the two parametrized modules (green).

where $\ell \in \{1,2\}$ and $G_{21,1} \neq G_{21,2}$. Since all of the parametrized modules are connected to node $j = 3$, it suffices to analyse the conditions of Theorem 7 and Theorem 11 for this node only.

Let us start by constructing the matrix $\breve{T}_{3,\ell}$, corresponding to the procedure described in Section III-A. This starts by formulation of the permutation matrices $P_3$ and $Q_3$, such that all the $\alpha_3 = 2$ parametrized entries of $(I - G_\ell(\theta))$ are on the left side, whereas $\beta_3 = 0$, since there are no parametrized entries in $U(\theta)$. The matrices from (28) are already correctly ordered, so $P_3 = I_3$ and $Q_3 = 1$. Using these ingredients the transfer matrix $\breve{T}_{3,\ell}$ is given by

$$\breve{T}_{3,\ell} = \begin{bmatrix} I_{\alpha_3} & 0 \end{bmatrix} T_\ell = \begin{bmatrix} 1 \\ G_{21,\ell} \end{bmatrix}, \tag{29}$$

with $T_\ell$ given by (10) and where $V_3$ does not exist, since $\beta_3 = 0$. Clearly, for $m = 1$, the conditions of Theorem 7 would not hold, since the matrix in (29) has more rows than columns. However, application of Theorem 7 for $m = 2$ yields

$$\text{rank} \begin{bmatrix} \breve{T}_{3,1} & \breve{T}_{3,2} \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 1 \\ G_{21,1} & G_{21,2} \end{bmatrix} = 2, \tag{30}$$

which does satisfy the conditions and renders the model set identifiable.

This result can be verified using the path-based conditions of Theorem 11, where it suffices again to consider node $j = 3$ only. The set of inputs to parametrized modules that connect to node 3 is given by $\mathcal{W}_3 = \{1,2\}$ and the set of nodes excited by an external signal is given by $\mathcal{U} = \{1\}$. Without including excitation by switching modules, this graph has only one path from $\mathcal{U}$ to $\mathcal{W}_3$, which does not satisfy the conditions in Lemma 10. Let us now include switching edges, for which the set of stimulated nodes by switching modules is given by $\mathcal{S} = \{2\}$, corresponding to (26). Since the paths $\mathcal{U} \to \mathcal{W}_3$ and $\mathcal{S} \to \mathcal{W}_3$ are vertex-disjoint, the conditions of Theorem 11 are satisfied and therefore the model set is generically identifiable.

## VI. Conclusion

In this paper, it has been shown that switching modules provide additional excitation in dynamic networks, such that remaining mode-invariant modules are identifiable. Verifiable conditions in terms of a transfer matrix composed from transfer matrices for all network modes are formulated. These conditions generalize existing results on network identifiability and identifiability of a plant in closed-loop with a switching controller, for which their relation is explicitly shown. These rank conditions are translated to path-based conditions for the notion of generic identifiability, which uses a standard graph formulation and therefore allows usage of standard algorithms to be evaluated.

## References


[1] D. Materassi and G. Innocenti, "Topological Identification in Networks of Dynamical Systems," *IEEE Trans. on Automatic Control*, vol. 55, no. 8, pp. 1860 – 1871, 2010.

[2] A. Chiuso and G. Pillonetto, "A Bayesian approach to sparse dynamic network identification," *Automatica*, vol. 48, no. 8, pp. 1553–1565, Aug. 2012.

[3] P. M. J. Van den Hof, A. G. Dankers, P. S. C. Heuberger, and X. Bombois, "Identification of dynamic models in complex networks with prediction error methods—Basic methods for consistent module estimates," *Automatica*, vol. 49, no. 10, pp. 2994–3006, 2013.

[4] K. R. Ramaswamy, G. Bottegal, and P. M. J. Van den Hof, "Local Module Identification in Dynamic Networks Using Regularized Kernel-Based Methods," in *IEEE Conf. on Decision and Control*, 2018, pp. 4713–4718.

[5] A. G. Dankers, "System identification in dynamic networks." Ph.D. dissertation, Delft University of Technology, 2014.

[6] J. Goncalves and S. Warnick, "Necessary and Sufficient Conditions for Dynamical Structure Reconstruction of LTI Networks," *IEEE Trans. on Automatic Control*, vol. 53, no. 7, pp. 1670–1674, 2008.

[7] H. H. M. Weerts, P. M. J. Van den Hof, and A. G. Dankers, "Prediction error identification of linear dynamic networks with rank-reduced noise," *Automatica*, vol. 98, pp. 256–268, 2018.

[8] ——, "Identifiability of linear dynamic networks," *Automatica*, vol. 89, pp. 247–258, 2018.

[9] A. S. Bazanella, M. Gevers, J. M. Hendrickx, and A. Parraga, "Identifiability of dynamical networks: which nodes need be measured?" in *IEEE 56th Conf. on Decision and Control*, 2017, pp. 5870–5875.

[10] J. M. Hendrickx, M. Gevers, and A. S. Bazanella, "Identifiability of Dynamical Networks With Partial Node Measurements," *IEEE Trans. on Automatic Control*, vol. 64, no. 6, pp. 2240–2253, 2019.

[11] A. L. Juloski, W. P. M. H. Heemels, G. Ferrari-Trecate, R. Vidal, S. Paoletti, and J. H. G. Niessen, "Comparison of Four Procedures for the Identification of Hybrid Systems," *European Journal of Control*, vol. 13, pp. 354–369, 2005.

[12] L. Bako, "Identification of switched linear systems via sparse optimization," *Automatica*, vol. 47, no. 4, pp. 668–677, Apr. 2011.

[13] F. Lauer and G. Bloch, *Hybrid System Identification*. Cham,: Springer Nature Switzerland AG, Oct. 2018.

[14] T. Söderström, L. Ljung, and I. Gustavsson, "Identifiability conditions for linear multivariable systems operating under feedback," *IEEE Transactions on Automatic Control*, vol. 21, no. 6, pp. 837–840, 1976.

[15] X. Cheng, S. Shi, and P. M. J. Van den Hof, "Allocation of Excitation Signals for Generic Identifiability of Linear Dynamic Networks," *arXiv:1910.04525 [math]*, 2019.

[16] J. W. van der Woude, "A graph-theoretic characterization for the rank of the transfer matrix of a structured system," *Mathematics of Control, Signals and Systems*, vol. 4, no. 1, pp. 33–40, 1991.

[17] L. R. Ford and D. R. Fulkerson, "Maximal flow through a network," *Canadian Journal of Mathematics*, vol. 8, pp. 399–404, 1956.

[18] S. Shi, X. Cheng, and P. M. J. Van den Hof, "Excitation allocation for generic identifiability of a single module in dynamic networks: A graphic approach," Tech. Rep., 2020, to appear in Proc. IFAC World Congress, 2020.


## Appendix

*Proof of Theorem 7:* This proof is inspired by [8], where the left hand side of implication (12) can be written for a single mode $\ell$ as

$$(I - G_\ell(\theta))T_\ell = U(\theta) \tag{31}$$

where shorthand notation is used for $T_\ell = T_\ell(\theta_0)$, $G_\ell(\theta) = G_\ell(\theta_1)$ and $U(\theta) = U(\theta_1)$. The following reasoning is then applied to every row of (31). Consider row $j$, for which permutation matrices $P_j$ and $Q_j$, defined in (13)-(14), are applied to obtain:

$$(I - G_\ell(\theta))_{j\star} P_j P_j^{-1} T_\ell Q_j = U(\theta)_{j\star} Q_j, \tag{32}$$

which leads to

$$-G(\theta)_{j\star}^{(1)} \breve{T}_{j,\ell} + (I - G_\ell)_{j\star}^{(2)} \bar{T}_{j,\ell} = \begin{bmatrix} U_{j\star}^{(1)} & 0 \end{bmatrix} + U(\theta)_{j\star}^{(2)} V_j \tag{33}$$

with $-G(\theta)_{j\star}^{(1)} = (I - G(\theta))_{j\star}^{(1)}$ and $P_j^{-1} T_\ell Q_j = \begin{bmatrix} \breve{T}_{j,\ell} \\ \bar{T}_{j,\ell} \end{bmatrix}$. This equation can be rewritten into

$$\begin{bmatrix} G(\theta)_{j\star}^{(1)} & U(\theta)_{j\star}^{(2)} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,\ell} \\ V_j \end{bmatrix} = \rho_\ell, \tag{34}$$

with $\rho_\ell = (I - G_\ell)_{j\star}^{(2)} \bar{T}_{j,\ell} - \begin{bmatrix} U_{j\star}^{(1)} & 0 \end{bmatrix}$, which has to hold for all $\ell \in \{1, \ldots, m\}$. All of these equations can be grouped for all $\ell \in \{1, \ldots, m\}$ as follows

$$\begin{bmatrix} G(\theta)_{j\star}^{(1)} & U(\theta)_{j\star}^{(2)} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix} = \begin{bmatrix} \rho_1 & \cdots & \rho_m \end{bmatrix}. \tag{35}$$

In (35) all the parametrized entries are contained in the left block $\begin{bmatrix} G(\theta)_{j\star}^{(1)} & U(\theta)_{j\star}^{(2)} \end{bmatrix}$, whereas the other parts of the equation are independent of $\theta$. Therefore, all of the parametrized elements are uniquely determined if the matrix $\begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix}$ has full row rank. A solution $G_\ell(\theta_0)$ and $U(\theta_0)$ exists to the equation (31) by Definition 6 of identifiability. Since the solution to the equation (35) is unique and $G_\ell(\theta_0)$ and $U(\theta_0)$ is a possible solution, the validity of (12) is proven.

The proof for situation II) is that the provided conditions should apply for all models $\theta \in \Theta$, meaning that one model for which the given transfer matrix loses rank, the model set is not identifiable. However, for the situation III), the provided conditions should hold for almost all $\theta \in \Theta$, meaning that a finite set of models may be excluded. ■

*Proof of Theorem 11:* The proof is based on showing that the rank condition on transfer matrices in Theorem 7 generically corresponds to path-based conditions on a specific set of nodes. Furthermore, is the following reasoning repeated for every node $j \in \mathcal{V}$, so the notation that indicates dependence on $j$ is omitted in the following reasoning.

First, we introduce a partitioning of the set of nodes $\mathcal{V}$ into disjoint sets. To this end, let $\mathcal{D}$ denote the *minimum disconnecting set* of the paths $\mathcal{U} \to \mathcal{W}$, where a disconnecting set refers to a set of nodes of which removal would lead to disconnection of the sets $\mathcal{U}$ and $\mathcal{W}$. In addition this set should be of minimal cardinality. Using this disconnecting set we define $\mathcal{A}$ as the set of nodes for which paths from $\mathcal{U}$ exist that do not intersect any node in $\mathcal{D}$, such that $\mathcal{U} \subseteq \mathcal{A}$. Furthermore, the remaining nodes are defined by $\mathcal{B} = \mathcal{V} \backslash (\mathcal{D} \cup \mathcal{A})$, i.e., the nodes that can only be reached from $\mathcal{U}$ by paths through $\mathcal{D}$. The proposed partitioning can also be used to partition the set of nodes excited by switching modules $\mathcal{S}$, which leads to two disjoint sets $\mathcal{S}_\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{D}$ and $\mathcal{S}_\mathcal{B} \subseteq \mathcal{B}$. The contribution for elements in both sets will be analyzed separately.

***The case $\mathcal{S} = \mathcal{S}_\mathcal{A}$ and $\mathcal{S}_\mathcal{B} = \emptyset$:*** Let us now consider the effect of switching modules before the disconnecting set $\mathcal{D}$ only, i.e., $\mathcal{S} = \mathcal{S}_\mathcal{A}$ and $\mathcal{S}_\mathcal{B} = \emptyset$. This is done through a transfer matrix decomposition, see e.g., [10], [18], which can be done using the disconnecting set. This decomposition is made through formulation of the following matrices in terms of the proposed partitioning of nodes

$$G_\ell = \begin{bmatrix} G_{\mathcal{B}\mathcal{B}} & G_{\mathcal{B}\mathcal{D}} & 0 \\ G_{\mathcal{D}\mathcal{B},\ell} & G_{\mathcal{D}\mathcal{D},\ell} & G_{\mathcal{D}\mathcal{A},\ell} \\ G_{\mathcal{A}\mathcal{B},\ell} & G_{\mathcal{A}\mathcal{D},\ell} & G_{\mathcal{A}\mathcal{A},\ell} \end{bmatrix},$$

$$U = \begin{bmatrix} 0 \\ U_{\mathcal{D}\mathcal{U}} \\ U_{\mathcal{A}\mathcal{U}} \end{bmatrix}, \; T_\ell = \begin{bmatrix} T_{\mathcal{B}\mathcal{U},\ell} \\ T_{\mathcal{D}\mathcal{U},\ell} \\ T_{\mathcal{A}\mathcal{U},\ell} \end{bmatrix},$$

where the subscript $\ell$ denotes the dependence on the network mode. This means that all transfer matrices that map to nodes in $\mathcal{A}$ or $\mathcal{D}$ might depend on the network mode $\ell$. In this formulation, the matrix $G_{\mathcal{B}\mathcal{A}} = 0$, since any path $\mathcal{A} \to \mathcal{B}$ goes through $\mathcal{D}$. Then, since $(I - G_\ell)T_\ell = U$, the top row of the equation yields

$$\begin{bmatrix} I - G_{\mathcal{B}\mathcal{B}} & -G_{\mathcal{B}\mathcal{D}} & 0 \end{bmatrix} \begin{bmatrix} T_{\mathcal{B}\mathcal{U},\ell} \\ T_{\mathcal{D}\mathcal{U},\ell} \\ T_{\mathcal{A}\mathcal{U},\ell} \end{bmatrix} = 0, \tag{36}$$

such that

$$\begin{aligned} T_{\mathcal{B}\mathcal{U},\ell} &= (I - G_{\mathcal{B}\mathcal{B}})^{-1} G_{\mathcal{B}\mathcal{D}} T_{\mathcal{D}\mathcal{U},\ell} \\ &= T_{\mathcal{B}\mathcal{D}} T_{\mathcal{D}\mathcal{U},\ell} \end{aligned} \tag{37}$$

with $T_{\mathcal{B}\mathcal{D}} = (I - G_{\mathcal{B}\mathcal{B}})^{-1} G_{\mathcal{B}\mathcal{D}}$ independent of the network mode $\ell$. This matrix $T_{\mathcal{B}\mathcal{D}}$ is a transfer matrix mapping from signals applied to the nodes in $\mathcal{D}$ to the nodes $\mathcal{B}$, to show this, consider $u(t)$ the external signal applied to nodes $\mathcal{U}$. The node signals of the set $\mathcal{B}$ are by (37) then, for every mode $\ell$, given by

$$w_\mathcal{D}(t) = T_{\mathcal{B}\mathcal{D}} T_{\mathcal{D}\mathcal{U},\ell} \; u(t) = T_{\mathcal{B}\mathcal{D}} \; \tilde{u}_\ell(t) \tag{38}$$

with $\tilde{u}_\ell(t) = T_{\mathcal{D}\mathcal{U},\ell} \; u(t)$. This shows that $T_{\mathcal{B}\mathcal{D}}$ should be interpreted as the mapping from external signals applied to $\mathcal{D}$ to internal node signals of $\mathcal{B}$. This interpretation holds for all the following matrices denoted by $T$ in this proof.

Since $\mathcal{W} \subseteq \mathcal{D} \cup \mathcal{B}$, the desired transfer matrix from $\mathcal{U}$ to $\mathcal{W}$ is a selection of rows from $T_{\mathcal{B}\mathcal{U},\ell}$ and $T_{\mathcal{D}\mathcal{U},\ell}$, given by

$$T_{\mathcal{W}\mathcal{U},\ell} = C_\mathcal{A} \begin{bmatrix} T_{\mathcal{B}\mathcal{U},\ell} \\ T_{\mathcal{D}\mathcal{U},\ell} \end{bmatrix} \tag{39}$$

with $C_\mathcal{A} \in \{0,1\}^{|\mathcal{B}| \times |\mathcal{W}|}$ that has a one in every row in the column corresponding to the nodes in $\mathcal{W}$. This can be further specified by

$$T_{\mathcal{W}\mathcal{U},\ell} = T_{\mathcal{W}\mathcal{D}} T_{\mathcal{D}\mathcal{U},\ell} \tag{40}$$

with $T_{\mathcal{W}\mathcal{D}} = C_\mathcal{A} \begin{bmatrix} T_{\mathcal{B}\mathcal{D}}^\top & I \end{bmatrix}^\top$.

Then, application of the rank condition of Theorem 7 for the defined matrix in (40) using (25) leads to checking

$$\text{rank} \; T_{\mathcal{W}\mathcal{D}} \begin{bmatrix} T_{\mathcal{D}\mathcal{U},1} & \cdots & T_{\mathcal{D}\mathcal{U},m} \end{bmatrix} \tag{41}$$

where the right-most matrix is always full row rank, since $\text{rank}\ T_{\mathcal{D}\mathcal{U},\ell} = b_{\mathcal{U}\to\mathcal{D}}$, for some $\ell \in \{1,\dots,m\}$ by the relation between generic rank of transfer matrices and vertex-disjoint paths, see e.g., [16]. Furthermore, $b_{\mathcal{U}\to\mathcal{D}} = |\mathcal{D}|$ by construction of $\mathcal{D}$, which means that the rank condition implies

$$\text{rank}\ T_{\mathcal{W}\mathcal{D}} = b_{\mathcal{U}\to\mathcal{W}}. \tag{42}$$

Furthermore, since $\mathcal{D}$ is also a disconnecting set that disconnects $\mathcal{A}$ and $\mathcal{B}$ by definition, adding excitation signals to nodes in $\mathcal{A}$ or $\mathcal{D}$, such as to $\mathcal{S}_\mathcal{A}$ does not increase the number of vertex-disjoint paths, i.e.,

$$b_{(\mathcal{U}\cup\mathcal{S}_\mathcal{A})\to\mathcal{W}} = b_{\mathcal{U}\to\mathcal{W}}. \tag{43}$$

This shows equivalence of the rank conditions in Theorem 7 and Theorem 11, which shows that excited nodes by switching modules in $\mathcal{A}$, i.e., $\mathcal{S}_\mathcal{A}$, do not influence identifiability.

***The case $\mathcal{S} = \mathcal{S}_\mathcal{B}$ and $\mathcal{S}_\mathcal{A} = \emptyset$:*** The other case is examined where only nodes excited by switching modules are present in $\mathcal{B}$, i.e., $\mathcal{S} = \mathcal{S}_\mathcal{B}$ and $\mathcal{S}_\mathcal{A} = \emptyset$. To address the dynamics related to the nodes $\mathcal{S}_\mathcal{B}$ directly, the set $\mathcal{B}$ is partitioned in $\mathcal{S}_\mathcal{B}$ and $\bar{\mathcal{B}} = \mathcal{B}\backslash\mathcal{S}_\mathcal{B}$, where the bar notation $\bar{(\cdot)}$ denotes absence of effects caused by switching modules. This partitioning yields the following matrices:

$$G_\ell = \begin{bmatrix} \bar{G}_{\bar{\mathcal{B}}\bar{\mathcal{B}}} & G_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}} & \bar{G}_{\bar{\mathcal{B}}\mathcal{D}} & 0 \\ G_{\mathcal{S}_\mathcal{B}\bar{\mathcal{B}},\ell} & G_{\mathcal{S}_\mathcal{B}\mathcal{S}_\mathcal{B},\ell} & G_{\mathcal{S}_\mathcal{B}\mathcal{D},\ell} & 0 \\ G_{\mathcal{D}\bar{\mathcal{B}}} & G_{\mathcal{D}\mathcal{S}_\mathcal{B}} & G_{\mathcal{D}\mathcal{D}} & G_{\mathcal{D}\mathcal{A}} \\ G_{\mathcal{A}\bar{\mathcal{B}}} & G_{\mathcal{A}\mathcal{S}_\mathcal{B}} & G_{\mathcal{A}\mathcal{D}} & G_{\mathcal{A}\mathcal{A}} \end{bmatrix},$$
$$U = \begin{bmatrix} 0 \\ 0 \\ U_{\mathcal{D}\mathcal{U}} \\ U_{\mathcal{A}\mathcal{U}} \end{bmatrix},\ T_\ell = \begin{bmatrix} T_{\bar{\mathcal{B}}\mathcal{U},\ell} \\ T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} \\ T_{\mathcal{D}\mathcal{U},\ell} \\ T_{\mathcal{A}\mathcal{U},\ell} \end{bmatrix}.$$

Then, since $(I - G_\ell)T_\ell = U$, the equation in the third row can be rewritten into the following decomposition

$$\begin{aligned} T_{\mathcal{D}\mathcal{U},\ell} &= (I - G_{\mathcal{D}\mathcal{D}})^{-1}(G_{\mathcal{D}\bar{\mathcal{B}}}T_{\bar{\mathcal{B}}\mathcal{U}} + G_{\mathcal{D}\mathcal{A}}T_{\mathcal{A}\mathcal{U}} + \\ &\quad U_{\mathcal{D}\mathcal{U}} + G_{\mathcal{D}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell}) \\ &= \bar{T}_{\mathcal{D}\mathcal{U}} + T_{\mathcal{D}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} \end{aligned} \tag{44}$$

with $\bar{T}_{\mathcal{D}\mathcal{U}} = (I - G_{\mathcal{D}\mathcal{D}})^{-1}(G_{\mathcal{D}\bar{\mathcal{B}}}T_{\bar{\mathcal{B}}\mathcal{U}} + G_{\mathcal{D}\mathcal{A}}T_{\mathcal{A}\mathcal{U}} + U_{\mathcal{D}\mathcal{U}})$ the transfer matrix that would occur when nodes in $\mathcal{S}_\mathcal{B}$ are removed, so independent of the network mode $\ell$, and $T_{\mathcal{D}\mathcal{S}_\mathcal{B}} = (I - G_{\mathcal{D}\mathcal{D}})^{-1}G_{\mathcal{D}\mathcal{S}_\mathcal{B}}$ which in product with $T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell}$ covers the excluded part by $\bar{T}_{\mathcal{D}\mathcal{U}}$. Similarly, using the decomposition in (44), $T_{\bar{\mathcal{B}}\mathcal{U},\ell}$ can be rewritten into

$$\begin{aligned} T_{\bar{\mathcal{B}}\mathcal{U},\ell} &= (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} + G_{\bar{\mathcal{B}}\mathcal{D}}T_{\mathcal{D}\mathcal{U},\ell}) \\ &= (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} + \\ &\quad G_{\bar{\mathcal{B}}\mathcal{D}}(\bar{T}_{\mathcal{D}\mathcal{U}} + T_{\mathcal{D}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell})) \\ &= (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}} + (G_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}} + \\ &\quad G_{\bar{\mathcal{B}}\mathcal{D}}T_{\mathcal{D}\mathcal{S}_\mathcal{B}})T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell}) \\ &= \bar{T}_{\bar{\mathcal{B}}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}} + \tilde{T}_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} \end{aligned} \tag{45}$$

with $\tilde{T}_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}} + G_{\bar{\mathcal{B}}\mathcal{D}}T_{\mathcal{D}\mathcal{S}_\mathcal{B}})$ and $\bar{T}_{\bar{\mathcal{B}}\mathcal{D}} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\mathcal{D}}$.

The set of in-neighbors $\mathcal{W} \subseteq \bar{\mathcal{B}} \cup \mathcal{S}_\mathcal{B} \cup \mathcal{D}$, for which the required transfer matrix for the conditions of Theorem 7 can be retrieved in a similar way as (39), which with substitution of (44) and (45) is given by

$$\begin{aligned} T_{\mathcal{W}\mathcal{U},\ell} &= C_\mathcal{B}\begin{bmatrix} T_{\bar{\mathcal{B}}\mathcal{U},\ell} \\ T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} \\ T_{\mathcal{D}\mathcal{U},\ell} \end{bmatrix} \\ &= C_\mathcal{B}\left(\begin{bmatrix} \bar{T}_{\bar{\mathcal{B}}\mathcal{D}} \\ 0 \\ I \end{bmatrix}\bar{T}_{\mathcal{D}\mathcal{U}} + \begin{bmatrix} \tilde{T}_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}} \\ I \\ \tilde{T}_{\mathcal{D}\mathcal{S}_\mathcal{B}} \end{bmatrix} T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell}\right) \\ &= \bar{T}_{\mathcal{W}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}} + \tilde{T}_{\mathcal{W}\mathcal{S}_\mathcal{B}}T_{\mathcal{S}_\mathcal{B}\mathcal{U},\ell} \end{aligned} \tag{46}$$

with $\bar{T}_{\mathcal{W}\mathcal{D}} = C_\mathcal{B}\left[\bar{T}_{\bar{\mathcal{B}}\mathcal{D}}^\top\ 0\ I\right]^\top$ the mapping from external signals on $\mathcal{D}$ to $\mathcal{W}$ excluding nodes in $\mathcal{S}_\mathcal{B}$ and $\tilde{T}_{\mathcal{W}\mathcal{S}_\mathcal{B}} = C_\mathcal{B}\left[\tilde{T}_{\bar{\mathcal{B}}\mathcal{S}_\mathcal{B}}^\top\ I\ \tilde{T}_{\mathcal{D}\mathcal{S}_\mathcal{B}}^\top\right]^\top$ the mapping from external signals on $\mathcal{S}_\mathcal{B}$ to $\mathcal{W}$. For this transfer matrix, the rank condition of Theorem 7 becomes

$$\text{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}_\mathcal{B}}\ \bar{T}_{\mathcal{W}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}}\right]\begin{bmatrix} T_{\mathcal{S}_\mathcal{B}\mathcal{U},1} & \cdots & T_{\mathcal{S}_\mathcal{B}\mathcal{U},m} \\ I & \cdots & I \end{bmatrix} \tag{47}$$

where the right-most matrix is always generically full row rank by combination of the construction of $\mathcal{S}$ that the dynamics of the switching modules are different at least for some network mode $\ell$, where the generic rank concept excludes particular realizations of the dynamics that would lose rank. This means that only the rank of the left-most matrix implies the row rank of the matrix in (47), for which the following relation can be concluded:

$$\text{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}_\mathcal{B}}\ \bar{T}_{\mathcal{W}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}}\right] = b_{(\mathcal{U}\cup\mathcal{S}_\mathcal{B})\to\mathcal{W}}, \tag{48}$$

by the relation between the rank of transfer matrices and the number of vertex-disjoint paths, see e.g., [16]. Since the columns of transfer matrices correspond to mappings from a particular external signal, the nodes in $\mathcal{S}_\mathcal{B}$ could be interpreted as being excited by virtual signals. Note that the transfer matrix defined in $\bar{T}_{\mathcal{W}\mathcal{D}}\bar{T}_{\mathcal{D}\mathcal{U}}$ specifies the path $\mathcal{U} \to \mathcal{W}$ that does not pass through $\mathcal{S}_\mathcal{B}$, which could have a lower rank than $b_{\mathcal{U}\to\mathcal{W}}$ when a path $\mathcal{D} \to \mathcal{W}$ passes through $\mathcal{S}_\mathcal{B}$. The loss of rank in this case will be corrected by the specification of the same path in the transfer matrix $\tilde{T}_{\mathcal{W}\mathcal{S}_\mathcal{B}}$. The combination of the results (43) and (48) leads to the following implication

$$b_{(\mathcal{U}\cup\mathcal{S}_\mathcal{B})\to\mathcal{W}} = b_{(\mathcal{U}\cup\mathcal{S}_\mathcal{B}\cup\mathcal{S}_\mathcal{A})\to\mathcal{W}} = b_{(\mathcal{U}\cup\mathcal{S})\to\mathcal{W}}. \tag{49}$$

This result shows that the following relation holds for elements in both $\mathcal{S}_\mathcal{A}$ and $\mathcal{S}_\mathcal{B}$

$$\text{rank}\begin{bmatrix} \breve{T}_{j,1} & \dots & \breve{T}_{j,m} \\ V_j & \dots & V_j \end{bmatrix} = b_{(\mathcal{U}\cup\mathcal{S})\to\mathcal{W}}, \tag{50}$$

from which it can be concluded that the excitation by switching modules on the nodes $\mathcal{S}$ contribute in the same way as excitation signals to identifiability of the parametrized modules. Since the rank of the matrix in (50) should be generically full row rank for identifiability, this also holds for the vertex-disjoint path condition, which concludes the proof. ■