

Frequency domain curve fitting with maximum amplitude criterion and guaranteed stability

RICHARD G. HAKVOORT† and PAUL M. J. VAN DEN HOF†

The problem is considered of fitting a stable rational transfer function to complex frequency response data minimizing a weighted maximum amplitude criterion. A solution for this problem is provided in both discrete and continuous time and for both SISO and MIMO systems by parametrizing the transfer function and solving a smooth nonlinear constrained optimization problem. The denominator is parametrized as a product of first and second-order polynomials. By adding linear constraints on the denominator parameters to the nonlinear programming problem the poles of the curve fit model are restricted to the stability region in the complex plane. To provide an initial estimate for the nonlinear optimization problem an iterative procedure is proposed where, in each step, a linear programming problem has to be solved.

1. Introduction

The problem is considered of fitting a discrete or continuous time stable rational transfer function of a specified order to a set of complex frequency response data. In the literature, reasonable attention has been paid to curve fit problems, however, without restricting the model set to the set of stable models. Levy (1959) introduced a curve fit problem with a sum of squares (l_2) criterion function and provided a solution for it. Sanathanan and Koerner (1963) proposed an iterative adjustment of the weighting in order to improve on the estimate. The asymptotic behaviour of this and other iterative linear schemes has been investigated by Whitfield (1987). Payne (1970) already noticed that if high order models are used in the curve fit procedure, the resulting estimate tends to be unstable.

Alternative criteria (other than l_2) were used by Spanos (1991) and Sidman *et al.* (1991). In the first paper, an iteration of weighted least squares problems is used to solve an unweighted least maximum amplitude (l_∞) optimization problem. In the latter paper the l_2 -criterion is adjusted such that it is well suited for logarithmically spaced frequency response data.

In the present paper a weighted l_∞ -criterion is considered. An advantage of an l_∞ -criterion, compared with an l_2 -criterion, is that in the l_∞ -case each frequency is basically equally important. The frequency distribution (linearly or logarithmically, etc) does not influence the resulting estimate as much as it does in the l_2 -case. Moreover, the weighting allows for an easy and effective shaping of the model error. A final motivation for a maximum amplitude criterion is that, in many applications, for example robust control design, it is desirable that the maximum (weighted) distance between system and model is minimal, in order to have maximum robustness.

Received 2 April 1993. Revised 26 July 1993.

†Mechanical Engineering Systems and Control Group, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands.

The l_∞ -optimization problem is solved by transforming it into a smooth nonlinear constrained optimization problem, which is a different approach from the one followed by Spanos (1991). The transfer function is guaranteed to be stable by parametrizing the denominator as a product of first and second-order polynomials and adding linear constraints on the denominator parameters to the optimization problem. In fact, in the discrete time case a bound can be specified for the amplitude of the transfer function poles. In the continuous time case, a bound can be specified for the real part of the poles. The nonlinear programming problem can be tackled using standard software, but requires the specification of a good initial estimate. A procedure is presented to arrive at such an initial estimate. It involves a Sanathanan–Koerner iteration in l_∞ -setting where, in each step, one linear programming problem has to be solved.

Extensions of the curve fit procedure to the MIMO case are presented. Frequency domain identification of MIMO systems has also been considered by, for example, Lin and Wu (1982) and Bayard (1992). In the first paper it has been proposed first to identify a set of SISO transfer functions from each input to each output separately, and then to compose the complete MIMO system by means of a minimal realization method. In the latter paper, a scalar denominator polynomial is used for the MIMO system. In the present paper, a different approach is taken by a parametrization with left and right diagonal denominator matrices.

A completely different approach for frequency domain identification is taken in the field of identification of H_∞ , see Helmick *et al.* (1991) for the original problem formulation, and also for example Gu and Khargonekar (1992) and Helmicki *et al.* (1992). There, the primary interest is in finding a stable transfer function with some worst-case optimality properties using frequency response data. For that purpose standard curve fit procedures do not apply as they do not guarantee stability of the resulting estimate. However, the algorithms for identification in H_∞ generally do not restrict the model order, extremely high order nominal models may result. Therefore, most of these algorithms include a model reduction step, see Mäkilä and Partington (1991). The present curve fit procedure seems to be an attractive alternative as stability is guaranteed and the model order is user-defined. Note that the maximum amplitude criterion is closely related to the H_∞ -norm.

The outline of the paper is as follows. In §2 the curve fit problem is formulated and a solution is derived in the form of a nonlinear programming problem. In §3 a procedure is derived to provide an initial estimate for the nonlinear programming problem. Section 4 discusses some practical implementation aspects of the linear and nonlinear programming problems involved. Section 5 extends the entire procedure to the MIMO case. Section 6 contains a simulation example which shows the applicability of the curve fit procedure. Finally, in §7 conclusions are drawn.

The present paper is an extended version of Hakvoort (1993).

2. The curve fit procedure

Consider complex-valued frequency response data $G(\omega_j)$, $j = 1, \dots, l$, and a positive real-valued frequency dependent weight $W(\omega_j)$, $j = 1, \dots, l$. The objective is to find a stable rational transfer function $\hat{G}(\xi)$ in a certain model set \mathcal{M} that optimally describes the frequency response data in a weighted l_∞ -sense

$$\hat{G}(\xi) = \arg \min_{\hat{G} \in \mathcal{M}} \max_{j=1, \dots, l} |(G(\omega_j) - \hat{G}'(\xi(\omega_j)))W(\omega_j)| \quad (1)$$

Following the notation in Bayard (1992) the complex variable ξ is used to present the continuous and discrete time case in a unified way. Hence, ξ can be thought of as the Laplace operator s or the z -transform operator z : and $\xi(\omega)$ represents the evaluation of the complex variable ξ as a function of frequency, hence $\xi(\omega) = i\omega$, $\xi(\omega) = e^{i\omega}$ respectively.

\mathcal{M} is chosen to be the model set consisting of all rational transfer functions with the numerator and denominator of specified degrees and the roots of the denominator in some *a priori* specified (stability) region \mathcal{R} in the complex plane.

$$\mathcal{M}: \hat{G}'(\xi) = \frac{n(\xi)}{d(\xi)}, \quad n(\xi) = \sum_{k=0}^n n_k \xi^k, \quad d(\xi) = \xi^d + \sum_{k=0}^{d-1} d_k \xi^k, \quad \xi_{d,k} \in \mathcal{R},$$

$$k = 1, \dots, d \quad (2)$$

where $\xi_{d,k}$, $k = 1, \dots, d$, are the roots of the polynomial $d(\xi)$, i.e. all solutions to the equation $d(\xi) = 0$. In the discrete time case the region \mathcal{R} is defined by $\mathcal{R} = \mathcal{R}_\rho$ with

$$\mathcal{R}_\rho = \{\xi_{d,k} \text{ subject to } |\xi_{d,k}| \leq \rho, k = 1, \dots, d\}$$

which corresponds to an upper bound ρ on the amplitude of the poles of the transfer function. In the continuous time case this region is defined by $\mathcal{R} = \mathcal{R}_r$ with

$$\mathcal{R}_r = \{\xi_{d,k} \text{ subject to } \text{Re}(\xi_{d,k}) \leq r, k = 1, \dots, d\}$$

which corresponds to an upper bound r on the real part of the poles of the transfer function. Here, ρ and r are user-defined constants. The choice $\rho = 1$ corresponds to stability in the discrete time case. The choice $r = 0$ corresponds to stability in the continuous time case.

In order to cope with the region \mathcal{R} for the roots of the denominator polynomial in a computationally attractive way, an alternative parametrization is introduced for the denominator by representing the polynomial as a product of first and second order polynomials

$$d(\xi) = \begin{cases} \prod_{k=1}^{d/2} (\xi^2 + a_k \xi + b_k), & d \text{ even} \\ (\xi + c) \prod_{k=1}^{(d-1)/2} (\xi^2 + a_k \xi + b_k), & d \text{ odd} \end{cases} \quad (3)$$

where the parameters a_k , b_k and c are real-valued. Then the following Lemma gives necessary and sufficient conditions for the location of the roots of $d(\xi)$ by means of simple linear constraints on the parameters.

Lemma 2.1: *Let $\xi_{d,k}$, $k = 1, \dots, d$, be the roots of $d(\xi)$, i.e. all solutions to $d(\xi) = 0$, then for any $\rho \geq 0$, $r \in \mathbb{R}$*

$$(i) \quad |\xi_{d,k}| \leq \rho, \forall k \Leftrightarrow \begin{cases} b_k \leq \rho^2, & \forall k \\ a_k \rho \leq \rho^2 + b_k, & \forall k \\ -a_k \rho \leq \rho^2 + b_k, & \forall k \\ -\rho \leq c \leq \rho, & \text{if } d \text{ is odd} \end{cases}$$

for nonlinear constrained optimization. In §4 some aspects of the practical implementation will be discussed. At this moment it is only noted that (4) is a smooth nonlinear problem, as the partial derivatives of the constraints and the objective function to the parameters are continuous functions.

3. The initial estimate

The nonlinear constrained optimization problem (4), which provides the solution for the curve fit problem (1), can generally only be solved satisfactorily if a good initial estimate is available. Such an initial estimate should satisfy two properties. First, it should be a feasible solution for the nonlinear optimization problem, which in fact means that its denominator roots should be in the region \mathcal{R} . Second it should be close to the optimal solution in the sense that a low, although not necessarily minimal, criterion value is achieved. To arrive at such an initial estimate a three-step procedure is proposed here. In the first step an iteration is performed in order to obtain a curve fit model which is expected to yield a low criterion value. However, the poles of the model are not restricted to the region \mathcal{R} . In the second step then, the poles are mirrored into the region \mathcal{R} , which ensures feasibility of the model. But this may happen at the price of an increase in criterion value. Therefore, in the third and final step a new optimization is performed with respect to the numerator, the denominator being fixed. This will then yield an initial model which satisfies the constraints in the nonlinear optimization problem (4) and also has a low criterion value h_∞^2 .

The first step consists of an iterative procedure using linear programming techniques. The following model structure is considered,

$$M: \hat{G}'(\xi) = \frac{n(\xi)}{d(\xi)}, \quad n(\xi) = \sum_{k=0}^n n_k \xi^k, \quad d(\xi) = \xi^d + \sum_{k=0}^{d-1} d_k \xi^k$$

hence stability of the model is not required or, more specifically, the location of the denominator roots is not restricted to the region \mathcal{R} . For some positive, real-valued weight $\tilde{W}(\omega_j)$ the following optimization problem is considered

$$\min_{n(\xi), d(\xi)} \max_{j=1, \dots, l} |(G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j)))\tilde{W}(\omega_j)| \quad (5)$$

or equivalently

$$\left. \begin{aligned} \min_{h_\infty, n(\xi), d(\xi)} h_\infty \text{ subject to } |(G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j)))\tilde{W}(\omega_j)| \leq h_\infty \\ j = 1, \dots, l \end{aligned} \right\} \quad (6)$$

Before continuing, an auxiliary lemma is presented.

Lemma 3.1: Consider the function $f_{m'}(x): \mathbb{C} \rightarrow \mathbb{R}$, defined by

$$f_{m'}(x) = \max_{k'=1, \dots, m'} \operatorname{Re}(c_{m', k'} x)$$

with $c_{m', k'} = e^{2\pi(k'/m')i}$, $k' = 1, 2, \dots, m' \geq 3$, where $\operatorname{Re}(\cdot)$ denotes the real part of \cdot , then

$$(i) \quad f_{m'}(x) \leq |x| \leq \frac{f_{m'}(x)}{\cos\left(\frac{\pi}{m'}\right)}$$

$$(ii) \quad \lim_{m' \rightarrow \infty} f_{m'}(x) = |x|$$

Proof: For any x and any $c_{m',k'}$ with $|c_{m',k'}| = 1$, $\operatorname{Re}(c_{m',k'}x) \leq |c_{m',k'}x| \leq |c_{m',k'}||x| = |x|$, which proves the left-hand inequality of (i). Further, for any x there exist an integer l' and

$$\delta \in \left[-\frac{\pi}{m'}, \frac{\pi}{m'} \right)$$

such that $x = |x|e^{(2\pi(l'/m')+\delta)i}$, yielding

$$\begin{aligned} \operatorname{Re}(c_{m',k'}x) &= \operatorname{Re}(e^{2\pi(k'/m')i}|x|e^{(2\pi(l'/m')+\delta)i}) = |x| \operatorname{Re}(e^{(2\pi(k'+l')/m'+\delta)i}) \\ &= |x| \cos\left(2\pi\frac{k'+l'}{m'} + \delta\right) \end{aligned}$$

If $k' = k^*$ is chosen such that $k^* + l' = n'm'$ for some integer n' , this gives

$$\begin{aligned} \operatorname{Re}(c_{m',k^*}x) &= |x| \cos(2\pi n' + \delta) = |x| \cos(\delta) \geq |x| \cos\left(\frac{\pi}{m'}\right) \\ \Leftrightarrow |x| &\leq \frac{\operatorname{Re}(c_{m',k^*}x)}{\cos\left(\frac{\pi}{m'}\right)} \end{aligned}$$

which proves the right-hand inequality of part (i). Finally, part (ii) immediately follows from part (i) for $m' \rightarrow \infty$. \square

Moreover, it is easy to show that the bounds in (i) are tight in the sense that there exists an x such that the lower bound becomes equality, and there is an x such that the upper bound becomes equality. The Lemma, in fact, says that the amplitude of a complex number can be calculated approximately by checking a number of different directions in the complex plane.

Using the $c_{m',k'}$ defined in Lemma 3.1 the optimization problem (6) can be approximated by the optimization problem

$$\begin{aligned} \min_{h_\infty, n(\xi), d(\xi)} h_\infty \text{ subject to } \operatorname{Re}(c_{m',k'}(G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j)))\widetilde{W}(\omega_j)) &\leq h_\infty \\ j &= 1, \dots, l, \quad k' = 1, \dots, m' \end{aligned}$$

which is equivalent to the linear programming problem

$$\begin{aligned} \min_{h_\infty, n_k, d_k} h_\infty \text{ subject to } &\widetilde{W}(\omega_j) \operatorname{Re}(c_{m',k'}G(\omega_j)\xi^d(\omega_j)) \\ &+ \sum_{k=0}^{d-1} d_k \widetilde{W}(\omega_j) \operatorname{Re}(c_{m',k'}G(\omega_j)\xi^k(\omega_j)) \\ &- \sum_{k=0}^n n_k \widetilde{W}(\omega_j) \operatorname{Re}(c_{m',k'}\xi^k(\omega_j)) - h_\infty \leq 0 \\ &j = 1, \dots, l, \quad k' = 1, \dots, m' \end{aligned} \quad (7)$$

The following theorem quantifies the accuracy of the approximation.

Theorem 3.2: Denote the optimal solution of (5) by h'_∞ , $n'(\xi)$, $d'(\xi)$ and the optimal solution of (7) by h''_∞ , $n''(\xi)$, $d''(\xi)$, then

$$(i) \quad h_\infty'' \leq \max_{j=1, \dots, l} |(G(\omega_j)d'(\xi(\omega_j)) - n'(\xi(\omega_j)))\tilde{W}(\omega_j)|$$

$$\leq \max_{j=1, \dots, l} |(G(\omega_j)d''(\xi(\omega_j)) - n''(\xi(\omega_j)))\tilde{W}(\omega_j)| \leq \frac{h_\infty''}{\cos\left(\frac{\pi}{m'}\right)}$$

$$(ii) \quad \lim_{m' \rightarrow \infty} \frac{n''(\xi)}{d''(\xi)} = \frac{n'(\xi)}{d'(\xi)}$$

Proof: The second inequality in part (i) arises from the definition of $n'(\xi)$, $d'(\xi)$. Next, it is noticed that the optimal solution to problem (7) has the property

$$h_\infty'' = \max_{j=1, \dots, l} f_{m'}((G(\omega_j)d''(\xi(\omega_j)) - n''(\xi(\omega_j)))\tilde{W}(\omega_j))$$

which yields the third inequality by applying Lemma 3.1. Finally, optimality of $n''(\xi)$, $d''(\xi)$ implies that for some j'

$$f_{m'}((G(\omega_{j'})d'(\xi(\omega_{j'})) - n'(\xi(\omega_{j'})))\tilde{W}(\omega_{j'})) \geq h_\infty''$$

which yields the first inequality of part (i) by again applying Lemma 3.1. For $m' \rightarrow \infty$ the right-hand side in (i) converges to the left-hand side, which proves part (ii). \square

The optimization problem (7) can be solved exactly using standard linear programming software. See Luenberger (1984) for an extensive treatment of the linear programming problem and algorithms to solve it. According to Theorem 3.2 the solution to the optimization problem (7) is an arbitrarily good approximation to problem (5) for m' sufficiently large.

Now, to obtain a curve fit model with a low criterion value h_∞^2 for the nonlinear problem (4) an iteration is proposed analogously to the Sanathanan-Koerner iteration for l_2 -curve fitting (Sanathanan and Koerner 1963). In each iteration the weight $\tilde{W}(\omega_j)$ is chosen such that

$$\tilde{W}(\omega_j) = W(\omega_j)|d_{\text{prev}}(\xi(\omega_j))|^{-1}, \quad j = 1, \dots, l$$

where $d_{\text{prev}}(\xi)$ denotes the denominator that resulted in the previous iteration. The idea is that the optimization problem (5) more and more resembles the nonlinear problem (1). It is emphasized that, on the one hand, convergence of the iterative procedure is not guaranteed and, on the other, if it converges the ultimate outcome will generally not be equal to the optimal result of the nonlinear problem (1). However, Sanathanan and Koerner (1963) reported good results obtained with this iteration in an l_2 -setting and extensive simulations show that the procedure also works adequately in the l_∞ -setting adopted here, i.e. generally there is rapid convergence and a low value of the nonlinear objective function is often achieved.

As mentioned above, the model resulting from the iteration described above need not have its poles in the region \mathcal{R} , as no constraints of the kind in Lemma 2.1 are imposed on the denominator. Therefore, a second step is performed which consists of mirroring the poles into the region \mathcal{R} . In the discrete time case, the proposed procedure is to replace a denominator root $\xi_{d,k} = a$, with $|a| > \rho$, by a root $\xi_{d,k} = 1/a$ if $1/a \leq \rho$, or by a root $\xi_{d,k} = \rho a/|a|$ if $1/a > \rho$.

Analogously, in the continuous time case, the proposed procedure is to replace a denominator root $\xi_{d,k} = a$, with $\text{Re}(a) > r$, by a root $\xi_{d,k} = -\text{Re}(a) + \text{Im}(a)$ if $-\text{Re}(a) \leq r$, or by a root $\xi_{d,k} = r + \text{Im}(a)$ if $-\text{Re}(a) > r$. An important feature of this procedure is that no new poles will be created which are closer to the unit circle (or imaginary axis) than the poles before mirroring. Hence, no ill-conditioning will occur when the model is evaluated over the unit circle (imaginary axis) in the nonlinear curve fit procedure.

In the second step described above, the poles are put into the region \mathcal{R} , but the achieved low criterion value h_∞^2 may be lost. Therefore, a third step is performed which consists of a final linear optimization. This optimization is basically identical to the optimization (5) of the first step, but now with the denominator $d(\xi)$ fixed to the result of the second step. Only the numerator $n(\xi)$ is adjusted by minimizing the criterion (5) with

$$d(\xi(\omega_j)) = d_{\text{fixed}}(\xi(\omega_j)), \quad \tilde{W}(\omega_j) = W(\omega_j) |d_{\text{fixed}}(\xi(\omega_j))|^{-1}, \quad j = 1, \dots, l$$

Note that in case a fixed denominator $d(\xi)$ is used, or equivalently the denominator degree d is chosen to be zero, then the optimization problem (5) is identical to the optimization problem (1). This, in turn, means that in this case, the solution of the linear programming problem (7) is arbitrarily close to the solution of the nonlinear programming problem (4). The consequence of this third and final step will generally be that a model is obtained which yields a low criterion value h_∞^2 in (4), whereas it is a feasible solution to the constraints in the nonlinear programming problem. Therefore, it can be expected that the resulting model is a suitable initial estimate for the nonlinear optimization problem (4). Extensive simulations confirm this expectation.

4. Implementation aspects

The curve fit problem (1) appears solvable by using linear and nonlinear programming routines. In practice, things like numerical accuracy play an important role. Therefore, some modifications to the programming problems (4) and (7) may be necessary to get well-conditioned optimization problems.

From (4) it is apparent that in the continuous time case, $\xi(\omega_j) = i\omega_j$, the expressions on the left and right-hand side can be very large if $\omega_j > 1$. This may cause numerical problems. The problem is dealt with by introducing a scaling factor. A proper scaling is achieved by multiplying all constraints corresponding to $\omega_j > 1$ in the continuous time case with the factor $f^2(\omega_j)$ where $f(\omega_j) = \omega_j^{-\max(n,d)}$. Hence, one such constraint is given by

$$f^2(\omega_j) W^2(\omega_j) |G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 \leq h_\infty^2 f^2(\omega_j) |d(\xi(\omega_j))|^2$$

Analogously the constraints in the linear programming problem (7) need to be scaled. In the continuous time case for $\omega_j > 1$ all terms appearing in these constraints should be multiplied with the factor $f(\omega_j)$ as given above. In the discrete time case and in the continuous time case for frequencies smaller than or equal to 1 no scaling is required.

A slight improvement of the nonlinear programming problem (4) is obtained by replacing h_∞^2 by a new parameter $h_{\infty,2}$. A consequence of this is that the objective function becomes linear, which is, in general, preferable to a nonlinear objective function.

For the simulations in § 6 the sequential quadratic programming method implemented in the FORTRAN NAG-library has been used, see Gill *et al.* (1981) for details about this method. This method requires the first partial derivatives of the nonlinear constraints. To compute these the fact can be used that for any complex-valued function $y(x_1, \dots, x_n)$ of the real variables x_1, \dots, x_n the first (partial) derivative of the squared magnitude is given by

$$\begin{aligned} \frac{\partial |y(x_1, \dots, x_n)|^2}{\partial x_k} &= \frac{\partial y' y}{\partial x_k} = y' \frac{\partial y}{\partial x_k} + \frac{\partial y'}{\partial x_k} y \\ &= y' \frac{\partial y}{\partial x_k} + \left(y' \frac{\partial y}{\partial x_k} \right)' = 2 \operatorname{Re} \left(y'(x_1, \dots, x_n) \frac{\partial y(x_1, \dots, x_n)}{\partial x_k} \right) \end{aligned}$$

where y' is the complex conjugate of y and $\operatorname{Re}(y)$ is the real part of y . Then the first partial derivatives of the constraints in (4) are straightforwardly computed as

$$\begin{aligned} &\frac{\partial W^2(\omega_j) |G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 - h_{\infty,2} |d(\xi(\omega_j))|^2}{\partial h_{\infty,2}} = -|d(\xi(\omega_j))|^2 \\ &\frac{\partial W^2(\omega_j) |G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 - h_{\infty,2} |d(\xi(\omega_j))|^2}{\partial n_k} \\ &\quad = -2W^2(\omega_j) \operatorname{Re} \left((G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' \frac{\partial n(\xi(\omega_j))}{\partial n_k} \right) \\ &\quad = -2W^2(\omega_j) \operatorname{Re} \left((G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' \xi^k(\omega_j) \right) \\ &\frac{\partial W^2(\omega_j) |G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 - h_{\infty,2} |d(\xi(\omega_j))|^2}{\partial a_k} \\ &= 2 \operatorname{Re} \left((W^2(\omega_j) (G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) - h_{\infty,2} d'(\xi(\omega_j))) \frac{\partial d(\xi(\omega_j))}{\partial a_k} \right) \\ &= 2 \operatorname{Re} \left((W^2(\omega_j) (G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) \right. \\ &\quad \left. - h_{\infty,2} d'(\xi(\omega_j))) \frac{d(\xi(\omega_j)) \xi(\omega_j)}{\xi^2(\omega_j) + a_k \xi(\omega_j) + b_k} \right) \\ &\frac{\partial W^2(\omega_j) |G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 - h_{\infty,2} |d(\xi(\omega_j))|^2}{\partial b_k} \\ &= 2 \operatorname{Re} \left((W^2(\omega_j) (G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) - h_{\infty,2} d'(\xi(\omega_j))) \frac{\partial d(\xi(\omega_j))}{\partial b_k} \right) \\ &= 2 \operatorname{Re} \left((W^2(\omega_j) (G(\omega_j) d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) \right. \\ &\quad \left. - h_{\infty,2} d'(\xi(\omega_j))) \frac{d(\xi(\omega_j))}{\xi^2(\omega_j) + a_k \xi(\omega_j) + b_k} \right) \end{aligned}$$

$$\begin{aligned}
& \frac{\partial W^2(\omega_j) |G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j))|^2 - h_{\infty,2} |d(\xi(\omega_j))|^2}{\partial c} \\
&= 2 \operatorname{Re} \left((W^2(\omega_j)(G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) - h_{\infty,2} d'(\xi(\omega_j))) \frac{\partial d(\xi(\omega_j))}{\partial c} \right) \\
&= 2 \operatorname{Re} \left((W^2(\omega_j)(G(\omega_j)d(\xi(\omega_j)) - n(\xi(\omega_j)))' G(\omega_j) \right. \\
&\quad \left. - h_{\infty,2} d'(\xi(\omega_j)) \frac{d(\xi(\omega_j))}{\xi(\omega_j) + c} \right)
\end{aligned}$$

If a scaling $f(\omega_j)$ is used the expressions should, of course, be changed accordingly.

Finally it is noticed that the parametrization (3) is not unique as there is freedom in the ordering of the second-order terms. It depends on the specific nonlinear programming routine that is used whether this non-uniqueness in the solution space will cause trouble. The routine applied for the simulations in § 6 has no difficulties with it. If a routine is used that has difficulties with the non-uniqueness, uniqueness may be enforced by specifying additional constraints which correspond to ordering the second-order terms uniquely. These constraints may, for example, be the following linear constraints,

$$a_1 > a_2 > \dots > a_p, \quad p = d/2 \text{ or } p = (d - 1)/2$$

yielding a unique parametrization which, however, excludes the possibility of identical a -parameters in two second-order polynomials.

5. The multivariable case

In this section, extensions of the curve fit procedure to the multivariable case are presented.

5.1. The curve fit procedure

Consider complex-valued frequency response data $G(\omega_j)$, $j = 1, \dots, l$, where $G(\omega_j)$ is an $m \times n$ matrix, and the real-valued $m \times n$ weighting matrix $W(\omega_j)$, $j = 1, \dots, l$. The objective is to find a stable rational transfer function $\hat{G}(\xi)$ in a certain model set \mathcal{M} that optimally describes the frequency response data in a weighted l_∞ -sense

$$\hat{G}(\xi) = \arg \min_{\hat{G} \in \mathcal{M}} \max_{i', j', j} |(G_{i', j'}(\omega_j) - \hat{G}_{i', j'}(\xi(\omega_j))) W_{i', j'}(\omega_j)| \quad (8)$$

For the model set \mathcal{M} a parametrization is chosen which is highly flexible but still easy to handle

$$\begin{aligned}
\mathcal{M}: \hat{G}'(\xi) &= D_L^{-1}(\xi) N(\xi) D_R^{-1}(\xi) \\
N(\xi) &= \begin{bmatrix} N_{11}(\xi) & \dots & N_{1n}(\xi) \\ \vdots & \dots & \vdots \\ N_{m1}(\xi) & \dots & N_{mn}(\xi) \end{bmatrix}, \quad N_{i', j'}(\xi) = \sum_{k=0}^{n_{i', j'}} n_{i', j'}(k) \xi^k
\end{aligned}$$

$$\begin{aligned}
 D_L(\xi) &= \text{diag} \{ D_{L1}(\xi), \dots, D_{Lm}(\xi) \} \\
 D_{Lj'}(\xi) &= \xi^{d_{Lj'}} + \sum_{k=0}^{d_{Lj'}-1} d_{Lj'}(k) \xi^k, \xi_{D_{Lj'},k} \in \mathcal{R} \\
 D_R(\xi) &= \text{diag} \{ D_{R1}(\xi), \dots, D_{Rn}(\xi) \} \\
 D_{Rj'}(\xi) &= \xi^{d_{Rj'}} + \sum_{k=0}^{d_{Rj'}-1} d_{Rj'}(k) \xi^k, \xi_{D_{Rj'},k} \in \mathcal{R}
 \end{aligned}$$

where $\xi_{D_{Lj'},k}$ and $\xi_{D_{Rj'},k}$ denote the roots of the polynomials $D_{Lj'}(\xi)$ and $D_{Rj'}(\xi)$ respectively. Consequently, the transfer function from input j to output i is given by

$$\hat{G}'_{ij'}(\xi) = \frac{N_{ij'}(\xi)}{D_{Lj'}(\xi) D_{Rj'}(\xi)}$$

The generic (McMillan) order of the transfer function $\hat{G}'(\xi)$ is $\sum_{i'=1}^m d_{Lj'} + \sum_{j'=1}^n d_{Rj'}$. The flexibility of this parametrization lies in the freedom of choosing values for the structure indices $n_{ij'}$, $d_{Lj'}$, $d_{Rj'}$. This is in contrast with, for example, Bayard (1992) where a parametrization has been chosen with a scalar denominator polynomial, i.e. $D_L(\xi) = \text{diag}(d(\xi), \dots, d(\xi))$, $D_R(\xi) = I$. The consequence of that choice is that the generic McMillan order of the MIMO system is $\min(m, n) \deg(d(\xi))$. With such a parametrization, in general, nominal models are estimated which have a (much) higher order than the underlying system. This becomes clear from a simple example.

Example 5.1: Consider a multivariable system with two inputs and two outputs, where each $G'_{ij'}(z)$ is a first-order transfer function, different from the other transfer functions (no common poles). Hence the order of the MIMO system is 4. If a parametrization with a scalar denominator polynomial is used to identify the system, a denominator polynomial of degree 4 is required (in the case of non-approximate identification). However, because of round-off errors and noise the resulting identified nominal model will, in general, have order $2 \times 4 = 8$! If the parametrization proposed above with diagonal left and right denominator matrices is used, it is, for example, possible to choose $d_{L1} = d_{L2} = d_{R1} = d_{R2} = 1$ or $d_{L1} = d_{L2} = 2$, $d_{R1} = d_{R2} = 0$ or $d_{L1} = d_{L2} = 0$, $d_{R1} = d_{R2} = 2$, each of these cases leading to fourth-order nominal models being estimated. \square

Of course, there are examples in which the parametrization proposed here also requires a higher order nominal to be estimated, but due to the higher flexibility in the parametrization this redundancy is less than in the case of a scalar denominator polynomial.

To determine appropriate values for $d_{Lj'}$ and $d_{Rj'}$, in practice, it is suggested first to fit SISO transfer functions to each set of frequency response data $G'_{ij'}(\omega_j)$, $j = 1, \dots, l$, using the techniques described in the previous sections. Now poles that are very close to each other (according to the user's judgement) can be assumed to be identical. Values for $d_{Lj'}$ and $d_{Rj'}$ can next be determined by observing if the poles repeat in the same row (output) or column (input) respectively. Next the MIMO curve fit problem can be solved using the techniques described below. An example of this procedure is given in § 6.

By choosing the alternative parametrization for the denominator

$$D_{L_{i'}}(\xi) = \begin{cases} \prod_{k=1}^{d_{L_{i'}/2}} (\xi^2 + a_{L_{i'}}(k)\xi + b_{L_{i'}}(k)), & d_{L_{i'}} \text{ even} \\ (\xi + c_{L_{i'}}) \prod_{k=1}^{(d_{L_{i'}}-1)/2} (\xi^2 + a_{L_{i'}}(k)\xi + b_{L_{i'}}(k)), & d_{L_{i'}} \text{ odd} \end{cases}$$

$$D_{R_{j'}}(\xi) = \begin{cases} \prod_{k=1}^{d_{R_{j'}/2}} (\xi^2 + a_{R_{j'}}(k)\xi + b_{R_{j'}}(k)), & d_{R_{j'}} \text{ even} \\ (\xi + c_{R_{j'}}) \prod_{k=1}^{(d_{R_{j'}}-1)/2} (\xi^2 + a_{R_{j'}}(k)\xi + b_{R_{j'}}(k)), & d_{R_{j'}} \text{ odd} \end{cases}$$

the roots of the denominator polynomials can be restricted to the (stability) region \mathcal{R} by means of linear constraints on the parameters $a_{L_{i'}}(k)$, $a_{R_{j'}}(k)$, $b_{L_{i'}}(k)$, $b_{R_{j'}}(k)$, $c_{L_{i'}}$ and $c_{R_{j'}}$ as given in Lemma 2.1, analogously to the SISO case.

Analogously to the derivation of (4) the objective (8) can be written as

$$\left. \begin{aligned} & \min_{\hat{G} \in \mathcal{M}} \max_{i', j', j} |(G_{i'j'}(\omega_j) - \hat{G}_{i'j'}(\xi(\omega_j)))W_{i'j'}(\omega_j)| \\ & = \min_{h_\infty, N, D_L, D_R} h_\infty^2 \text{ subject to} \\ & \left. \begin{aligned} & W_{i'j'}^2(\omega_j) |G_{i'j'}(\omega_j) D_{L_{i'}}(\xi(\omega_j)) D_{R_{j'}}(\xi(\omega_j)) - N_{i'j'}(\xi(\omega_j))|^2 \\ & \leq h_\infty^2 |D_{L_{i'}}(\xi(\omega_j)) D_{R_{j'}}(\xi(\omega_j))|^2, \quad i' = 1, \dots, m, j' = 1, \dots, n, j = 1, \dots, l \\ & \xi_{D_{L_{i'}.k}} \in \mathcal{R}, \quad i' = 1, \dots, m, \xi_{D_{R_{j'}.k}} \in \mathcal{R}, \quad j' = 1, \dots, n \end{aligned} \right\} \quad (9) \end{aligned}$$

which again is a smooth nonlinear constrained optimization problem.

5.2. The initial estimate

To determine the initial estimate required for the nonlinear programming problem, the three-step procedure of §3 is proposed again. In the first step, a different parametrization is used for the denominator

$$D_{L_{i'}}(\xi) = \xi^{d_{L_{i'}}} + \sum_{k=0}^{d_{L_{i'}}-1} d_{L_{i'}}(k)\xi^k, \quad D_{R_{j'}}(\xi) = \xi^{d_{R_{j'}}} + \sum_{k=0}^{d_{R_{j'}}-1} d_{R_{j'}}(k)\xi^k$$

hence without restricting the roots of the denominator to the region \mathcal{R} . Consider two subproblems, the first one given by

$$\min_{N(\xi), D_L(\xi)} \max_{i', j', j} (G_{i'j'}(\omega_j) D_{L_{i'}}(\xi(\omega_j)) F_{j'}(\omega_j) - N_{i'j'}(\xi(\omega_j))) \tilde{W}_{i'j'}(\omega_j) \quad (10)$$

for some weight $\tilde{W}(\omega_j)$ and constant $F_{j'}(\omega_j)$, $j' = 1, \dots, n$, to which will be come back later. The second subproblem is

$$\min_{N(\xi), D_R(\xi)} \max_{i', j', j} (G_{i'j'}(\omega_j) D_{R_{j'}}(\xi(\omega_j)) F_{i'}(\omega_j) - N_{i'j'}(\xi(\omega_j))) \tilde{W}_{i'j'}(\omega_j) \quad (11)$$

for some weight $\tilde{W}(\omega_j)$ and constant $F_{i'}(\omega_j)$, $i' = 1, \dots, m$, to which will be come back later as well. Consider the first subproblem. It is equivalent to the constrained optimization problem

$\min_{h_\infty, N, D_L} h_\infty$ subject to

$$|(G_{i'j'}(\omega_j)D_{L_{i'}}(\xi(\omega_j))F_{j'}(\omega_j) - N_{i'j'}(\xi(\omega_j)))\widetilde{W}_{i'j'}(\omega_j)| \leq h_\infty$$

$$i' = 1, \dots, m, \quad j' = 1, \dots, n, \quad j = 1, \dots, l$$

which can be split up into a set of m independent optimization problems

$$\left. \begin{array}{l} \min_{h_\infty, N_{i'j'}, D_{L_{i'}}} h_\infty \text{ subject to} \\ |(G_{i'j'}(\omega_j)D_{L_{i'}}(\xi(\omega_j))F_{j'}(\omega_j) - N_{i'j'}(\xi(\omega_j)))\widetilde{W}_{i'j'}(\omega_j)| \leq h_\infty \\ j' = 1, \dots, n, \quad j = 1, \dots, l \end{array} \right\} \quad (12)$$

for $i' = 1, \dots, m$. Similar to the SISO case, this optimization problem can be solved within any accuracy desired using linear programming techniques. The details of the linear programming problem are not presented here as they are quite technical though straightforward to derive.

The second subproblem (11) can be tackled similarly to the first (10). In fact, both subproblems are equivalent and can be recast into each other by interchanging the roles of i' and j' . One routine to solve one can also be applied to the other.

A comment is needed on how to choose the constants $F_{i'}(\omega_j)$, $F_{j'}(\omega_j)$ and the weight $\widetilde{W}(\omega_j)$. It is proposed to set these iteratively to

$$F_{i'}(\omega_j) = D_{L_{i'}}^{\text{prev}}(\xi(\omega_j)), \quad F_{j'}(\omega_j) = D_{R_{j'}}^{\text{prev}}(\xi(\omega_j))$$

$$\widetilde{W}_{i'j'}(\omega_j) = W_{i'j'}(\omega_j) |D_{L_{i'}}^{\text{prev}}(\xi(\omega_j))D_{R_{j'}}^{\text{prev}}(\xi(\omega_j))|^{-1}$$

where the superscript *prev* denotes the outcome of a previous iteration. It is then proposed to solve the optimization problems (10) and (11) in turn. Again, the idea is that these optimization problems (10) and (11) more and more resemble the nonlinear problem (9).

The second step of the three-step procedure to arrive at an initial estimate consists of mirroring all roots of the denominators $D_{L_{i'}}$, $i' = 1, \dots, m$, and $D_{R_{j'}}$, $j' = 1, \dots, n$, into the region \mathcal{R} . This is entirely similar to the SISO case, for details one is referred to §3. The model resulting from this step will be a feasible solution to the nonlinear problem (9).

The third step consists of a final linear optimization of the numerator matrix $N(\xi)$, the denominator matrices $D_L(\xi)$ and $D_R(\xi)$ being fixed to the results of the second step. Hence, subproblem (10) is solved with

$$D_{L_{i'}}(\xi(\omega_j)) = D_{L_{i'}}^{\text{fixed}}(\xi(\omega_j)), \quad F_{j'}(\omega_j) = D_{R_{j'}}^{\text{fixed}}(\xi(\omega_j))$$

$$\widetilde{W}_{i'j'}(\omega_j) = W_{i'j'}(\omega_j) |D_{L_{i'}}^{\text{fixed}}(\xi(\omega_j))D_{R_{j'}}^{\text{fixed}}(\xi(\omega_j))|^{-1}$$

Note that, in this case of fixed denominator matrices, the optimization problems (10), (11) and (8) become identical. The model resulting from this third step is expected to be a good initial estimate for the nonlinear optimization problem (9).

5.3. Implementation aspects

Analogously to the SISO case a proper scaling of (12) is obtained by multiplying a constraint corresponding to i' , j' and $\omega_j > 1$ in the continuous

time case with $f_{i'j'}(\omega_j)$ given by

$$f_{i'j'}(\omega_j) = \omega_j^{-\max(n_{i'j'}, d_{L'} + d_{R'})}$$

A proper scaling of (9) is achieved by multiplying the constraint corresponding to i', j' and $\omega_j > 1$ in the continuous time case with $f_{i'j'}^2(\omega_j)$.

In addition to this scaling, it is advisable to scale the frequency response data $G_{i'j'}(\omega_j)$ by means of multiplication with a constant scaling $s_{i'j'}$ such that each transfer function is approximately of the same order of magnitude. This will improve the conditioning of the linear and nonlinear programming problems. The optimization problem itself remains identical if the weighting $W_{i'j'}(\omega_j)$ is adjusted accordingly by dividing it by the constant scaling $s_{i'j'}$. Afterwards, the resulting numerators should be corrected by dividing each $N_{i'j'}(\xi)$ by $s_{i'j'}$.

Also in the multivariable case, it is possible to replace the parameter h_∞^2 in (9) by a new parameter $h_{\infty,2}$.

The computation of the partial derivatives of the constraints to the parameters is completely analogous to the SISO case and is not performed here.

6. Example

For the simulation example consider the fifth order discrete time MIMO system given by

$$G(z) = \begin{bmatrix} z - 0.9 & 0 \\ 0 & z^2 - 0.5z + 0.8 \end{bmatrix}^{-1} \\ \times \begin{bmatrix} 0.1z^2 - 0.2 & 0.1z^2 - 0.2z + 0.2 \\ -0.5z^2 + 0.2z + 1 & z^2 + 0.4z - 0.3 \end{bmatrix} \begin{bmatrix} z - 0.8 & 0 \\ 0 & z - 0.7 \end{bmatrix}^{-1}$$

from which noisy frequency response measurements $G(\omega_j)$ have been taken at 50 frequencies logarithmically distributed between $10^{-1.5}$ and π . At each frequency, the noise was the sum of a real and imaginary part, and these parts were realizations of independent gaussian stochastic processes with zero mean and variance 0.1. The weighting was chosen to be equal to

$$W(\omega_j) = \begin{bmatrix} (e^{i\omega_j} - 0.8)/(e^{i\omega_j} - 0.5) & (e^{i\omega_j} - 0.8)/(e^{i\omega_j} - 0.5) \\ (e^{i\omega_j} - 0.8)/(e^{i\omega_j} - 0.5) & (e^{i\omega_j} - 0.8)/(e^{i\omega_j} - 0.5) \end{bmatrix}$$

The identification procedure was as follows. First SISO curve fitting has been performed on the basis of each of the four frequency responses. An initial estimate has been determined with the three-step procedure of § 3 with $m' = 4$. The iteration started with $d_{\text{prev}}(q)$ set to 1 and was stopped when the maximum parameter change in both numerator and denominator was smaller than 0.05. This never took more than ten iterations. In fact, the second and third step appeared unnecessary as the resulting models were already stable. After this, the nonlinear problem of § 2 was solved with $\rho = 1$. This has been done for different model orders in order to select an appropriate model order. The correct model orders could, in this case, be determined by looking at the Hankel singular values. If the model order had been chosen too high, small Hankel singular values occurred. The resulting estimated model orders were $d_{11} = 2$, $d_{12} = 2$, $d_{21} = 3$, $d_{22} = 3$.

Next, for these models a closer look was taken at the precise location of the denominator roots. It appeared that the transfer functions $\hat{G}_{11}(z)$ and $\hat{G}_{12}(z)$

had a pole at 0.94 in common. The transfer functions $\hat{G}_{21}(z)$ and $\hat{G}_{22}(z)$ had a complex pole pair $0.25 \pm 0.86i$ in common. The transfer functions $\hat{G}_{11}(z)$ and $\hat{G}_{21}(z)$ approximately had a pole at 0.73 in common. Finally, the transfer functions $\hat{G}_{12}(z)$ and $\hat{G}_{22}(z)$ approximately had a pole at 0.51 in common. This resulted in the MIMO structure $d_{L1} = 1, d_{L2} = 1, d_{R1} = 1, d_{R2} = 1$. The numerators were chosen to be fully parametrized, i.e. $n_{11} = 2, n_{12} = 2, n_{21} = 3, n_{22} = 3$. Next the multivariable curve fit problem was solved with the procedure described in § 5. The initial estimate was determined with $m' = 4$ and $D_L^{\text{prev}}(z)$ and D_R^{prev} set to

$$D_L^{\text{prev}}(z) = \begin{bmatrix} z - 0.94 & 0 \\ 0 & z^2 - 0.5z + 0.8021 \end{bmatrix}$$

$$D_R^{\text{prev}}(z) = \begin{bmatrix} z - 0.73 & 0 \\ 0 & z - 0.51 \end{bmatrix}$$

The maximum amplitudes of the weighted fit errors $\max_j |(G_{i'j'}(\omega_j) - \hat{G}_{i'j'}(e^{i\omega_j}))W_{i'j'}(\omega_j)|, i' = 1, 2, j' = 1, 2$, are, for the resulting initial model, given by

$$\begin{bmatrix} 0.3158 & 0.3019 \\ 0.3491 & 0.3305 \end{bmatrix}$$

Finally, the nonlinear problem was solved with $\rho = 1$. The identified fifth-order MIMO model is given by

$$\hat{G}(z) = D_L^{-1}(z)N(z)D_R^{-1}(z)$$

$$D_L(z) = \begin{bmatrix} z - 0.91 & 0 \\ 0 & z^2 - 0.49z + 0.78 \end{bmatrix}, \quad D_R(z) = \begin{bmatrix} z - 0.77 & 0 \\ 0 & z - 0.61 \end{bmatrix}$$

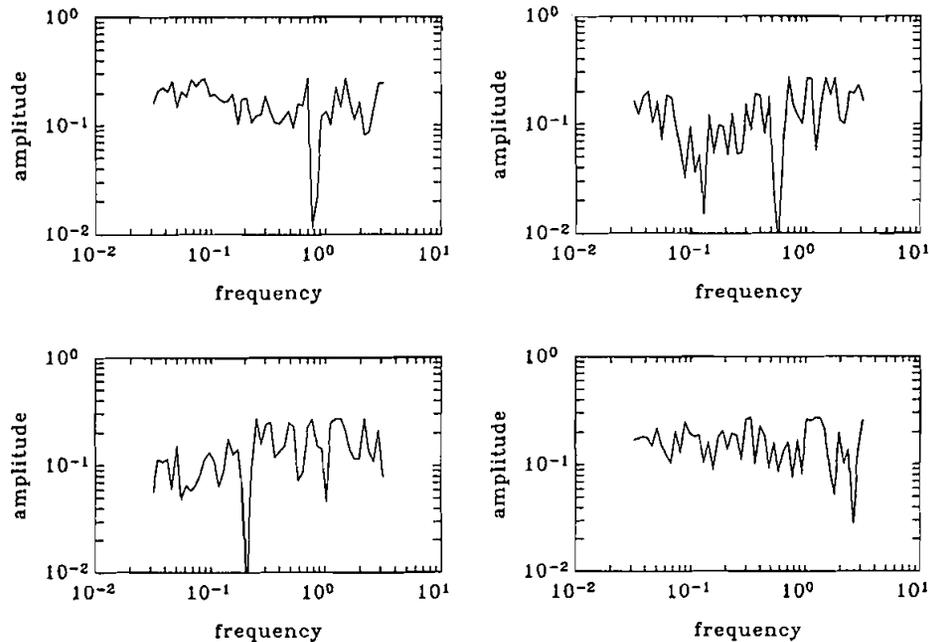
$$N(z) = \begin{bmatrix} 0.070z^2 + 0.086z - 0.25 & 0.11z^2 - 0.22z + 0.24 \\ -0.007z^3 - 0.46z^2 + 0.11z + 1.1 & -0.015z^3 + 1.0z^2 + 0.51z - 0.25 \end{bmatrix}$$

In the Figure an amplitude Bode diagram is presented of the frequency dependent weighted fit error for the optimal model. From this plot it is very clear that the optimization objective has been to minimize the maximum amplitude over all frequencies and for all transfer functions. The maximum amplitudes of the weighted fit errors $\max_j |(G_{i'j'}(\omega_j) - \hat{G}_{i'j'}(e^{i\omega_j}))W_{i'j'}(\omega_j)|, i' = 1, 2, j' = 1, 2$, are for this optimal model given by

$$\begin{bmatrix} 0.2713 & 0.2709 \\ 0.2713 & 0.2709 \end{bmatrix}$$

7. Conclusions

A procedure has been developed to fit a discrete or continuous time rational transfer function to a set of frequency response data minimizing a weighted l_∞ -criterion. The procedure consists of two steps. First, an initial estimate is calculated using linear programming techniques. Then the optimal model is estimated by solving a smooth nonlinear constrained optimization problem. An important feature of the present curve fit procedure is that the resulting nominal model has its poles in a user-defined region in the complex plane. It is, for example, straightforward to restrict the model set to the set of stable models.



Amplitude Bode diagram weighted fit error $(G_{i'j'}(\omega_j) - \hat{G}_{i'j'}(e^{i\omega_j}))W_{i'j'}(\omega_j)$.

The consequence of this then is that the resulting curve fit model will always be stable. The procedure works for SISO as well as MIMO systems. The example in the previous section showed the applicability of the proposed curve fit procedure. The weighting $W(\omega_j)$ can be tuned in order to shape the model error as a function of frequency.

The curve fit procedure can be used to construct a parametric model from frequency response measurements taken from some system. However, this is not the only application that one may think of. Because of the close connection of the l_∞ -norm and the H_∞ -norm the curve fit procedure may, for example, also be applied to perform model reduction in the H_∞ -norm. Moreover, it can also be used to perform a transformation from discrete to continuous time or vice versa in a weighted H_∞ -optimal way. Note that also in the MIMO case, the norm used in the curve fit problem (8) is closely related to the H_∞ -norm as, for any $m \times n$ matrix A , the following bound can easily be established

$$\max_{i,j} |A_{ij}| \leq \|A\|_2 = \sigma_{\max}(A) \leq \sqrt{mn} \max_{i,j} |A_{ij}|$$

The only problem that remains is the intersample behaviour of the curve fit model. As only a finite number of frequencies is used in the curve fit criterion function, the model may theoretically be arbitrarily bad between two subsequent frequencies. In Hakvoort (1992) some conditions are given which guarantee that the intersample behaviour is not arbitrarily bad, but bounded. In practice, it is often a matter of taking enough frequencies compared with the curve fit model order, and afterwards checking if the frequency response of the estimated model is satisfactory for all frequencies.

REFERENCES

- BAYARD, D. S., 1992, Multivariable frequency domain identification via 2-norm minimization. *Proceedings of the American Control Conference*, Chicago, pp. 1253–1257.
- GILL, P. E., MURRAY, W., and WRIGHT, M. H., 1981, *Practical Optimization* (London, U.K.: Academic Press).
- GU, G., and KHARGONEKAR, P. P., 1992, A class of algorithms for identification in H_∞ . *Automatica*, **28**, 299–312.
- HAKVOORT, R. G., 1992, Worst-case system identification in H_∞ : error bounds and optimal models. *Selected Topics in Identification, Modelling and Control*, Vol. 5, edited by O. H. Bosgra and P. M. J. Van den Hof (The Netherlands: Delft University Press), pp. 73–84. Also in Preprints of the Proceedings of the 12th IFAC World Congress, Sydney, Australia, Vol. 8, pp. 161–164 (1993).
- HAKVOORT, R. G., 1993, Frequency domain curve fitting with maximum amplitude criterion and guaranteed stability. *Proceedings of the European Control Conference*, Groningen, The Netherlands, pp. 252–257.
- HELMICKI, A. J., JACOBSON, C. A., and NETT, C. N., 1991, Control-oriented system identification: a worst-case/deterministic approach in H_∞ . *IEEE Transactions on Automatic Control*, **36**, 1163–1176; 1992, Worst-case/deterministic identification in H_∞ : the continuous-time case. *IEEE Transactions on Automatic Control*, **37**, 604–610.
- JACOBS, O. L., 1974, *Introduction to Control Theory* (Oxford, U.K.).
- LEVY, E. C., 1959, Complex curve fitting. *IRE Transactions on Automatic Control*, **4**, 37–43.
- LIN, P. L., and WU, Y. C., 1982, Identification of multi-input multi-output linear systems from frequency response data. *Journal of Dynamic Systems, Measurement and Control*, **104**, 58–64.
- LUENBERGER, D. G., 1984, *Linear and Nonlinear Programming*, second edition (Addison-Wesley).
- MÄKILÄ, P. M., and PARTINGTON, J. R., 1991, Robust approximation and identification in H_∞ . *Proceedings of the American Control Conference*, pp. 70–76.
- PAYNE, P. A., 1970, An improved technique for transfer function synthesis from frequency response data. *IEEE Transactions on Automatic Control*, **15**, 480–483.
- SANATHANAN, C. K., and KOERNER, J., 1963, Transfer function synthesis as a ratio of two complex polynomials. *IEEE Transactions on Automatic Control*, **8**, 56–58.
- SIDMAN, M. D., DEANGELIS, F. E., and VERGHESE, G. C., 1991, Parametric system identification on logarithmic frequency response data. *IEEE Transactions on Automatic Control*, **36**, 1065–1070.
- SPANOS, J. T., 1991, Algorithms for l_2 and l_∞ transfer function curve fitting. *Proceedings of the AIAA Guidance Navigation and Control Conference*, New Orleans, Louisiana, U.S.A.
- WHITFIELD, A. H., 1987, Asymptotic behaviour of transfer function synthesis methods. *International Journal of Control*, **45**, 1083–1092.