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# Instrumental Variable Methods for Closed-loop Continuous-time Model Identification

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## 1.1 Introduction

For many industrial production processes, safety and production restrictions are often strong reasons for not allowing identification experiments in open-loop. In such situations, experimental data can only be obtained under closed-loop conditions. The main difficulty in closed-loop identification is due to the correlation between the disturbances and the control signal, induced by the loop. Several alternatives are available to cope with this problem, broadly classified into three main approaches: direct, indirect and joint input/output [9, 14]. Some particular versions of these methods have been developed more recently in the area of control-relevant identification as e.g. the two-stage, the coprime factor and the dual-Youla methods. An overview of these recent developments can be found in [2] and [19].

When considering methods that can be used to identify models of systems operating in closed-loop, instrumental variable (IV) techniques are rather attractive because they are normally simple or iterative modifications of the linear regression algorithm. For instance, when dealing with highly complex processes that are high dimensional in terms of inputs and outputs, it can be attractive to rely on methods, such as these, that do not require non-convex optimization algorithms. In addition to this computationally attractive property, IV methods also have the potential advantage that they can yield consistent and asymptotically unbiased estimates of the plant model parameters if the noise does not have rational spectral density or the noise model is misspecified; or even if the control system is non-linear and/or time-varying.

For closed-loop identification, a basic IV estimator was first suggested assuming knowledge of the controller [20]; and the topic was later discussed in more detail in [15]. More recently a so-called 'tailor-made' IV algorithm has been suggested [6], where the closed-loop plant is parameterized using (open-loop) plant parameters. The class of algorithms denoted by BELS (bias eliminated least squares), e.g. [27], is also directed towards the use of only simple linear regression-like algorithms. Recently, it has been shown that these algorithms are, in fact, particular forms of IV estimation [6, 16].

When comparing the various available IV algorithms that all lead to consistent plant estimates, it is important to ask how the algorithm can be made statistically efficient: i.e. how is it possible to achieve the smallest variance of the parameter estimates. Concerning extended IV methods, an optimal variance result has been developed in the open-loop identification case, showing consequences for the choice of weights, filters, and instruments [9, 14, 17]. Similar enhancements of the basic IV approach are also the basis of the optimal refined instrumental variable (RIV) method [8, 21, 26] which is designed specifically for the Box-Jenkins transfer function model in discrete (RIV) or continuous (RIVC) time. For the closed-loop case, a statistical analysis has been provided [13, 15]; and this analysis has been used to compare several closed-loop identification methods [1]. More recently, some attention has been given to a characterization of the properties of the several (extended) IV methods [7].

All of the above methods, except RIVC, focus on the identification of discrete-time (DT) models. Recently, however, there has been renewed interest in the relevance of continuous-time (CT) model identification methods: see e.g. papers at the 15th IFAC World Congress in 2002, such as [23], which compares RIVC with another optimal approach; and [11], [12], where the advantages of direct CT approaches are illustrated by extensive simulation examples. Unfortunately, however, closed-loop model CT identification is still an issue that has not so far received adequate attention. Indeed, there are only a few recent publications that deal with closed-loop CT identification: amongst these, a bias eliminated least squares method [3], some basic instrumental variable methods [5], and a two-step algorithm using the RIVC algorithm [24], appear to be the most successful (see also **??** in the present book).

This chapter aims at filling this gap: it describes and evaluates statistically more efficient IV methods for the closed-loop identification of 'hybrid' CT transfer function models from discrete-time, sampled data (based, in part, on the analysis of the optimal open-loop approach in Chapter **??** of this book). Here, the model of the basic dynamic system is estimated in continuous-time, differential equation form, while the associated additive noise model is estimated as a discrete-time process. Several IV and IV-related methods are presented and they are unified in an extended IV framework. Then, the minimum variance closed-loop IV estimation approach is introduced, with the consequences of this formulation on the several design variables. Since such an optimal statistical approach requires the concurrent estimation of a model for the noise process, several bootstrap methods are proposed for accomplishing this. A comparison between these different proposed methods is made with the help of simulation examples, showing that more statistically efficient estimation can be achieved by an appropriate choice of the design parameters.

## 1.2 Problem Formulation

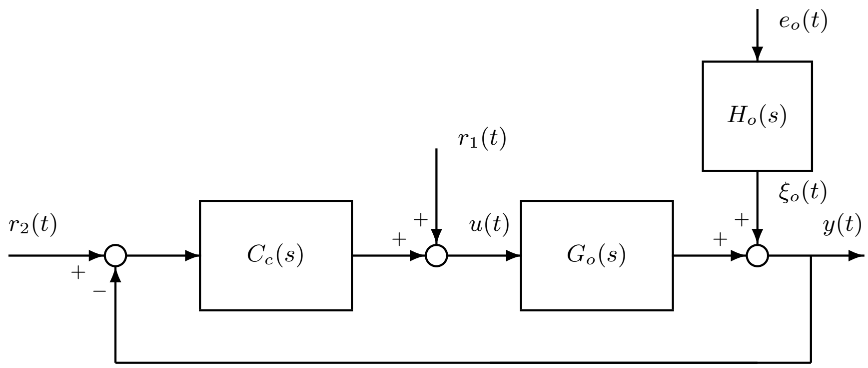


**Fig. 1.1.** Closed-loop configuration

Consider a stable, linear, SISO, closed-loop system of the form shown in Figure 1.1. The data generating system is assumed to be given by the relations

$$\mathcal{S} : \begin{cases} y(t) = G_o(s)u(t) + H_o(s)e_o(t) \\ u(t) = r(t) - C_c(s)y(t) \end{cases} \tag{1.1}$$

The process is denoted by $G_o(s) = B_o(s)/A_o(s)$ and the controller by $C_c(s)$ where $s$ is the differential operator ($s = d/dt$). $u(t)$ describes the process input signal, $y(t)$ the process output signal. For ease of notation we introduce an external signal $r(t) = r_1(t) + C_c(s)r_2(t)$. Moreover, it is also assumed that the CT signals $u(t)$ and $y(t)$ are uniformly sampled at $T_s$. A coloured disturbance is assumed to affect the closed-loop: bearing in mind the spectral factorisation theorem, this noise term, $\xi_o(t) = H_o(s)e_o(t)$, is modelled as linearly filtered white noise. The external signal $r(t)$ is assumed to be uncorrelated with the noise disturbance $\xi_o(t)$.

The CT model identification problem is to find estimates of $G_o(s)$ from finite sequences $\{r(t_k)\}_{k=1}^N$, $\{u(t_k)\}_{k=1}^N$, $\{y(t_k)\}_{k=1}^N$ of, respectively, the external signal, the process input and output DT data. The model is then described by the following hybrid equation,

$$\mathcal{M} : \; y(t_k) = G(s, \boldsymbol{\theta})u(t_k) + H(q^{-1}, \boldsymbol{\theta})e(t_k), \tag{1.2}$$

where $q^{-i}$ is the backward shift operator, i.e. $q^{-i}y(t_k) = y(t_{k-i})$; $e(t_k)$ is a discrete-time white noise, with zero mean and variance $\sigma_e^2$. The conventional notation $w(t_k)$ is used here to denote the sampled value of $w(t)$ at time-instant $t_k$.

The hybrid form (1.2) of the continuous-time transfer function model is considered here for two reasons. First, the approach is simple and straightforward: the theoretical and practical problems associated with the estimation of purely stochastic, continuous-time noise models are avoided by formulating the problem in this manner. Second, one of the main functions of the noise estimation is to improve the statistical efficiency of the parameter estimation by introducing appropriately defined prefilters into the estimation procedure. And, as we shall see in this chapter, this can be achieved adequately by assuming that the additive coloured noise $\xi(t_k)$ has rational spectral density, so that it can be represented in the form of a discrete-time, autoregressive moving average (ARMA) model (see below).
With the above assumptions, the parameterized CT hybrid process model takes the form,

$$\mathcal{G} : G(s, \boldsymbol{\rho}) = \frac{B(s, \boldsymbol{\rho})}{A(s, \boldsymbol{\rho})} = \frac{b_0 s^{n_b} + b_1 s^{n_b - 1} + \cdots + b_{n_b}}{s^{n_a} + a_1 s^{n_a - 1} + \cdots + a_{n_a}} \tag{1.3}$$

where $n_b$, $n_a$ denote the degrees of the process numerator and denominator polynomials, respectively, with the pair $(A, B)$ assumed to be coprime. The process model parameters are stacked columnwise in the parameter vector

$$\boldsymbol{\rho} = [a_1 \ \cdots \ a_{n_a} \ b_0 \ \cdots \ b_{n_b}]^T \in \mathbb{R}^{n_a + n_b + 1}. \tag{1.4}$$

The numerator and denominator orders $n_b$ and $n_a$ are to be identified from the data and the parameterized DT noise model is assumed to be in the form of the following discrete-time ARMA process,

$$\xi(t_k) = H(q^{-1}, \boldsymbol{\eta}) e(t_k) \tag{1.5a}$$

$$\mathcal{H} : H(q^{-1}, \boldsymbol{\eta}) = \frac{C(q^{-1}, \boldsymbol{\eta})}{D(q^{-1}, \boldsymbol{\eta})} = \frac{1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}}{1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d}} \tag{1.5b}$$

$$e(t_k) \sim \mathcal{N}(0, \sigma_e^2) \tag{1.5c}$$

where the associated noise model parameters are stacked columnwise in the parameter vector,

$$\boldsymbol{\eta} = \left[d_1 \cdots d_{n_d} \ c_1 \cdots c_{n_c}\right]^T \in \mathbb{R}^{n_c + n_d}, \tag{1.6}$$

where, as shown, $e(t_k)$ is a zero-mean, normally distributed, discrete-time white noise sequence.
The model structure $\mathcal{M}$ (1.2) is chosen so that the process and noise models do not have common factors; these models can therefore be parameterized independently. More formally, there exists the following decomposition of the parameter vector $\boldsymbol{\theta}$ for the whole hybrid model,

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\eta} \end{pmatrix}. \tag{1.7}$$

Additionally, the controller $C_c(s)$ is given by

$$C_c(s) = \frac{Q(s)}{P(s)} = \frac{q_0 s^{n_q} + q_1 s^{n_q - 1} + \cdots + q_{n_q}}{s^{n_p} + p_1 s^{n_p - 1} + \cdots + p_{n_p}}, \tag{1.8}$$

with the pair $(P, Q)$ assumed to be coprime. Of course, in this hybrid context, the continuous-time controller could be replaced by a DT alternative if this is required (see e.g. [24] where the continuous-time process is estimated within a DT, non-minimal state-space control loop [18]). In the following, the closed-loop system is assumed to be asymptotically stable and $r(t)$ is an external signal that is persistently exciting of sufficient high order.

With these notations, the closed-loop system can be described as

$$\begin{cases} y(t_k) = \dfrac{G_o(s)}{1 + C_c(s)G_o(s)} r(t_k) + \dfrac{1}{1 + C_c(s)G_o(s)} \xi_o(t_k) \\ u(t_k) = \dfrac{1}{1 + C_c(s)G_o(s)} r(t_k) - \dfrac{C_c(s)}{1 + C_c(s)G_o(s)} \xi_o(t_k) \end{cases} \tag{1.9}$$

In the following instrumental variable algorithms, use is made of the noise-free input/output signals deduced from (1.9) and denoted from hereon as

$$\begin{cases} x(t_k) = \dfrac{G_o(s)}{1 + C_c(s)G_o(s)} r(t_k) \\ \nu(t_k) = \dfrac{1}{1 + C_c(s)G_o(s)} r(t_k) \end{cases} \tag{1.10}$$

Now consider the relationship between the process input and output signals in (1.1),

$$y(t) = G_o(s)u(t) + H_o(s)e_o(t) \tag{1.11}$$

This latter can also be written in the vector form at time-instant $t = t_k$

$$y^{(n_a)}(t_k) = \boldsymbol{\varphi}^T(t_k)\boldsymbol{\rho}_o + v_o(t_k) \tag{1.12}$$

where $\boldsymbol{\rho}_o$ denotes the true process parameter vector,

$$\boldsymbol{\varphi}^T(t_k) = [-y^{(n_a - 1)}(t_k) \cdots - y(t_k) \; u^{(n_b)}(t_k) \cdots u(t_k)] \tag{1.13}$$

$w^{(i)}(t_k)$ denotes the $i$th time-derivative of the CT signal $w(t)$ at time-instant $t_k$ and

$$v_o(t_k) = A_o(s)H_o(s)e_o(t_k). \tag{1.14}$$

Note that the noise-free signals $x(t_k)$ and $\nu(t_k)$ in (1.10) are not available from measurements, therefore the several closed-loop methods presented in this chapter use of this noisy regressor $\boldsymbol{\varphi}(t_k)$.

There are two main time-domain approaches to estimate a CT model in this form. The first, indirect approach, is to estimate an initial DT model from the sampled data and then convert this into a CT model. The second, direct approach, that we consider in the present chapter, is to identify a CT model directly from the DT data.

## 1.3 Basic Instrumental Variable Estimators

The process model parameters $\boldsymbol{\rho}$ can be estimated using a basic instrumental variable (IV) estimator. By assuming that the time-derivatives of the input, output and external signals are available (see Section 1.5.2), the CT version of the basic IV estimate of $\boldsymbol{\rho}$ is given by

$$\hat{\boldsymbol{\rho}}_{iv} = sol \left\{ \frac{1}{N} \sum_{k=1}^{N} \boldsymbol{\zeta}(t_k)[y^{(n_a)}(t_k) - \boldsymbol{\varphi}^T(t_k)\boldsymbol{\rho}] = 0 \right\} \tag{1.15}$$

where $N$ denotes the number of data and $\boldsymbol{\zeta}(t_k)$ is a vector of instrumental variables.

There is a considerable amount of freedom in the choice of the instruments. A first solution is to adapt the closed-loop IV method developed for DT models in [15] to the CT model identification case. This method is referred to as CLIVC and was first presented in [5]. It involves using the external signal time-derivatives as instruments. The so-called basic IV estimate for closed-loop CT models is then given by

$$\hat{\boldsymbol{\rho}}_{clivc} = \left[ \sum_{k=1}^{N} \boldsymbol{\zeta}(t_k)\boldsymbol{\varphi}^T(t_k) \right]^{-1} \sum_{k=1}^{N} \boldsymbol{\zeta}(t_k) y^{(n_a)}(t_k) \tag{1.16}$$

$$\text{with } \boldsymbol{\zeta}^T(t_k) = \left[ r^{(n_a+n_b)}(t_k) \cdots r(t_k) \right] \in \mathbb{R}^{n_a+n_b+1} \tag{1.17}$$

In contrast with the basic IV for DT model identification which uses a difference equation model, the CT version makes use of an instrument built up from the time-derivatives of the external signals.

### 1.3.1 Consistency Properties

By inserting (1.12) into (1.15), the following equation is obtained

$$\hat{\boldsymbol{\rho}}_{iv} = \boldsymbol{\rho}_o + \left[ \sum_{k=1}^{N} \boldsymbol{\zeta}(t_k)\boldsymbol{\varphi}^T(t_k) \right]^{-1} \left[ \sum_{k=1}^{N} \boldsymbol{\zeta}(t_k) v_o(t_k) \right] \tag{1.18}$$

where $\boldsymbol{\varphi}^T(t_k)$ and $v_o(t_k)$ are given by (1.13) and (1.14) respectively. It can be deduced from (1.18) that $\hat{\boldsymbol{\rho}}_{iv}$ is a consistent estimate of $\boldsymbol{\rho}$ if[4]

$$\begin{cases} \bar{\mathsf{E}}[\boldsymbol{\zeta}(t_k)\boldsymbol{\varphi}^T(t_k)] \text{ is nonsingular} \\ \bar{\mathsf{E}}[\boldsymbol{\zeta}(t_k)v_o(t_k)] = 0 \end{cases} \tag{1.19}$$

Several IV variants can be obtained by different choices of the instruments $\boldsymbol{\zeta}(t_k)$ in (1.15), respecting the conditions given by (1.19).

[4] The notation $\bar{\mathsf{E}}[.] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathsf{E}[.]$ is adopted from the prediction error framework of [9].

### 1.3.2 Accuracy Analysis

The asymptotic distribution of the parameter estimate $\hat{\boldsymbol{\rho}}_{iv}$ in (1.15) has been investigated extensively in the open-loop DT context (e.g. [9, 13, 14]). More recently, this work also has been extended to the closed-loop DT model identification framework [7]. By considering (1.15), these previous results can be applied to the case of the CT hybrid model given by (1.2). As a result, under the assumptions formulated in Section 1.2 and $G_o \in \mathcal{G}$, $\hat{\boldsymbol{\rho}}_{iv}$ is asymptotically Gaussian distributed

$$\sqrt{N}(\hat{\boldsymbol{\rho}}_{iv} - \boldsymbol{\rho}^*) \overset{dist}{\rightarrow} \mathcal{N}(0, \mathbf{P}_{iv}) \tag{1.20}$$

where $\boldsymbol{\rho}^*$ represents the limit of $\hat{\boldsymbol{\rho}}_{iv}$ when $N \to \infty$ and where the covariance matrix is given by

$$\mathbf{P}_{iv} = \sigma_{e_o}^2 \left[\bar{\mathsf{E}}\boldsymbol{\zeta}(t_k)\boldsymbol{\varphi}^T(t_k)\right]^{-1} \left[\bar{\mathsf{E}}\tilde{\boldsymbol{\zeta}}(t_k)\tilde{\boldsymbol{\zeta}}^T(t_k)\right] \left[\left(\bar{\mathsf{E}}\boldsymbol{\zeta}(t_k)\boldsymbol{\varphi}^T(t_k)\right)^{-1}\right]^T \tag{1.21}$$

with $\tilde{\boldsymbol{\zeta}}(t_k) = H_o(s)A_o(s)\boldsymbol{\zeta}(t_k)$ and $\sigma_{e_o}^2$ denotes the intensity of $\{e_o(t_k)\}$.

## 1.4 Extended Instrumental Variable Estimators

There are various ways of considering IV estimation from an optimal standpoint. One such approach is to consider an extended IV solution (see Introduction section). In CT model identification, if the time-derivatives signals are assumed to be known, the extended IV estimate of $\boldsymbol{\rho}$ is obtained by pre-filtering the input/output data appearing in (1.15) and by generalizing the basic IV estimates $\hat{\boldsymbol{\rho}}_{iv}$ using an augmented instrument $\boldsymbol{\zeta}(t_k) \in \mathbb{R}^{n_\zeta}$ $(n_\zeta \geq n_a + n_b + 1)$ so that an over-determined system of equations is obtained in the form,

$$\hat{\boldsymbol{\rho}}_{xiv} = \arg\min_{\boldsymbol{\rho}} \left\| \left[\sum_{k=1}^{N} f(s)\boldsymbol{\zeta}(t_k) f(s)\boldsymbol{\varphi}^T(t_k)\right] \boldsymbol{\rho} - \left[\sum_{k=1}^{N} f(s)\boldsymbol{\zeta}(t_k) f(s) y^{(n_a)}(t_k)\right] \right\|_W^2 \tag{1.22}$$

where $f(s)$ is a stable pre-filter, and $\|x\|_W^2 = x^T W x$, with $W$ a positive definite weighting matrix. This extended IV gives a parameter estimator that requires more computations than the basic IV. However, the enlargement of the IV vector can be used for improving the accuracy of the parameter estimates [14]. Note that, when $f(s) = 1$ and $n_\zeta = n_a + n_b + 1$ $(W = I)$, the basic IV estimate (1.15) is obtained.

### 1.4.1 Consistency Properties

The consistency conditions are easily obtained by generalizing (1.19) to the estimator (1.22)

$$\begin{cases} \bar{\mathsf{E}}[f(s)\boldsymbol{\zeta}(t_k)f(s)\boldsymbol{\varphi}^T(t_k)] \text{ is full column rank} \\ \bar{\mathsf{E}}[f(s)\boldsymbol{\zeta}(t_k)f(s)v_o(t_k)] = 0 \end{cases} \tag{1.23}$$

### 1.4.2 Accuracy Analysis

The asymptotic distribution of parameter vector (1.22) is obtained by following the same reasoning as in Section 1.3.2. Therefore, by considering the results given in Section 1.2, under the assumption that $G_o \in \mathcal{G}$, $\hat{\boldsymbol{\rho}}_{xiv}$ is asymptotically Gaussian distributed,

$$\sqrt{N}(\hat{\boldsymbol{\rho}}_{xiv} - \boldsymbol{\rho}^*) \stackrel{dist}{\rightarrow} \mathcal{N}(0, \mathbf{P}_{xiv}) \tag{1.24}$$

where the covariance matrix is given by

$$\mathbf{P}_{xiv} = \sigma^2_{e_{of}} \left[\mathbf{R}^T\mathbf{W}\mathbf{R}\right]^{-1} \mathbf{R}^T\mathbf{W} \left[\bar{\mathsf{E}}\tilde{\boldsymbol{\zeta}}(t_k)\tilde{\boldsymbol{\zeta}}^T(t_k)\right] \mathbf{W}\mathbf{R} \left[\mathbf{R}^T\mathbf{W}\mathbf{R}\right]^{-1}$$

with

$$\tilde{\boldsymbol{\zeta}}(t_k) = f(s)H_o(s)A_o(s)\boldsymbol{\zeta}(t_k) \text{ and } \mathbf{R} = \bar{\mathsf{E}} f(s)\boldsymbol{\zeta}(t_k)f(s)\mathring{\boldsymbol{\varphi}}^T(t_k)$$

where $\mathring{\boldsymbol{\varphi}}(t_k)$ is the noise-free part of the regressor $\boldsymbol{\varphi}(t_k)$ (1.13), built up from the noise-free input/ouput signals $\nu(t_k)$ and $x(t_k)$ (1.10) as

$$\mathring{\boldsymbol{\varphi}}^T(t_k) = [-x^{(n_a-1)}(t_k) \cdots - x(t_k) \;\; \nu^{(n_b)}(t_k) \cdots \nu(t_k)] \tag{1.25}$$

Note that the noise-free part of the regressor is partly defined by the noise-free output variable $x(t_k)$ in (1.10) and its derivatives. It is well-known in open-loop estimation that an estimate of this variable, generated as the output of an 'auxiliary model', is normally used as the prime source of instrumental variable for the output variable. In the closed-loop context, however, the measured regression vector also contains the filtered process input and its derivatives, it is clear, therefore, that a suitable estimate of the noise-free process input $\nu(t_k)$ will also be required for accurate IV estimation.

## 1.5 Optimal Instrumental Variable Estimators

### 1.5.1 Main Results

The choice of the instruments $\boldsymbol{\zeta}(t)$, the number of IV components $n_\zeta$, the weighting matrix $W$ and the prefilter $f(s)$ may have a considerable effect on the covariance matrix $\mathbf{P}_{xiv}$. In the open-loop DT situation, the lower bound of $\mathbf{P}_{xiv}$ for any unbiased identification method is given by the Cramér-Rao lower bound [9], [13]. Optimal choices of the above mentioned design variables exist so that $\mathbf{P}_{xiv}$ reaches the Cramér-Rao lower bound. These results cannot be applied to the closed-loop IV case because of the correlation between the process input signal $u(t_k)$ and the noise. In this regard, it has been shown in [15] that, for a model given by (1.12), there exists a minimum value of the covariance matrix $\mathbf{P}_{xiv}$ as a function of the design variables $\boldsymbol{\zeta}(t_k)$, $f(s)$ and $W$, under the restriction that $\boldsymbol{\zeta}(t_k)$ is a function of the external signal $r(t_k)$ only. Although these results have been obtained for the case of DT models, a similar analysis applies in the CT case and the covariance matrix can be optimized with respect to the design variables. The optimal covariance matrix (different from the Cramér-Rao lower bound) for a data generating closed-loop system given by (1.2) where $u(t_k)$ and $y(t_k)$ are correlated by noise, is then

$$\mathbf{P}_{xiv} \geq \mathbf{P}_{xiv}^{opt} \quad \text{and}$$
$$\mathbf{P}_{xiv}^{opt} = \sigma^2_{e_{ofopt}} \left\{ \bar{\mathsf{E}} \left[ [A_o(s)H_o(s)]^{-1} \mathring{\boldsymbol{\varphi}}^T(t_k) \right]^T \left[ [A_o(s)H_o(s)]^{-1} \mathring{\boldsymbol{\varphi}}^T(t_k) \right] \right\}^{-1} \tag{1.26}$$

$\mathbf{P}_{xiv}^{opt}$ is then obtained by taking,

$$\begin{aligned} \boldsymbol{\zeta}(t_k) &= f^{opt}(s)\mathring{\boldsymbol{\varphi}}(t_k), \\ f^{opt}(s) &= [A_o(s)H_o(s)]^{-1}, \\ n_\zeta &= n_a + n_b + 1, \\ W &= I. \end{aligned} \tag{1.27}$$

Therefore, the only difference between open-loop and closed-loop cases is that in the latter, the input process signal is correlated with the noise, so that the instruments must be correlated with the noise-free part of $u(t)$ but uncorrelated with the noisy part of $u(t)$ (due to the feedback loop).
Moreover, when defined in this manner, it would appear that the optimal IV estimator can only be obtained if, first, the true noise (and process) model is exactly known and secondly the noise-free part of the regressor is available. However, this is a probabilistic estimation problem and therefore the statistically optimal estimates can be obtained if these TF model polynomials are replaced by their optimal estimates. Moreover, practically useful sub-optimal solutions can be obtained by utilizing good, if not optimal, estimates. This is discussed in the next sub-sections.

### 1.5.2 Implementation Issues

#### Handling of the Unmeasurable Time-derivative Signals

In comparison with the DT counterpart, direct CT model identification raises several technical issues. The first is related to implementation. Unlike the difference equation model, the differential equation model is not a linear combination of the sampled process input and output signals but contains time-derivatives signals. The theoretical study presented in the previous section assumes that these time-derivatives signals are available, and therefore, the parameter estimation procedure can be directly applied on them. However, these input, output and external time-derivatives signals are not available as measurement data in most practical cases. A standard approach used in CT model identification is to introduce a low-pass stable filter $f_c(s)$, i.e. define

$$y_{f_c}(t_k) = f_c(s)y(t_k), \quad u_{f_c}(t_k) = f_c(s)u(t_k) \tag{1.28}$$

where the subscript $f_c$ is used to denote the prefiltered forms of the associated variables. The filtered time-derivatives can then be obtained by sending both input/output signals to a bench of filters of the form $f_c(s)s^i$

$$y^{(i)}_{f_c}(t_k) = f_c(s)s^i y(t_k), \quad i \leq n_a \tag{1.29}$$

$$u^{(i)}_{f_c}(t_k) = f_c(s)s^i u(t_k), \quad i \leq n_b \tag{1.30}$$

The motivation is that the filtered signals $u_{f_c}(t)$ and $y_{f_c}(t)$ satisfy

$$y_{f_c}(t_k) = G_o(s)u_{f_c}(t_k) + f_c(s)H_o(s)e_o(t_k) \tag{1.31}$$

i.e. the process transfer function is not changed but the noise transfer function is modified by the introduction of the filter. Equation (1.31) can be rewritten under the following linear regression form

$$y^{(n_a)}_{f_c}(t_k) = \boldsymbol{\varphi}^T_{f_c}(t_k)\boldsymbol{\rho}_o + v_{of_c}(t_k) \tag{1.32}$$

with

$$\boldsymbol{\varphi}^T_{f_c}(t_k) = [-y^{(n_a-1)}_{f_c}(t_k) \ \cdots \ -y_{f_c}(t_k) \ u^{(n_b)}_{f_c}(t_k) \ \cdots \ u_{f_c}(t_k)] \tag{1.33}$$

$$v_{of_c}(t_k) = f_c(s)v_o(t_k) \tag{1.34}$$

Various types of CT filters have been devised to deal with the need to reconstruct the time-derivatives [4] and the CONtinuous-Time System IDentification (CONTSID) toolbox has been developed on the basis of these methods (see Chapter **??** in the present book). Four usual filters that have been used in simple IV methods are as follows [4]

$$f_{c_1}(s) = \left(\frac{\beta}{s+\lambda}\right)^{n_a} \qquad f_{c_2}(s) = \left(\frac{\beta}{s+\lambda}\right)^{n_a+1} \tag{1.35a}$$

$$f_{c_3}(s) = \left(\frac{1}{s}\right)^{n_a} \qquad f_{c_4}(s) = \left(\frac{1-e^{-lT_s s}}{s}\right)^{n_a} \tag{1.35b}$$

where $f_{c_1}(s)$ and $f_{c_2}(s)$ represent the filters used in the case of the minimal order multiple filter (also referred to as state-variable filter) method and generalized Poisson moment functional approach respectively; $f_{c_3}(s)$ denotes the

usual multiple integral operation while $f_{c_4}(s)$ is referred to as the linear integral filter. Note however that this list is not exhaustive. Moreover, it is clearly possible to select the prefilter $f_c(s)$ in order to achieve some form of optimal IV estimation and this is considered later in Section 1.5.
For simplicity, it has been assumed above that the differential equation model (1.2) is initially at rest. However, note that, in the general case, the initial condition terms do not vanish in (1.32). Whether they require estimation or they can be neglected depends upon the selected signal pre-filtering method.

### Noise Modelling and Hybrid Filtering

The choice of the instruments and prefilter in the IV method affects the asymptotic variance, while consistency properties are generically secured by the IV implementation. It has been found that minor deviations from the optimal estimates of the polynomials required for the implementation of the auxiliary model and prefilters will normally only cause second-order effects in the resulting accuracy. Therefore, a reasonable IV estimator can be obtained if consistent, but not necessarily efficient estimates of the polynomials are utilized (see [9] for a discussion in the DT case). In addition, the computational procedures can be kept simple and tractable if linear regression estimates are used in the preliminary stages of the estimation.

Several bootstrap IV methods have been proposed, in an attempt to approximate the optimal IV method (see e.g. [9, 13, 21] for the open-loop situation and [7] for the closed-loop one). As explained in Section 1.5, the difference between open-loop and closed-loop cases lies in the input process signal which is correlated with the noise in the latter. Therefore, the instrumental variable vector must include IVs associated with the input as well as the output signal, and these must be correlated with the noise-free part of $u(t)$ but uncorrelated with the noise on $u(t)$ arising from the feedback loop.

Following the discussion in Section 1.2, CT models are estimated to represent the transfer between the external signal and the output, as well as for the transfer between the external signal and the input. And according to the hybrid model (1.2) we are using here, DT models are used to estimate the noise contribution.
From (1.27) and Section 1.5.2, the IV filter involves a filter $f(s)$ required for handling the time-derivatives along with the CT process TF denominator polynomial and noise model contributions. As a result, the IV estimation will require hybrid filtering operations involving:

- a CT filter $f(s) = f_c(s)$ needed to compute the time-derivatives (see (1.28) and also Chapter **??** in the present book).

- a DT filtering operation needed to approximate the inverse of the CT process TF denominator polynomial and noise model contributions (see (1.27)), denoted from hereon as $f_d(q^{-1}, \boldsymbol{\eta})$.

To realize the optimal choices for the instruments, two alternatives are developed in the following sections: the first relies on multi-step algorithms while the second is based on iterative (adaptive) solutions. As we will see, the form of the CT and DT filters will differ according to the assumed true CT system model structures.

### 1.5.3 Multi-step Approximate Implementations of the Optimal IV Estimate

#### Two-step CLIVC2 algorithm

The two-step IV algorithm, denoted as CLIVC2, is based on the following CT ARX model

$$\begin{cases} A_o(s)y(t_k) = B_o(s)u(t_k) + e_o(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s)y(t_k) \end{cases} \tag{1.36}$$

or its filtered version

$$\begin{cases} y_{f_c}(t_k) = \frac{B_o(s)}{A_o(s)} u_{f_c}(t_k) + f_c(s)\frac{1}{A_o(s)} e(t_k) \\ \text{with } u_{f_c}(t_k) = r_{f_c}(t_k) - C_c(s)y_{f_c}(t_k) \end{cases} \tag{1.37}$$

where we see that the noise model is constrained to include the process TF denominator polynomial $A_o(s)$.
In this particular case, the approximate optimal filter $f_{clivc2}$ is composed of:

- the CT filter $f_c(s)$, which is chosen amongst the several options given in (1.35a)-(1.35b),
- the DT filter $f_d(q^{-1}) = 1$ since the noise model of the assumed CT ARX data generating system is $H_o(s) = 1/A_o(s)$.

Since the CT filter $f_c(s)$ is chosen amongst several non-optimal filters, the resulting CLIVC2 algorithm is an approximate implementation of the optimal IV solution presented Section 1.5.1.

The outline of the CLIVC2 algorithm is then the following

1. Choose a CT pre-filter $f_c(s)$ to compute $y^{(i)}_{f_c}(t_k)$, $u^{(i)}_{f_c}(t_k)$ and $r^{(i)}_{f_c}(t_k)$, for $i \leq n_a$.
   Write the filtered CT ARX model structure as a linear regression
   $$y^{(n_a)}_{f_c}(t_k) = \boldsymbol{\varphi}^T_{f_c}(t_k)\boldsymbol{\rho} \tag{1.38}$$
   and obtain an estimate $\hat{\boldsymbol{\rho}}_1$ of $\boldsymbol{\rho}$ by the least squares method.

2. Use this estimate $\hat{\boldsymbol{\rho}}_1$ along with the process model, as defined by,

$$G(s, \hat{\boldsymbol{\rho}}_1) = \frac{B(s, \hat{\boldsymbol{\rho}}_1)}{A(s, \hat{\boldsymbol{\rho}}_1)},$$

to generate the instruments $\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)$ using the following closed-loop auxiliary models

$$\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) = \frac{G(s, \hat{\boldsymbol{\rho}}_1)}{1 + C_c(s)G(s, \hat{\boldsymbol{\rho}}_1)} r_{f_c}(t_k) \tag{1.39}$$

$$\hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) = \frac{1}{1 + C_c(s)G(s, \hat{\boldsymbol{\rho}}_1)} r_{f_c}(t_k) \tag{1.40}$$

$$\begin{aligned}\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) = [&-\hat{x}_{f_c}^{(n_a-1)}(t_k, \hat{\boldsymbol{\rho}}_1) \;\cdots\; -\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) \\ &\hat{\nu}_{f_c}^{(n_b)}(t_k, \hat{\boldsymbol{\rho}}_1) \;\cdots\; \hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)]^T\end{aligned} \tag{1.41}$$

$\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)$ represents an estimate of the noise-free part of the regressor $\boldsymbol{\varphi}_{f_c}(t_k)$ and according to the notations used in Chapter **??**, it will be denoted from hereon as $\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) = \hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)$.
Using the instrument $\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)$ and the prefilter $f_d(q^{-1}, \boldsymbol{\eta}) = 1$, determine the IV estimate in (1.38) as

$$\hat{\boldsymbol{\rho}}_{clivc2} = \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clivc2}}(t_k, \hat{\boldsymbol{\rho}}_1)\boldsymbol{\varphi}_{f_{clivc2}}^T(t_k)\right]^{-1}\left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clivc2}}(t_k, \hat{\boldsymbol{\rho}}_1) y_{f_{clivc2}}^{(n_a)}(t_k)\right], \tag{1.42}$$

where $f_{clivc2}(s) = f_c(s)$ here.

*Remark 1.1.* In contrast to the discrete-time case, a high-order least squares estimator should not be used in the first step of the continuous-time system identification procedure because of the numerical errors induced by the simulation method required for the generation of the filtered variables in (1.33) and (1.41).

### Four-step CLIVC4 algorithm

Although the process parameter estimates from the CLIVC2 algorithm are consistent, it is worthwhile considering improved noise model estimation in order to construct an estimator with a smaller variance (closer to the optimal solution). One improvement is to assume the following CT ARARX model structure

$$\begin{cases} A_o(s)y(t_k) = B_o(s)u(t_k) + \frac{1}{D_o(s)} e_o(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s)y(t_k) \end{cases} \tag{1.43}$$

or its filtered version

$$\begin{cases} y_{f_c}(t_k) = \dfrac{B_o(s)}{A_o(s)} u_{f_c}(t_k) + f_c(s) \dfrac{1}{A_o(s)D_o(s)} e_o(t_k) \\ \text{with } u_{f_c}(t_k) = r_{f_c}(t_k) - C_c(s) y_{f_c}(t_k) \end{cases} \tag{1.44}$$

where we see that the noise model is also constrained to include the TF denominator polynomial $A_o(s)$.
In this particular case, the approximate optimal filter $f_{clivc4}$ is composed of:

- the CT filter $f_c(s)$, which is chosen amongst the several options given in (1.35a)-(1.35b),
- the DT filter $f_d(q^{-1}, \boldsymbol{\eta}) = 1/D(q^{-1}, \boldsymbol{\eta})$ (AR model of order to be chosen or identified) since the noise model of the assumed CT ARARX data generating CT system is $H_o(s) = 1/A_o(s)D_o(s)$.

As a result, the proposed CLIVC4 algorithm is then based on the following CT hybrid ARARX model structure [7]

$$\begin{cases} A(s, \boldsymbol{\rho}) y(t_k) = B(s, \boldsymbol{\rho}) u(t_k) + \dfrac{1}{D(q^{-1}, \boldsymbol{\eta})} e(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s) y(t_k) \end{cases} \tag{1.45}$$

Note that in the above equation, we are mixing discrete and continuous-time operators somewhat informally in order to indicate the hybrid computational nature of the estimation problem being considered here. Thus, operations such as,

$$\frac{B(s, \boldsymbol{\rho})}{A(s, \boldsymbol{\rho})} u(t_k)$$

imply that the input variable $u(t_k)$ is interpolated in some manner. This is to allow for the intersample behaviour that is not available from the sampled data and so has to be inferred in order to allow for the continuous-time numerical integration of the associated differential equations. For such integration, the discretization interval will be varied, dependent on the numerical method employed, but it will usually be much smaller than the sampling interval $T_s$ (see Chapter **??** in the present book).

This proposed solution may be seen as an extension of the four-step IV technique for open-loop DT model identification (IV4) [9] to the CT hybrid closed-loop framework. The difference between both algorithm is that in the CT version, a filter is needed to handle the time-derivatives problem. As previously, since it is carried out by a CT filter $f_c(s)$ chosen amongst several non-optimal filters, the resulting CLIVC4 algorithm is an approximate implementation of the optimal IV solution presented in Section 1.5.1.

The outline of the CLIVC4 algorithm is as follows:

1. Choose a CT pre-filter $f_c(s)$ to compute $y_{f_c}^{(i)}(t_k)$, $u_{f_c}^{(i)}(t_k)$ and $r_{f_c}^{(i)}(t_k)$, for $i \leq n_a$.
   Write the filtered model structure as a linear regression

$$y_{f_c}^{(n_a)}(t_k) = \boldsymbol{\varphi}_{f_c}^T(t_k)\boldsymbol{\rho}. \tag{1.46}$$

Obtain an estimate $\hat{\boldsymbol{\rho}}_1$ of $\boldsymbol{\rho}$ by the least squares method and use this to define the corresponding CT transfer function $G(s, \hat{\boldsymbol{\rho}}_1)$.

2. Generate the instruments $\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) = \hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)$ using the closed-loop auxiliary models as in (1.41). $\hat{\boldsymbol{\varphi}}_{fc}(t_k, \hat{\boldsymbol{\rho}}_1)$ represents an estimate of the noise-free part of the filtered regressor $\boldsymbol{\varphi}_{f_c}(t_k)$. Determine the IV estimate of $\boldsymbol{\rho}$ in (1.46) as

$$\hat{\boldsymbol{\rho}}_2 = \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1)\boldsymbol{\varphi}_{f_c}^T(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_1) y_{f_c}^{(n_a)}(t_k)\right] \tag{1.47}$$

and use this to define the corresponding CT transfer function $G(s, \hat{\boldsymbol{\rho}}_2)$.

3. Let $\hat{w}(t_k) = y_{f_c}^{(n_a)}(t_k) - \boldsymbol{\varphi}_{f_c}^T(t_k)\hat{\boldsymbol{\rho}}_2$. Now, an AR model[5] of order $2n_a$ can be postulated for $\hat{w}(t_k)$:

$$f_d(q^{-1}, \hat{\boldsymbol{\eta}})\hat{w}(t_k) = e(t_k)$$

and then $f_d(q^{-1}, \hat{\boldsymbol{\eta}})$ can be estimated using the least squares method.

4. Generate the instruments $\boldsymbol{\zeta}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2) = \hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2)$ as

$$\begin{aligned}\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2) = [-\hat{x}_{f_c}^{(n_a-1)}(t_k, \hat{\boldsymbol{\rho}}_2) \ \cdots \ -\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2) \\ \hat{\nu}_{f_c}^{(n_b)}(t_k, \hat{\boldsymbol{\rho}}_2) \ \cdots \ \hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2)]^T\end{aligned} \tag{1.48}$$

where $\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2)$ and $\hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2)$ are the estimated noise-free output of the closed-loop auxiliary models computed as in (1.39)-(1.40) on the basis of $G(s, \hat{\boldsymbol{\rho}}_2)$.

Using these instruments $\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2)$ and the prefilter $f_d(q^{-1}, \hat{\boldsymbol{\eta}})$, determine the IV estimate of $\boldsymbol{\rho}$ in (1.46) as

$$\hat{\boldsymbol{\rho}}_{clivc4} = \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clivc4}}(t_k, \hat{\boldsymbol{\rho}}_2)\boldsymbol{\varphi}_{f_{clivc4}}^T(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clivc4}}(t_k, \hat{\boldsymbol{\rho}}_2) y_{f_{clivc4}}^{(n_a)}(t_k)\right] \tag{1.49}$$

where

$$\hat{\boldsymbol{\varphi}}_{f_{clivc4}}(t_k, \hat{\boldsymbol{\rho}}_2) = f_d(q^{-1}, \hat{\boldsymbol{\eta}})\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}_2), \tag{1.50}$$

$$\boldsymbol{\varphi}_{f_{clivc4}}(t_k) = f_d(q^{-1}, \hat{\boldsymbol{\eta}})\boldsymbol{\varphi}_{f_c}(t_k), \tag{1.51}$$

$$\text{and} \quad y_{f_{clivc4}}^{(n_a)}(t_k) = f_d(q^{-1}, \hat{\boldsymbol{\eta}}) y_{f_c}^{(n_a)}(t_k). \tag{1.52}$$

[5] Or the AR order can be identified using a model order identification method, such as the Akaike information criterion (AIC).

### 1.5.4 Iterative Implementations of the Optimal IV Estimate

In the previous algorithms, the filter $f_c(s)$ used to compute the time-derivative signals, is fixed *a priori* by the user and is not included into the design variables of the method. Furthermore, the CLIVC2 and CLIVC4 approaches rely on a noise model that is constrained to include the process TF denominator polynomial (see (1.43)).
An alternative approach is to consider instead, a CT Box-Jenkins (BJ) transfer function (TF) model defined as follows

$$\begin{cases} y(t_k) = \dfrac{B_o(s)}{A_o(s)} u(t_k) + \dfrac{C_o(s)}{D_o(s)} e_o(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s) y(t_k) \end{cases} \tag{1.53}$$

For most practical purposes, this model is the most natural one to use since it does not constrain the process and the noise models to have common denominator polynomials. It also has the advantage that the maximum likelihood estimates of the process model parameters are asymptotically independent of the noise model parameter estimates (see Chapter **??** in this book and [10]). The problem introduced by considering (1.53), however, is that the model is non-linear-in-the-parameters so that simple IV estimation cannot be directly applied.

Fortunately, this problem of nonlinear estimation can be overcome by designing an iterative estimation algorithm on the basis of the procedures used in the refined instrumental variable (RIV) algorithm [8, 21, 22, 25] and its CT equivalent, the refined instrumental variable for continuous system (RIVC) algorithm [26], as discussed fully in Chapter **??**, suitably extended to handle the closed-loop identification case.
Following the usual *Prediction Error Minimization* (PEM) approach in the present hybrid situation (which is ML estimation because of the Gaussian assumptions on $e(t_k)$), a suitable error function $\varepsilon(t_k)$, at the $k$th sampling instant, is given by

$$\varepsilon(t_k) = \frac{D_o(s)}{C_o(s)} \left\{ y(t_k) - \frac{B_o(s)}{A_o(s)} u(t_k) \right\}$$

which can be written as

$$\varepsilon(t_k) = \frac{D_o(s)}{C_o(s)} \left\{ \frac{1}{A_o(s)} \left[ A_o(s) y(t_k) - B_o(s) u(t_k) \right] \right\} \tag{1.54}$$

where the CT prefilter $D_o(s)/C_o(s)$ will be recognized as the inverse of the continuous-time auto-regressive moving average (CARMA) noise model in 1.53.
Minimization of a least squares criterion function in $\varepsilon(t_k)$, measured at the sampling instants, provides the basis for stochastic estimation. However, since

the polynomial operators commute in this linear case, (1.54) can be considered in the alternative form,

$$\varepsilon(t_k) = A_o(s)y_f(t_k) - B_o(s)u_f(t_k) \tag{1.55}$$

where $y_f(t_k)$ and $u_f(t_k)$ represent the *sampled* outputs of the complete CT prefiltering operation

$$y_f(t_k) = \frac{1}{A_o(s)}\frac{D_o(s)}{C_o(s)}y(t_k), \tag{1.56}$$

$$u_f(t_k) = \frac{1}{A_o(s)}\frac{D_o(s)}{C_o(s)}u(t_k). \tag{1.57}$$

In this particular case, the optimal filter $f_{clrivc}$ is composed of:

- $f_c(s,\boldsymbol{\rho}) = 1/A(s,\boldsymbol{\rho})$ which is used to generate the time-derivatives,
- $f_d(q^{-1},\boldsymbol{\eta}) = D(q^{-1},\boldsymbol{\eta})/C(q^{-1},\boldsymbol{\eta})$ since the noise model of the assumed CT BJ data generating system is $H_o(s) = C_o(s)/D_o(s)$.

As a result, the proposed CLRIVC algorithm is then based on the following CT hybrid Box-Jenkins model structure

$$\begin{cases} y(t_k) = \dfrac{B(s,\boldsymbol{\rho})}{A(s,\boldsymbol{\rho})}u(t_k) + \dfrac{C(q^{-1},\boldsymbol{\eta})}{D(q^{-1},\boldsymbol{\eta})}e(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s)y(t_k) \end{cases} \tag{1.58}$$

It involves an iterative (or relaxation) algorithm in which, at each iteration, the auxiliary model (see previously Section 1.4.2) used to generate the instrumental variables, as well as the associated prefilters, are updated, based on the parameter estimates obtained at the previous iteration.

### Iterative CLRIVC Algorithm

The outline of the CLRIVC algorithm is as follows:

1. Set $C(q^{-1},\hat{\boldsymbol{\eta}}^0) = D(q^{-1},\hat{\boldsymbol{\eta}}^0) = 1$. Choose an initial CT pre-filter $f_c(s)$ to compute $y^{(i)}_{f_c}(t_k)$, $u^{(i)}_{f_c}(t_k)$ and $r^{(i)}_{f_c}(t_k)$, for $i \leq n_a$.
   From the linear model structure (1.55), generate an initial estimate $\hat{\boldsymbol{\rho}}^0$ of $\boldsymbol{\rho}$ using e.g. the CLSRIVC algorithm (see next section): the corresponding TF is denoted by $G(s,\hat{\boldsymbol{\rho}}^0)$. Use this initial estimate to define the CT pre-filter $f_c(s,\hat{\boldsymbol{\rho}}^0) = 1/A(s,\hat{\boldsymbol{\rho}}^0)$, and set $j = 1$.
2. Iterative estimation.
   for $j = 1 :$ *convergence*
   a) Generate the filtered instrumental variables $\boldsymbol{\zeta}_{f_c}(t_k,\hat{\boldsymbol{\rho}}^{j-1}) = \hat{\boldsymbol{\varphi}}_{f_c}(t_k,\hat{\boldsymbol{\rho}}^{j-1})$ from the estimates of the noise-free input and output variables using the following closed-loop auxiliary models

$$\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = \frac{G(s, \hat{\boldsymbol{\rho}}^{j-1})}{1 + C_c(s)G(s, \hat{\boldsymbol{\rho}}^{j-1})} r_{f_c}(t_k) \tag{1.59}$$

$$\hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = \frac{1}{1 + C_c(s)G(s, \hat{\boldsymbol{\rho}}^{j-1})} r_{f_c}(t_k) \tag{1.60}$$

$$\begin{aligned}\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) =& [-\hat{x}_{f_c}^{(n_a-1)}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \ \cdots \ -\hat{x}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \\ & \hat{\nu}_{f_c}^{(n_b)}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \ \cdots \ \hat{\nu}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1})]^T\end{aligned} \tag{1.61}$$

where the CT filter is given as

$$f_c(s, \hat{\boldsymbol{\rho}}^{j-1}) = \frac{1}{A(s, \hat{\boldsymbol{\rho}}^{j-1})}.$$

Use this filter to compute $y_{f_c}^{(i)}(t_k, \hat{\boldsymbol{\rho}}^{j-1})$ and $u_{f_c}^{(i)}(t_k, \hat{\boldsymbol{\rho}}^{j-1})$, for $i \leq n_a$ and update the filtered regression filter

$$\begin{aligned}\boldsymbol{\varphi}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) =& [-y_{f_c}^{(n_a-1)}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \ \cdots \ -y_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \\ & u_{f_c}^{(n_b)}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \ \cdots \ u_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1})]^T\end{aligned} \tag{1.62}$$

b) Obtain an optimal estimate of the noise model parameter vector $\boldsymbol{\eta}^j$ based on the estimated noise sequence

$$\hat{\boldsymbol{\xi}}(t_k) = y(t_k) - \hat{x}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \tag{1.63}$$

using a selected ARMA estimation algorithm and use this to define the corresponding TF: $H(q^{-1}, \hat{\boldsymbol{\eta}}^j)$.

c) Use the estimated noise model parameters in $\hat{\boldsymbol{\eta}}^j$ to define the DT filter $f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j)$, which takes the form

$$f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j) = \frac{D(q^{-1}, \hat{\boldsymbol{\eta}}^j)}{C(q^{-1}, \hat{\boldsymbol{\eta}}^j)}$$

Then, sample the filtered derivative signals at the discrete-time sampling interval $T_s$ and prefilter these by the discrete-time filter $f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j)$.

d) Based on these prefiltered data, generate an updated estimate $\hat{\boldsymbol{\rho}}^j$ of the process model parameter vector as

$$\begin{aligned}\hat{\boldsymbol{\rho}}^j =& \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \boldsymbol{\varphi}_{f_{clrivc}}^T(t_k, \hat{\boldsymbol{\rho}}^{j-1})\right]^{-1} \\ & \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) y_{f_{clrivc}}^{(n_a)}(t_k, \hat{\boldsymbol{\rho}}^{j-1})\right]\end{aligned} \tag{1.64}$$

where

$$\hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j)\hat{\boldsymbol{\varphi}}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}), \tag{1.65}$$

$$\boldsymbol{\varphi}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j)\boldsymbol{\varphi}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}), \tag{1.66}$$

$$y^{(n_a)}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = f_d(q^{-1}, \hat{\boldsymbol{\eta}}^j)y^{(n_a)}_{f_c}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) \tag{1.67}$$

Together with the estimate $\hat{\boldsymbol{\eta}}^j$ of the noise model parameter estimate from step (2b), this provides the estimate $\hat{\boldsymbol{\theta}}^j$ of the composite parameter vector at the $j$th iteration.

3. After the convergence of the iterations is complete, compute the estimated parametric error covariance matrix $\hat{\mathbf{P}}_\rho$, associated with the converged estimate $\hat{\boldsymbol{\rho}}$ of the system model parameter vector, from the expression (see Chapter **??** in this book),

$$\hat{\mathbf{P}}_\rho \;=\; \hat{\sigma}_e^2 \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}})\hat{\boldsymbol{\varphi}}^T_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}})\right]^{-1} \tag{1.68}$$

where $\hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}})$ is the IV vector obtained at convergence and $\hat{\sigma}_e^2$ is the estimated residual variance.

### Simplified Iterative CLSRIVC Algorithm

It will be noted that the above formulation of the CLRIVC estimation problem is considerably simplified if it is assumed in the CT BJ model structure that the additive noise is white, i.e. $C_o(s) = D_o(s) = 1$. In this case, the assumed model structure is a CT hybrid OE model given as

$$\begin{cases} y(t_k) = \dfrac{B(s, \boldsymbol{\rho})}{A(s, \boldsymbol{\rho})}u(t_k) + e(t_k) \\ \text{with } u(t_k) = r(t_k) - C_c(s)y(t_k) \end{cases} \tag{1.69}$$

The simplified CLRIVC (denoted as CLSRIVC) algorithm may be used here; the estimation only involves the parameters in the $A(s, \boldsymbol{\rho})$ and $B(s, \boldsymbol{\rho})$ polynomials and the optimal filter $f_{clsrivc}$ involves:

- the CT filter $f_s(s, \boldsymbol{\rho}) = 1/A(s, \boldsymbol{\rho})$,
- the DT filter $f_d(q^{-1}, \boldsymbol{\eta}) = 1$ since the noise model of the associated CT OE data generating system is $H_o(s) = 1$.

Consequently, the main steps in the CLSRIVC algorithm are the same as those in the CLRIVC algorithm, except that the noise model estimation and subsequent discrete-time prefiltering in steps (2b) and (2c) of the iterative procedure are no longer required and are omitted.

**Comments**

1. Note that the IV vector used in (1.64) should be written as
$$\hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}) = \hat{\boldsymbol{\varphi}}_{f_{clrivc}}(t_k, \hat{\boldsymbol{\rho}}^{j-1}, \hat{\boldsymbol{\eta}}^{j}) \tag{1.70}$$
because the instrumental variables are prefiltered and therefore are a function of both the system parameter estimates at the previous iteration and the most recent noise model parameter estimates (see algorithm). For simplicity, however, these additional arguments are omitted in the algorithm.
2. The fact that the ARMA noise model estimation is carried out separately on the basis of the estimated noise signal $\hat{\xi}(t_k)$ obtained from the IV part of the estimation algorithm in (1.63), implies that the system and noise model parameters are statistically independent (see Chapter **??** for a thorough analysis).
3. The initial selection of $A(s, \hat{\boldsymbol{\rho}}^0)$ does not have to be particularly accurate provided the prefilter $f_c(s, \hat{\boldsymbol{\rho}}^0)$ based on it does not seriously attenuate any signals within the pass-band of the system being modelled (see Chapter **??**).
4. These bootstrap algorithms (CLIVC2, CLIVC4, CLRIVC, CLSRIVC) require knowledge of the controller. However, when it is unknown, another solution may be used to build up the instrumental vector which satisfies the optimal conditions (1.27). Indeed, the noise-free estimation of this instrumental vector can be achieved by using the two closed-loop transfers between $r(t_k), u(t_k)$ and between $r(t_k), y(t_k)$ instead of the open-loop one (between $u(t_k)$ and $y(t_k)$). The second step consists then in identifying the two closed-loop transfers $G_{yr}(s, \boldsymbol{\rho})$ and $G_{ur}(s, \boldsymbol{\rho})$ to compute the instruments as
$$\begin{aligned}\hat{x}(t_k, \hat{\boldsymbol{\rho}}) &= G_{yr}(s, \hat{\boldsymbol{\rho}}) r(t_k)\\ \hat{\nu}(t_k, \hat{\boldsymbol{\rho}}) &= G_{ur}(s, \hat{\boldsymbol{\rho}}) r(t_k)\end{aligned} \tag{1.71}$$
5. Another solution is to estimate the closed-loop TF $G_{ur}(s, \hat{\boldsymbol{\rho}})$ by SRIVC or RIVC and then this can be used to obtain an estimate of the noise-free input for use as the input in the direct RIVC estimation of the process TF. This solution is close to the two-step method [19]; it is not optimal but yields good results with reasonable, albeit not minimum, variance parameter estimates [24].

## 1.6 Summary

The theoretical optimal choices for the design variables of the two multi-step and two iterative algorithms for complete CT modelling are summarized in Table 1.1, while the CT and DT filter forms required, for implementation, in each optimal IV version for CT hybrid modelling are given in Table 1.2.

**Table 1.1.** Optimal choices for the design variables of the proposed IV methods for complete CT modelling

| Method | Assumed filtered data generating CT system (1.72) | Model structure | Shaping noise model $f(s)H_o(s)$ | $f^{opt}(s) = f_c(s)f_d(s)$ see (1.27) $f_c(s)$ | $f_d(s)$ |
|---|---|---|---|---|---|
| CLIVC2 | $y_f(t_k) = \frac{B_o(s)}{A_o(s)}u_f(t_k) + \frac{f(s)}{A_o(s)}e_o(t_k)$ | CT ARX | $\frac{f(s)}{A_o(s)}$ | $f(s)$ | 1 |
| CLIVC4 | $y_f(t_k) = \frac{B_o(s)}{A_o(s)}u_f(t_k) + \frac{f(s)}{A_o(s)D_o(s)}e_o(t_k)$ | CT ARARX | $\frac{f(s)}{A_o(s)D_o(s)}$ | $f(s)$ | $D_o(s)$ |
| CLSRIVC | $y_f(t_k) = \frac{B_o(s)}{A_o(s)}u_f(t_k) + e_o(t_k)$ | CT OE | 1 | $\frac{1}{A_o(s)}$ | 1 |
| CLRIVC | $y_f(t_k) = \frac{B_o(s)}{A_o(s)}u_f(t_k) + \frac{C_o(s)}{D_o(s)}e_o(t_k)$ | CT BJ | $\frac{C_o(s)}{D_o(s)}$ | $\frac{1}{A_o(s)}$ | $\frac{D_o(s)}{C_o(s)}$ |

It will be noticed that, in Table 1.1, the assumed filtered data generating CT system is given as

$$\begin{cases} y_f(t_k) = G_o(s)u_f(t_k) + f(s)H_o(s)e_o(t_k) \\ u_f(t_k) = r_f(t_k) - C_c(s)y_f(t_k) \end{cases} \tag{1.72}$$

**Table 1.2.** Implemented filter forms in the multi-step and iterative IV methods for CT hybrid modelling

| Method | Model structure | $f_c(s)$ | $f_d(q^{-1})$ |
|---|---|---|---|
| CLIVC2 | CT ARX | $\left(\frac{\lambda}{s+\lambda}\right)^{n_a}$ | 1 |
| CLIVC4 | CT hybrid ARARX | $\left(\frac{\lambda}{s+\lambda}\right)^{n_a}$ | AR($2n_a$) |
| CLSRIVC | CT hybrid OE | $\frac{1}{A(s,\hat{\boldsymbol{\rho}})}$ | 1 |
| CLRIVC | CT hybrid BJ | $\frac{1}{A(s,\hat{\boldsymbol{\rho}})}$ | ARMA($n_d, n_c$) |

## 1.7 Numerical Examples

The following numerical example is used to illustrate and compare the performances of the proposed approaches. The process to be identified is described by (1.1), where

$$G_o(s) = \frac{s+1}{s^2+0.5s+1}, \tag{1.73}$$

$$C_c(s) = \frac{10s+15}{s}. \tag{1.74}$$

An external signal is added to $r_1(t_k)$ (see Figure 1.1) and chosen to be a pseudo-random binary signal of maximum length generated from a shift register with 4 stages and a clock period of 500 ($N = 7500$ data points). The sampling period $T_s$ is chosen equal to 5 ms.
From the comparative studies presented recently in [4], the state-variable filter (SVF) approach can be considered as one of the simplest methods to handle the time-derivative problem. This latter approach has been used here with the basic (CLIVC) and multi-step estimators (CLIVC2, CLIVC4). It is not required in the case of CLRIVC because the continuous-time part of the optimal hybrid prefilter is used to generate the filtered derivatives.

### 1.7.1 Example 1: White Noise

Firstly, a Gaussian white noise disturbance ($H_o(q^{-1}) = 1$) is considered in order to illustrate the performance of the CLIVC, CLIVC2 and CLSRIVC algorithms. The process model parameters are estimated on the basis of closed-loop data sequences. A Monte Carlo simulation of 100 experiments is performed for a signal-to-noise (SNR) ratio given as

$$\text{SNR} = 10 \log \left( \frac{P_x}{P_e} \right) = 15 \quad \text{dB}, \tag{1.75}$$

where $P_e$ represents the average power of the zero-mean additive noise on the system output (e.g. the variance) while $P_x$ denotes the average power of the noise-free output fluctuations.
The Monte Carlo simulation (MCS) results are presented in Table 1.3 where the mean and standard deviation of the estimated parameters are displayed. It can be seen that the three IV methods deliver similar unbiased estimates of the model parameters with reasonable standard deviations. However, as expected, note that the basic CLIVC estimates are not as statistically efficient as the CLIVC2 and CLSRIVC estimates, where the standard deviations are smaller and, in the case of $b_0$, the standard deviation is some 7 times smaller.
Furthermore, the 2-norm of the difference between the true ($G(e^{i\omega}, \boldsymbol{\rho}_o)$) and estimated ($G(e^{i\omega}, \hat{\boldsymbol{\rho}}_j)$) transfer functions is also computed for each method

$$\text{Norm} = \frac{1}{N_{exp}} \sum_{j=1}^{N_{exp}} \int |G(e^{i\omega}, \boldsymbol{\rho}_o) - G(e^{i\omega}, \hat{\boldsymbol{\rho}}_j))|^2 d\omega \tag{1.76}$$

where $N_{exp}$ is the number of Monte Carlo simulation runs. The results are given in Table 1.3 and confirm the previous results: the three IV methods lead to accurate results; moreover, the bootstrap methods provide slightly better results than the basic IV technique.

### 1.7.2 Example 2: Colored Noise

A second example is used to analyse the performance of the proposed methods in the case of a colored noise, with

**Table 1.3.** Mean and standard deviations of the open-loop parameter estimates for 100 Monte Carlo runs - White Noise

| Method | $\hat{b}_0 \pm \sigma_{\hat{b}_0}$ | $\hat{b}_1 \pm \sigma_{\hat{b}_1}$ | $\hat{a}_1 \pm \sigma_{\hat{a}_1}$ | $\hat{a}_2 \pm \sigma_{\hat{a}_2}$ | Norm |
|---|---|---|---|---|---|
| True value | 1 | 1 | 0.5 | 1 | |
| CLIVC | 0.995 ± 0.036 | 1.050 ± 0.047 | 0.536 ± 0.021 | 1.015 ± 0.030 | 0.936 |
| CLIVC2 | 0.994 ± 0.005 | 1.018 ± 0.046 | 0.520 ± 0.021 | 1.012 ± 0.028 | 0.875 |
| CLSRIVC | 0.995 ± 0.003 | 0.990 ± 0.050 | 0.518 ± 0.020 | 1.013 ± 0.030 | 0.910 |

$$H(q^{-1}, \boldsymbol{\eta}_o) = \frac{1 - 0.98q^{-1}}{1 - 1.9747q^{-1} + 0.9753q^{-2}}$$

The process parameters are estimated on the basis of closed-loop data sequences described previously. A Monte Carlo simulation of 100 experiments is performed for a SNR = 15dB. The first 100 points of the external signal are forced to zeros in order to be free of the prefiltering initial conditions. The external signal, input and output data are plotted in Figure 1.2. The process model parameters are estimated by using methods CLIVC, CLIVC4, CLRIVC. Moreover, the direct closed-loop approach (see [19]) is used as well in this example, in order to illustrate the difficulties of identifying a closed-loop model in a colored noise situation and to see how much bias is introduced into the parameter estimates in this direct approach, when the closed-loop operation is not really taken into account. The open-loop SRIVC algorithm (see Chapter **??**) is used for this purpose.

The mean and standard deviation of the 100 sets of estimated model parameters from the MCS analysis are given in Table 1.4. The Bode diagrams of the 100 identified models are displayed in Figures 1.3 to 1.6. As expected, the direct closed-loop approach using the open-loop SRIVC method clearly leads to biased results: however, it will be noticed that, although the SRIVC estimates are biased, the inherent pre-filtering introduced by CT estimation allows us to obtain better results than those obtained from indirect DT estimation. Furthermore, the three closed-loop IV methods provide similar unbiased estimates of the model parameters with reasonable standard deviations. However, again as expected, the CLIVC estimates are not as statistically efficient as the estimates produced by the multi-step CLIVC4 and iterative CLRIVC algorithms, where the standard deviation are always smaller, thanks to the prefiltering and associated noise model estimation. Furthermore, thanks to its iterative structure and its prefilter updating operation, the CLRIVC algorithm leads to better results than the CLIVC4 method. Moreover, it is interesting to note that, from our experience, the basic CLIVC method provides better results than the DT version (using the sampled external signal), thanks to the inherent CT prefiltering (see [7]).

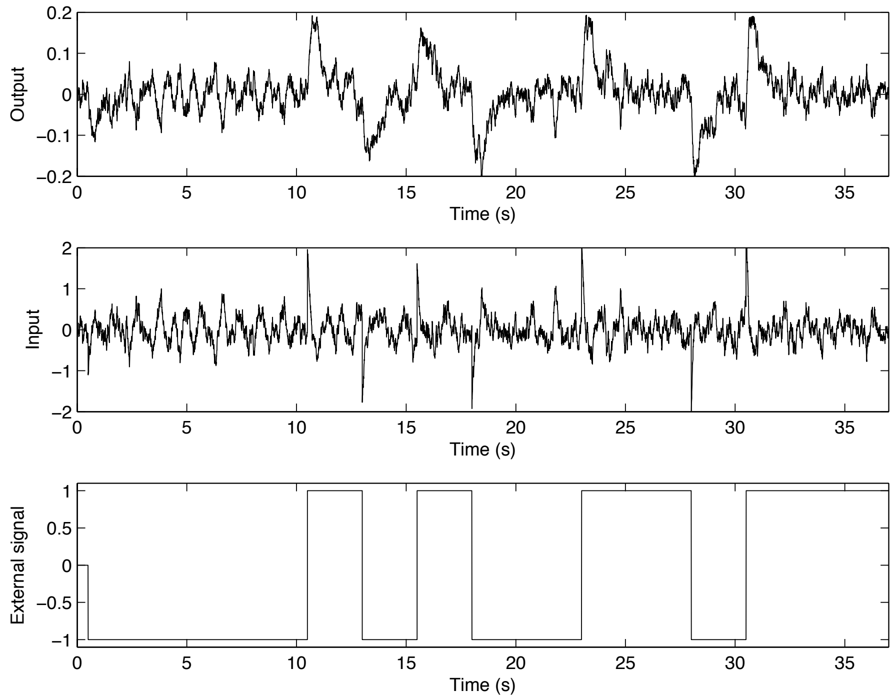


**Fig. 1.2.** Closed-loop data used in Example 2 - Colored noise

**Table 1.4.** Mean and standard deviations of the open-loop parameter estimates for 100 Monte Carlo runs - Colored noise

| Method | $\hat{b}_0 \pm \sigma_{\hat{b}_0}$ | $\hat{b}_1 \pm \sigma_{\hat{b}_1}$ | $\hat{a}_1 \pm \sigma_{\hat{a}_1}$ | $\hat{a}_2 \pm \sigma_{\hat{a}_2}$ | Norm |
|---|---|---|---|---|---|
| True value | 1 | 1 | 0.5 | 1 | |
| SRIVC | 0.449 ± 0.050 | 1.254 ± 0.247 | 0.636 ± 0.140 | 1.354 ± 0.156 | 0.855 |
| CLIVC | 1.011 ± 0.278 | 0.812 ± 0.299 | 0.546 ± 0.099 | 0.963 ± 0.231 | 0.784 |
| CLIVC4 | 0.960 ± 0.131 | 0.977 ± 0.240 | 0.563 ± 0.104 | 1.015 ± 0.119 | 0.767 |
| CLRIVC | 0.972 ± 0.112 | 0.973 ± 0.191 | 0.557 ± 0.083 | 1.007 ± 0.094 | 0.779 |

## 1.8 Conclusions

This chapter has addressed the problem of estimating the parameters of continuous-time transfer functions models for linear dynamic systems operating in closed-loop using instrumental variable techniques. Several closed-loop IV estimators have been described, including the development of explicit expressions for the parametric error covariance matrix. In particular, the chapter has shown that reduced values of this covariance matrix can be achieved for

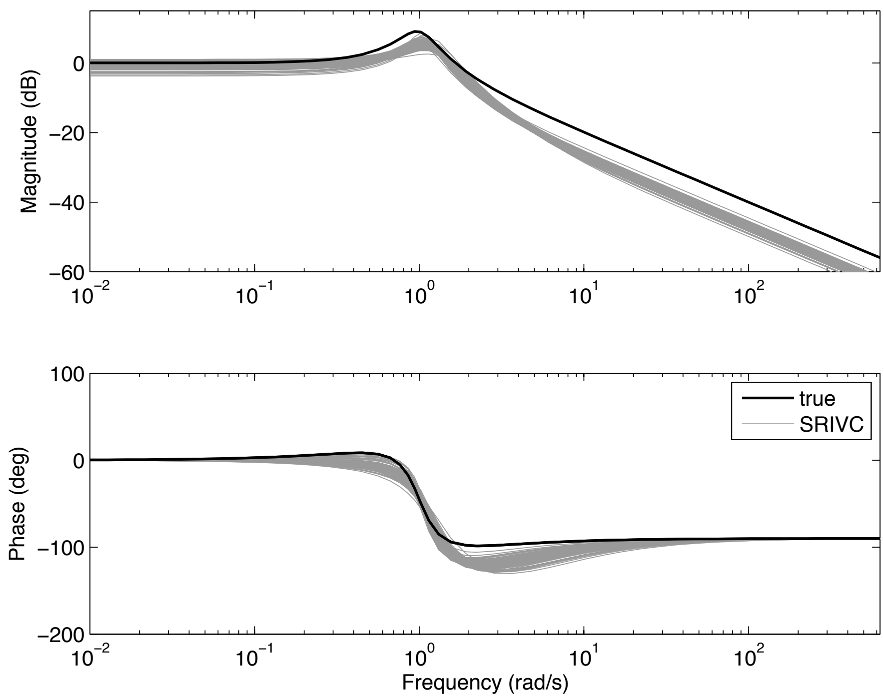


**Fig. 1.3.** Bode plots of the 100 identified SRIVC models - Colored noise

a particular choice of instruments and prefilter; and both multi-step and iterative solutions have been developed to determine the design parameters that allow for such improved closed-loop IV estimation.

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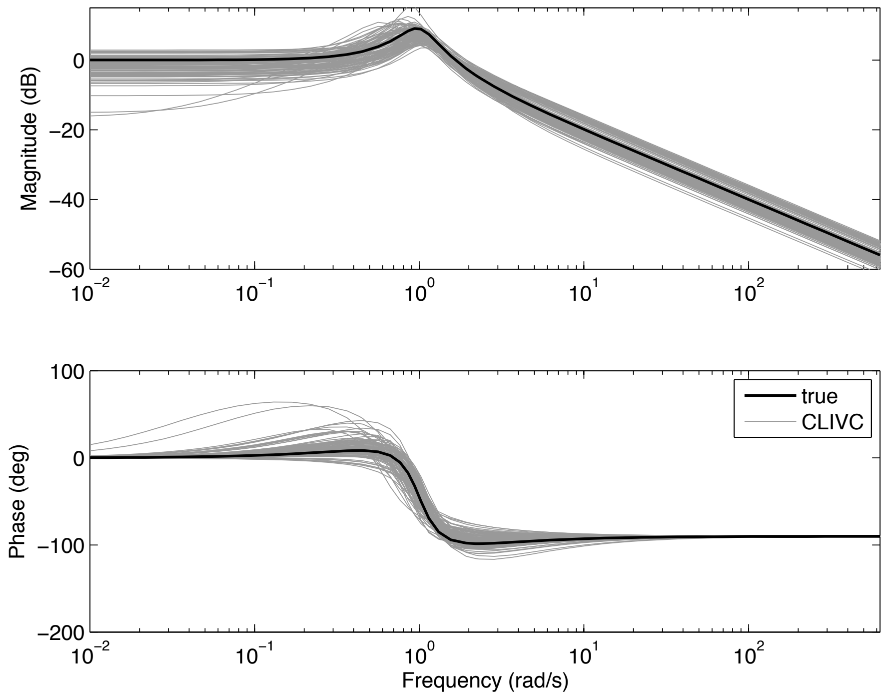


**Fig. 1.4.** Bode plots of the 100 identified CLIVC models - Colored noise

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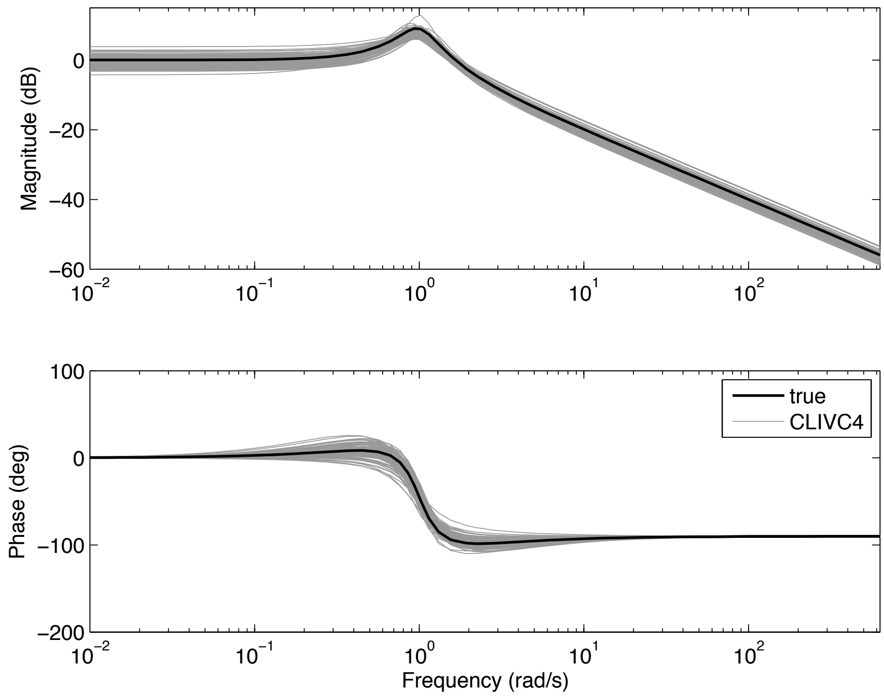


**Fig. 1.5.** Bode plots of the 100 identified CLIVC4 models - Colored noise

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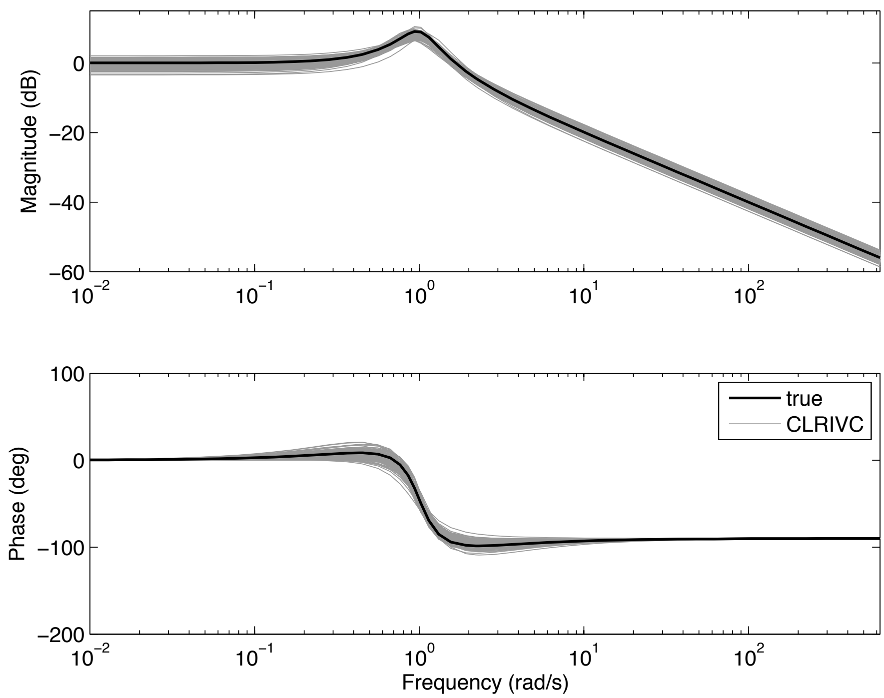


**Fig. 1.6.** Bode plots of the 100 identified CLRIVC models - Colored noise

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