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Minimal Partial Realization From Orthonormal Basis Function Expansions

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Abstract

Given an expansion of a dynamical system in terms of a generalized orthonormal (Hambo) basis, the problem of realizing a state-space model of minimal McMillan degree such that its first N expansion coefficients match the given ones is addressed and solved. For the solution use is made of the properties of the Hambo operator transform theory. The resulting realization algorithms can be applied in an exact and approximative sense and can also be applied to solve a related interpolation problem.

1 Introduction

The idea of decomposing a system in terms of basis functions is widely applied in system theory and related problems such as system approximation and identification. It is for instance common to represent a stable discrete-time system $G(z)$ in the form of its Laurent expansion as

$$G(z) = \sum_{k=1}^{\infty} g_k z^{-k}, \tag{1}$$

where the functions $\{z^{-k}\}$ form an orthonormal basis for the space of strictly-proper, stable transfer functions, denoted as H_2. The associated expansion coefficients g_k, also known as the Markov parameters, play an important role in systems theory, realization theory, system approximation and identification. In this context systems are often represented in terms of a finite set of Markov parameters, which is known as FIR (finite impulse response) modeling.

Generalized orthonormal basis constructions have been proposed that offer the flexibility to tune them so as to perform better than FIR models in particular situations. These are expansions of the general form

$$G(z) = \sum_{k=1}^{\infty} c_k f_k(z), \tag{2}$$

where the functions $f_k(z)$ represent general orthonormal basis functions while $c_k \in \mathbb{R}$ are the corresponding expansion coefficients. Examples of such expansions are the well-known Laguerre and two-parameter Kautz basis constructions [10, 8, 18, 19]. Further generalizations were proposed in [4, 6, 11]. Their general form is given by

$$f_k(z) = \frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k} \prod_{i=1}^{k-1} \frac{1-\xi_i^* z}{z-\xi_i} \tag{3}$$

where $\{\xi_i\}$ is a collection of poles to be chosen by the user. The origin of these constructions lies in the work of Takenaka and Malmquist in the 1920s [20]. The functions constitute a complete orthonormal set in H_2 provided that $\sum_{k=1}^{\infty}(1-|\xi_k|) = \infty$. Typically the rate of convergence of the series expansion (2) is higher when the pre-chosen poles ξ_i are closer to the poles of the underlying system. This paper considers the basis construction that was proposed in [6], also denoted as the Hambo basis, which in terms of (3) is equivalent to a finite pole selection $\{\xi_i\}$, $i = 1,..,n_b$ which is repeated periodically, i.e. $\xi_{k+n_b} = \xi_k, \forall k$.

The problem considered is as follows: given a partial expansion $\{\tilde{c}_k\}_{k=1,\cdots N}$, find a minimal state-space realization (A,B,C,D) of a system $G(z)$ of smallest order such that $G(z) = \sum_{k=1}^{\infty} c_k f_k(z)$ and $c_k = \tilde{c}_k$, $k = 1,\cdots N$. This is a generalization of the classical minimal partial realization problem that was solved in [7, 16]. This problem has been addressed for the Laguerre case and the Hambo basis case with full information ($N \to \infty$) in [12, 13, 15]. In order to deal with finite N a different approach has to be followed. It will be shown that a solution for this case can be constructed by exploiting the Hambo transform theory (see [17, 5, 1]).
The presented results will be limited to scalar transfer functions. The generalization to multivariable systems presents no great difficulties. Throughout this paper it will be assumed that all state-space realizations and expansion coefficients are real-valued.
The outline of the paper is as follows. First some preliminaries about the Hambo basis and Hambo transform theory are recalled in section 2. In section 3 the Hankel operator framework is presented in which the realization problem is solved for the case where one has knowledge of the full expansion and for the case where the McMillan degree of the

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system is known. In section 4 this approach is combined with results from Hambo basis theory to derive the main results. In section 5 application of the results in the context of system identification is discussed and illustrated.
For the proofs of all results see [2, 3]. We will use the following notational conventions:

- $L_2^{p\times m}$ Hilbert space of complex matrix functions of dimension $p\times m$ that are square integrable on the unit circle. The superscript is suppressed if $p = m = 1$.
- $H_2^{p\times m}$ Subspace of $L_2^{p\times m}$ of functions analytic outside the unit disc, and zero at infinity.
- RH_2 Subspace of *rational* transfer functions of H_2.
- $\langle X,Y\rangle_M$ Matrix "inner" product between $X \in L_2^{p\times 1}$ and $Y \in L_2^{m\times 1}$ defined as $\langle X,Y\rangle_M = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{i\omega})Y(e^{i\omega})^* d\omega$. For $p = m = 1$ the subscript M will be suppressed.
- $H_2^{\perp}$ The orthogonal complement of H_2 in L_2.
- $H_{2,0}^{p\times m}$ The space of discrete-time, stable, *proper* transfer functions of output/input dimension $p\times m$.
- $RH_{2,0}^{p\times m}$ Subspace of *rational* transfer functions of $H_{2,0}^{p\times m}$.
- $\ell_2^n(J)$ The space of square summable vector sequences, of vector dimension n, where J denotes the index set. The superscript n will be omitted if $n = 1$.
- $\mathbf{e}_i$ i-th canonical Euclidean basis (column) vector.

2 Preliminaries

2.1 Orthogonal basis functions—Hambo basis

One way to construct the Hambo basis functions [5] is by considering a finite set of poles $\{\xi_i\}_{i=1,\cdots n_b}$ that are stable, i.e. $|\xi_i| < 1$, generating an all-pass transfer function

$$G_b(z) = \prod_{i=1}^{n_b} \frac{(1-\xi_i^* z)}{(z-\xi_i)}$$

having a balanced realization (A_b, B_b, C_b, D_b) that satisfies

$$\begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}^T \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} = I. \tag{4}$$

It follows that the input-to-states transfer functions of G_b:

$$\phi_i(z) := \mathbf{e}_i^T (zI - A_b)^{-1} B_b, \quad i = 1,..,n_b,$$

form an orthonormal set. An orthogonal basis for H_2 is created by introducing

$$\phi_{i,k}(z) = \phi_i(z) G_b(z)^{k-1}, \quad k = 1,\cdots\infty.$$

For convenience we also denote: $V_k = [\phi_{1,k}\ \phi_{2,k}\ \cdots\ \phi_{n_b,k}]^T$, leading to $V_1(z) = (zI - A_b)^{-1}B_b$ and vector functions

$$V_k(z) = V_1(z) G_b(z)^{k-1}. \tag{5}$$

Since the functions $\{\phi_{i,k}\}_{i=1,\cdots n_b; k=1,\cdots\infty}$ form an orthonormal basis, any element G of H_2 can be written as

$$G(z) = \sum_{k=1}^{\infty}\sum_{i=1}^{n_b} l_{i,k}\phi_{i,k}(z), \text{ with } l_{i,k} = \left\langle G, \phi_{i,k}\right\rangle. \tag{6}$$

with $l_{i,k}$ the expansion coefficients. Similarly we can write

$$G(z) = \sum_{k=1}^{\infty} L_k^T V_k(z) \tag{7}$$

with $L_k^T := [\ l_{1,k}\ l_{2,k}\ \cdots\ l_{n_b,k}] = \langle G, V_k\rangle_M$.

The function $G_b(z)$ also generates a basis for $H_2^{\perp}$(and thus for the entire L_2 space), by repeatedly multiplying $V_1(z)$ with $G_b(z)^{-1} = G_b(1/z)$, as is shown e.g. in [14]. We will denote the basis functions of $H_2^{\perp}$ in vector form by

$$U_k(1/z) = V_1(z) G_b(1/z)^{k+1} = V_1(z) G_b(z)^{-(k+1)}, \tag{8}$$

for $k \in \mathbb{N}_0$. In the sequel we merely discuss systems in H_2. It is important however that most of the results can be translated to $H_2{}^{\perp}$ by a simple mirror operation.

2.2 Signal and operator transforms

By the isomorphic property of the z-transform there exist equivalent time-domain representations of $\phi_{i,k}$ that form an orthonormal basis of the signal space $\ell_2(\mathbb{N})$. They are written as $\phi_{i,k}(t)$, or $V_k(t)$, where the index $t \in \mathbb{N}$ denotes time. The Hambo signal transform [5] of a signal x in $\ell_2(\mathbb{N})$ is then defined by

$$\mathcal{X}(\lambda) = \sum_{k=1}^{\infty} \mathcal{X}(k)\lambda^{-k}, \quad \text{with } \mathcal{X}(k) = \langle V_k, x\rangle_M$$

and λ a complex indeterminate. This signal transform gives rise to a transform operation on a dynamical system, as formulated next.

Proposition 1 *Suppose that $u \in \ell_2(\mathbb{N})$, $G \in RH_{2,0}$ and let $y(z) = G(z)u(z)$. Define $\mathcal{U}$ and $\mathcal{Y}$ as the Hambo signal transforms of u respectively y. Then,*

$$\mathcal{Y}(m) = \sum_{j=1}^{m} M_{m-j}\,\mathcal{U}(j), \tag{9}$$

with the Markov parameters M_k given by

$$M_k = \left\langle V_1(z) G_b(z)^k, V_1(z) G(z)\right\rangle_M. \tag{10}$$

The resulting dynamical system $\tilde{G} \in RH_{2,0}^{n_b\times n_b}$ determined by

$$\tilde{G}(\lambda) = \sum_{k=0}^{\infty} M_k \lambda^{-k}, \tag{11}$$

is referred to as the Hambo operator transform of G.

This proposition shows that the *Hambo operator transform* of the scalar system G is a causal, linear time-invariant $n_b \times n_b$ system. An important property of this transform is that $\tilde{G}(\lambda)$ and $G(z)$ have the same McMillan degree.
In [2, 3] algorithms are given for directly computing a minimal state-space realization of $\tilde{G}(\lambda)$ on the basis of a minimal state-space realization of $G(z)$ and vice versa.

Remark 2 *It is important to note that the image of $RH_{2,0}$ is only a subspace of $RH_{2,0}^{n_b\times n_b}$, i.e. the latter space contains systems that are not a Hambo transform.*

One can expand any $G(z) \in RH_2$ in terms of the Hambo basis functions as in (7). We now recall from [15, 1] the connection between the coefficient sequence $\{L_k\}$ and the sequence of Markov parameters $\{M_k\}$ of the Hambo operator transform $\tilde{G}(\lambda)$. The relation will prove to be essential for the solution of the generalized realization problems.

Proposition 3 *Let $G \in H_2$ have an expansion as in* (6).
(a) the Markov parameters M_k of the Hambo operator transform $\tilde{G}(\lambda)$ satisfy

$$M_k = \begin{cases} \sum_{i=1}^{n_b} l_{i,k+1}P_i^T + l_{i,k}Q_i^T, & k \geq 1, \\ \sum_{i=1}^{n_b} l_{i,1}P_i^T & k = 0. \end{cases} \tag{12}$$

(b) $\widetilde{zG(z)}(\lambda) = \sum_{k=1}^{\infty} M_k^{\leftarrow}\lambda^{-k}$ *with*

$$M_k^{\leftarrow} = L_{k+1}^T B_b \cdot I + \sum_{i=1}^{n_b}\{L_{k+1}^T A_b\}_i P_i^T + \{L_k^T A_b\}_i Q_i^T, \;\; k \geq 1 \tag{13}$$

where $\{\cdot\}_i$ denotes the i-th element of the corresponding vector. The matrices P_i and Q_i are obtained as the unique solutions to the following Sylvester equations.

$$A_b P_i A_b^T + B_b \mathbf{e}_i^T A_b^T = P_i, \tag{14}$$

$$A_b^T Q_i A_b + C_b^T \mathbf{e}_i^T = Q_i. \tag{15}$$

The expression for $\widetilde{zG(z)}(\lambda)$ will turn out to be useful when constructing the realization algorithm in the sequel. The main implication of part (a) of proposition 3 is that the Markov parameters of $\tilde{G}(\lambda)$ can be derived directly from the expansion coefficients. More precisely, M_k solely depends on the coefficient vectors L_k and L_{k+1}. In the next section this fact is used to solve the realization problem.

3 Realization

The solution to the classical minimal realization problem [7], is based on the representation of a system in Hankel operator form, reflecting the mapping from past input signals $u \in \ell_2(-\infty, 0]$ to future output signals $y \in \ell_2[1,\infty)$. This operator is represented by an infinite Hankel matrix $\mathbf{H}$ that operates on the infinite vectors $\mathbf{u}$ and $\mathbf{y}$, as in:

$$\mathbf{y} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & \cdots \\ g_2 & g_3 & \\ \vdots & & \ddots \end{bmatrix} \begin{bmatrix} u(0) \\ u(-1) \\ \vdots \end{bmatrix} = \mathbf{Hu} \tag{16}$$

The Ho-Kalman realization algorithm employs the property that any full-rank decomposition of $\mathbf{H}$

$$\mathbf{H} = \Gamma\Delta \tag{17}$$

corresponds to a minimal realization (A,B,C). The B and C matrices of this realization are obtained by extracting the first column of Δ and the first row of Γ, while the A matrix is obtained by solving the equation

$$\mathbf{H}^{\leftarrow} = \Gamma A \Delta. \tag{18}$$

Here $\mathbf{H}^{\leftarrow}$ is the Hankel matrix that is obtained by removing the first column of $\mathbf{H}$ and can be viewed as the Hankel matrix associated with the system $zG(z)$. This algorithm yields an exact realization provided that an underlying finite dimensional system exists.

In our situation the problem is to find this system not on the basis of $\{g_k\}$, but by starting with $\{L_k\}$. Thereto we formulate the Hankel operator of the system (16) in terms of a matrix representation that considers the signals to be decomposed in terms of the generalized basis functions. We then define the vectors $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ containing the expansion coefficient sequences according to

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathcal{Y}(1)^T & \mathcal{Y}(2)^T & \cdots \end{bmatrix}^T, \text{ and } \tilde{\mathbf{u}} = \begin{bmatrix} \mathcal{U}(0)^T & \mathcal{U}(-1)^T & \cdots \end{bmatrix}^T \tag{19}$$

Since the coefficients satisfy $\mathcal{Y}(k) = \langle V_k, y\rangle_M$ and $\mathcal{U}(-k) = \langle U_k, u\rangle_M$ one can express the vectors $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ as

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \end{bmatrix} \mathbf{y} = \mathbf{T}_1\mathbf{y}, \text{ and } \tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \end{bmatrix} \mathbf{u} = \mathbf{T}_2\mathbf{u}, \tag{20}$$

where $\mathbf{v}_k$ and $\mathbf{u}_k$ are given by

$$\mathbf{v}_k = \begin{bmatrix} V_k(1) & V_k(2) & \cdots \end{bmatrix}, \text{ and } \mathbf{u}_k = \begin{bmatrix} U_k(0) & U_k(-1) & \cdots \end{bmatrix}. \tag{21}$$

The matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ consist of inverse z-transforms of the orthonormal basis functions $V_k(z)$ and $U_k(z)$. Hence they are unitary (orthogonal) matrices: $\mathbf{T}_1^T\mathbf{T}_1 = \mathbf{T}_2^T\mathbf{T}_2 = I$. From equations (16) and (20) it then follows that one can write

$$\tilde{\mathbf{y}} = \mathbf{T}_1\mathbf{H}\mathbf{T}_2^T\tilde{\mathbf{u}} = \tilde{\mathbf{H}}\tilde{\mathbf{u}}. \tag{22}$$

The matrix $\tilde{\mathbf{H}}$ is the Hankel operator representation associated with expansions of signals in terms of Hambo basis functions. See [2, 3] for the exact formulas of $\mathbf{T}_1$ and $\mathbf{T}_2$.

Proposition 4 *With $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ as defined in equation* (19) *it holds that $\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{u}}$ with $\tilde{\mathbf{H}}$ given by*

$$\tilde{\mathbf{H}} = \begin{bmatrix} M_1 & M_2 & \cdots & \\ M_2 & M_3 & & \\ \vdots & & & \ddots \end{bmatrix}, \tag{23}$$

where M_k are the Markov parameters of the Hambo operator transform of $G(z)$ as defined by equation (10)*.*

It follows that $\tilde{\mathbf{H}}$ coincides with the block Hankel matrix associated with the system $\tilde{G}(\lambda)$, the Hambo operator transform of $G(z)$. This matrix can thus be constructed from the expansion coefficients L_k using the result of proposition 3.

The construction of a minimal realization according to (17) and (18) requires a full rank decomposition of $\mathbf{H}$ and the availability of $\mathbf{H}^{\leftarrow}$.
A full rank decomposition of $\mathbf{H}$ is obtained by any full rank decomposition of $\tilde{\mathbf{H}} = \tilde{\Gamma}\tilde{\Delta}$, because from (22) it follows that $\mathbf{H} = \mathbf{T}_1^T\tilde{\Gamma}\cdot\tilde{\Delta}\mathbf{T}_2$ is a full rank decomposition. The shifted Hankel matrix $\mathbf{H}^{\leftarrow}$ is obtained by observing that it is the Hankel matrix related to the shifted system $zG(z)$, satisfying $\mathbf{H}^{\leftarrow} = \mathbf{T}_1^T\tilde{\mathbf{H}}^{\leftarrow}\mathbf{T}_2$, where $\tilde{\mathbf{H}}^{\leftarrow}$ is the Hankel matrix related to $\widetilde{zG(z)}$. The Markov parameters of this latter system are specified by proposition 3(b), and so $\tilde{\mathbf{H}}^{\leftarrow}$ can be constructed. With these ingredients it is straightforward to formulate a realization algorithm [15, 2, 3].
Unfortunately this algorithm has limited practical value as it requires knowledge of the expansion coefficients of G up to infinity. The situation of a given finite expansion is considered next.

For the classical basis, it is well-known that when a finite sequence $\{g_k\}_{k=1,\cdots N}$ is given, the Ho-Kalman algorithm can be applied to a finite submatrix $\mathbf{H}_{N_1,N_2}$ of the full matrix $\mathbf{H}$ (with $N_1 + N_2 = N$), leading to an exact realization of the underlying system if N is sufficiently large.

We first treat the (intermediate) problem that a finite number of expansion coefficients $\{L_k\}_{k=1,\cdots N}$ is given of a system $G \in RH_2$ with known McMillan degree n. In the next section the situation with unknown McMillan degree will be considered.
Given a finite number of coefficients $\{L_k\}_{k=1,\cdots N}$ of a system $G \in RH_2$, this information can be translated to a finite number of Markov parameters $\{M_k\}_{k=0,\cdots N-1}$ of $\tilde{G}$ according to (12). If N is sufficiently large, allowing the construction of a finite matrix $\tilde{\mathbf{H}}_{N_1,N_2}$ with $N_1 + N_2 = N - 1$ that has the same rank as $\tilde{\mathbf{H}}$, a standard Ho-Kalman algorithm (in the transform domain) can be applied to arrive at a minimal realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ of $\tilde{G}$. Applying the inverse Hambo transform [2, 3] then yields a minimal realization of G.

We recall from [1, 2] an important property of the expansion coefficients of a system $G(z)$, that will be essential for the solution of the partial realization problem, when no knowledge about the McMillan degree is available.

Proposition 5 *Given a system $G(z) \in RH_2$, let $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ be a minimal state space realization of the Hambo transform $\tilde{G}(\lambda)$. Then there exists a matrix X such that the expansion coefficients $\{L_k\}$ of G satisfy*

$$L_k = \tilde{C}\tilde{A}^{k-1}X, \tag{24}$$

4 Minimal partial realization

This section provides an algorithm that solves the minimal partial realization problem as formulated in the introduction. In [16] it was shown that a unique solution (modulo similarity transformation) to the classical minimal partial realization problem is obtained through application of the Ho-Kalman algorithm, provided that a certain rank condition is satisfied by the sequence of Markov parameters $\{g_k\}_{k=1,..,N}$.
A similar result can be derived for the generalized case. Given the coefficients L_k for $k = 1,..,N$ one can calculate the Markov parameters M_k for $k = 0,..,N-1$ as described in section 2. This might suggest that the problem can be solved as in the previous section under the condition that the sequence $\{M_k\}_{k=1,..,N-1}$ satisfies the realizability condition given in [16]. This condition is however not sufficient to guarantee that the resulting realization $(\tilde{A},\tilde{B},\tilde{C},M_0)$ constitutes a valid Hambo transform (see also Remark 2). We require a realizability criterion that is specifically tuned to our problem.
The key to find this condition is provided by Proposition 5, which shows that for a system $G \in RH_2$ the sequences $\{L_k\}$ and $\{M_k\}$ are realized by state-space realizations that share the state transition matrix $\tilde{A}$. This leads us consider the sequence of concatenated matrices

$$K_k = \left[\begin{array}{c|c|c} M_k & L_k & L_{k+1} \end{array}\right], \quad k = 1,..,N-1. \tag{25}$$

Parameter L_{k+1} is included in view of the fact that M_k is obtained on the basis of L_k and L_{k+1}.
The following lemma provides the conditions under which the minimal partial realization problem can be solved.

Lemma 6 *Let $\{L_k\}_{k=1,..,N}$ be an arbitrary sequence of $n_b \times 1$ vectors and let M_k and K_k for $k = 1,..N-1$ be derived from L_k via relations* (12) *and* (25)*. Then there exists a unique minimal realization (modulo similarity transformation) $(\tilde{A},\tilde{B},\tilde{C})$ with McMillan degree n, and an $n \times 1$ vector B such that*

(a) $M_k = \tilde{C}\tilde{A}^{k-1}\tilde{B}$ for $k = 1,..,N-1$, and $L_k = \tilde{C}\tilde{A}^{k-1}B$ for $k = 1,..,N$,
(b) the infinite sequences $[\ M_k \quad L_k\ ] := \tilde{C}\tilde{A}^{k-1}[\ \tilde{B} \quad B\ ]$ satisfy relation (12) *for all $k \in \mathbb{N}$,*

if and only if there exist positive N_1, N_2 such that $N_1 + N_2 = N - 1$ and

$$\text{rank}\,\hat{\mathbf{H}}_{N_1,N_2} = \text{rank}\,\hat{\mathbf{H}}_{N_1+1,N_2} = \text{rank}\,\hat{\mathbf{H}}_{N_1,N_2+1} = n, \tag{26}$$

where $\hat{\mathbf{H}}_{i,j}$ is the Hankel matrix built from the matrices $\{K_k\}$ with block-dimensions $i \times j$.

Condition (26) is in fact equal to the condition given by [16] applied to the sequence $\{K_k\}$. Note that it can only be checked for $N > 2$. On the basis of lemma 6 we can formulate the following proposition that also provides an algorithm to solve the minimal partial realization problem.

Proposition 7 *Let $\{L_k\}_{k=1,..,N}$ be an arbitrary sequence of $n_b \times 1$ vectors, then there exists a minimal realization (A,B,C) of McMillan degree n, such that $\{L_k\}_{k=1,..,N}$ are the first N expansion coefficients of $G(z) = C[zI-A]^{-1}B$, if*

1. *there exist positive N_1, N_2 such that $N_1 + N_2 = N-1$ and condition (26) of lemma 6 holds,*
2. *the minimal realization $(\tilde{A}, \begin{bmatrix} \tilde{B} & X_2 & X_3 \end{bmatrix}, \tilde{C}, \tilde{D})$, resulting from application of the Ho-Kalman algorithm to the sequence $\{K_k\}_{k=1,..,N-1}$, is stable.*

Furthermore, the matrices A, B, C are derived by application of the inverse Hambo transform to the realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$.

Remark 8 *The requirement that $\tilde{A}$ is stable assures that the Ho-Kalman algorithm yields a valid Hambo transform of a stable system. However it is straightforward to extend the algorithm of proposition 7 to the case where $\tilde{A}$ has no poles on the unit circle. If $\tilde{G}$ has unstable poles, it has to be separated in a stable and unstable part. The unstable part is transformed by mirroring it to a stable function and after transformation, mirroring the transform back to an unstable function. Hence the only case which is actually not covered by this algorithm is when $\tilde{A}$ has poles on the unit circle.*

Remark 9 (Interpolation) *It is well known that approximating a transfer function $G(z)$ in terms of a finite set of rational basis functions interpolates to $G(z)$ and/or its derivatives in the points $1/\xi_i$, with ξ_i the poles of the basis functions [20]. It is not surprising that there exists a one-to-one correspondence between the coefficient vector sequence $\{L_k\}_{k=1,..,N}$ and the interpolation data $\{\frac{d^{k-1}G}{dz^{k-1}}(1/\xi_i)\}_{k=1,..,N}$. Explicit expressions for this relation are given in [2, 3]. One can hence solve the following interpolation problem, using the algorithm of proposition 7.*

Problem 10 *Given the interpolation conditions*

$$\frac{d^{k-1}G}{dz^{k-1}}(1/\xi_i) = c_{i,k},\ c_{i,k} \in \mathbb{C}$$

for $i = 1,..,n_b$ and $k = 1,..,N$ $(N > 2)$, with $\xi_i \neq 0$ distinct points, inside the unit disc, find the RH_2 transfer function of least possible degree that interpolates these points.

5 Approximate realization

The classical partial realization algorithm can be applied as a system identification method [21, 9], building a Hankel matrix with (possibly noise corrupted) expansion coefficients and by applying rank reduction through singular value truncation. This can be applied similarly to the generalized situation. Here an example is given in which this method is compared with the classical approximate realization method. Besides in an identification context, the approximate realization procedure can also be applied as a model reduction method.

In comparison with the classical case, approximate realization in the generalized case has one additional difficulty, due to the fact that not every system in $RH_{2,0}^{n_b \times n_b}$ is the Hambo transform of a system in H_2. Although the inverse transform can be applied to any system in $RH_{2,0}^{n_b \times n_b}$, the resulting system in H_2 will not have a one-to-one correspondence with the original. In the exact realization setting, this problem does not arise. The full implications of this phenomenon are not fully understood yet and will be the subject of further research.

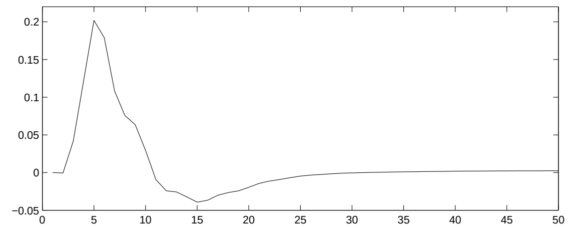

Figure 1: Impulse response of the example system.

As example we compare the application of the generalized approximate realization method with the method of [9]. We consider a 6-th order SISO transfer function $G(z)$, given by

$$10^{-3}\frac{-0.564z^5 + 43.9z^4 - 21.67z^3 - 1.04z^2 - 95.7z + 75.2}{z^6 - 3.35z^5 + 4.84z^4 - 4.44z^3 + 3.11z^2 - 1.48z + 0.318}.$$

Fig. 1 shows the impulse response of $G(z)$, revealing that the system incorporates a mix of fast and slow dynamics. 10 simulations are carried out in which the response of the system $G(z)$ to a Gaussian white noise input with unit standard deviation is determined. An independent Gaussian noise disturbance with standard deviation 0.05 is added to the output. This amounts to a signal to noise ratio (in terms of RMS values) of about 17 dB. The length of the input and output data signals is taken to be 1000 samples. For each of the 10 data sets two basis function models of the form

$$\hat{G}(z) = \sum_{k=1}^{N} \hat{L}_k^T V_k(z), \qquad (27)$$

are estimated using the least-squares method described in [17]. The first model is a 40-th order FIR model. Hence in this case $V_k(z) = z^{-k}$ and $N = 40$. The second model uses a generalized basis that is generated by a second order all-pass function with poles 0.5 and 0.9. For this model 20 coefficient vectors are estimated. Hence the number of estimated coefficients is equal for both models. We now apply the approximate realization method using the estimated expansion coefficients of both models, for all 10 simulations.

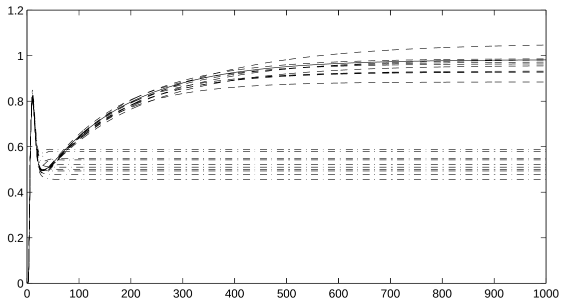

Figure 2: Step response plots of the example system (solid) and the models obtained in 10 simulations with approximate realization using the standard basis (dash-dotted) and the generalized basis (dashed).

In either case a 6-th order model is computed, through truncation of the SVD of the finite Hankel matrix. In figure 2 step response plots of the resulting models are shown. It is seen that approximate realization using the standard basis, results in a model that only fits the first samples of the response well. This is a known drawback of this method. Employing the generalized basis, with poles 0.5 and 0.9 results in models that better capture the transient behavior. Apparently, a sensible choice of basis can considerably improve the performance of the Kung algorithm [9].

6 Conclusions

In this paper an algorithm is derived that solves the minimal partial realization problem for expansions in terms of generalized orthonormal basis functions, generated according to the Hambo basis construction. The realization problem is solved by linking it to the classical realization problem formulated in the Hambo operator transform domain. The resulting algorithm can also be used in an approximate sense, e.g. for the purpose of model reduction or in a system identification setting.

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