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Identifiability of diffusively coupled linear networks with partial instrumentation ⋆

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Abstract: This paper presents identifiability conditions for identifying the complete dynamics of diffusively coupled linear networks. These conditions are derived by exploiting the uniqueness of the nonmonic polynomial network description, given the locations of the actuators and sensors. The analysis is performed under a more relaxed instrumentation setup than the typical restriction to a full set of sensors (full measurement) or a full set of actuators (full excitation). This leads to more general identifiability conditions, including more flexible instrumentation requirements.

Keywords: Identifiability, diffusive couplings, linear dynamic networks, experiment design.

1. INTRODUCTION

In recent years, large-scale interconnected systems are receiving increasingly more attention. Diffusively coupled linear networks model interconnected systems with symmetric cause-effect relationships in the links. Examples are physical linear networks, which can describe many processes from different domains, such as electrical circuits, mechanical systems, and chemical and biological processes.

In literature there are several methods available for identifying the complete dynamics of diffusively coupled networks from data. For example, black-box state-space models can be estimated from which the model parameters can be derived using eigenvalue decompositions (Friswell et al., 1999; Luş et al., 2003). De Angelis et al. (2002) concluded that the model parameters of a 2nd order model can be extracted from an (identified) state-space model if all nodes contain either a sensor or an actuator with at least one colocated sensor–actuator pair. Mukhopadhyay et al. (2014) made the same observation and further analyzed instrumentation conditions and identifiability issues for shear-type systems. These methods are restricted to 2nd order models and typically do not consider disturbances. They also do not have any guarantees on the statistical accuracy of the estimates and lack a consistency analysis.

Van Waarde et al. (2018) considered undirected state-space models, which can be seen as first order diffusively coupled linear networks with the states as nodes. Their identifiability analysis based on Markov parameters resulted in a specific subset of nodes that is required to have a colocated sensor-actuator pair. For higher order diffusively coupled linear networks, the Markov parameters become more complex, which hinders the analysis.

Diffusively coupled networks can also be modeled as directed dynamic networks with specific structural properties (Kivits and Van den Hof, 2019). These networks can be modeled as interconnections of transfer function modules (Gonçalves and Warnick, 2008; Van den Hof et al., 2013), for which an identification framework has been developed by Van den Hof et al. (2013). In this framework, identifiability of the complete dynamics or a subset of the dynamics is analyzed under partial instrumentation conditions (Bazanella et al., 2019; Cheng et al., 2022; Shi et al., 2023). However, the specific network model structure is generally lost, resulting in conservative conditions.

Hannan and Deistler (2012) analyzed the identifiability of polynomial models. These models have the typical assumption of monicity and therefore do not fit the diffusively coupled linear network model, where monicity does not hold.

In this paper, we follow the modeling approach of Kivits and Van den Hof (2023), who discuss the identification of the full diffusively coupled network dynamics in the case of full measurement, including detailed identifiability and consistency results. The objective of this paper is to derive conditions for identifiability of the full diffusively coupled network dynamics in the case that only some nodes are excited and only some node signals are measured. We do this by reviewing the identifiability results for the full measurement case, which are based on non-monic matrix fraction descriptions (MFDs) (Kivits and Van den Hof, 2022, 2023), combine it with the dual situation of the full excitation case, and then formulate the conditions for the generalized case, involving MFDs with three polynomials.

The networks that will be considered are defined in Section 2. Section 3 defines identifiability. Section 4 recaps the identifiability conditions for full measurement and Section 5 describes the dual conditions for full excitation. Section 6 presents the main result: identifiability conditions for partial instrumentation. Finally, section 7 concludes

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the paper. For simplicity, we restrict to representations in the discrete-time domain.

We consider the following notation throughout the paper. A polynomial matrix $A(q^{-1})$ consists of matrices A_ℓ and (j,k)th polynomial elements $a_{jk}(q^{-1})$ such that $A(q^{-1}) = \sum_{\ell=0}^{n_a} A_\ell q^{-\ell}$ and $a_{jk}(q^{-1}) = \sum_{\ell=0}^{n_a} a_{jk,\ell} q^{-\ell}$. Hence, the (j,k)th element of the matrix A_ℓ is denoted by $a_{jk,\ell}$. Let $A_{\mathcal{C}\mathcal{Z}}(q^{-1})$, $A_{\mathcal{C}\bullet}(q^{-1})$, and $A_{\bullet\mathcal{Z}}(q^{-1})$ indicate all $a_{jk}(q^{-1})$, $a_{jm}(q^{-1})$, and $a_{mk}(q^{-1})$ with $j \in \mathcal{C}$ and $k \in \mathcal{Z}$, respectively. Let $det(A)$ and $adj(A)$ denote the determinant and adjugate of $A(q^{-1})$, respectively.

2. DIFFUSIVELY COUPLED NETWORK

2.1 Diffusive couplings

Diffusive couplings describe an interaction that depends on the difference between the signals of interest (or nodes). The nodes can also have diffusive couplings with a zero node (or ground node). In line with Kivits and Van den Hof (2023), the behavior of each node signal $w_j(t)$ can be described by

$$\sum_{\ell=0}^{n_x} \mathsf{x}_{jj,\ell} w_j^{(\ell)}(t) + \sum_{k\in\mathcal{N}_j} \sum_{\ell=0}^{n_y} \mathsf{y}_{jk,\ell} \big[w_j^{(\ell)}(t) - w_k^{(\ell)}(t) \big] = u_j(t), \tag{1}$$

with n_x and n_y the order of the dynamics in the network; with $\mathcal{N}_j$ the set of indices of all neighbor nodes of $w_j(t)$; with real-valued coefficients $\mathsf{x}_{jj,\ell} \geq 0$ and $\mathsf{y}_{jk,\ell} = \mathsf{y}_{kj,\ell} \geq 0$; where $w^{(\ell)}(t)$ is the ℓth derivative of $w_j(t)$; and where $u_j(t)$ is the external signal entering the jth node. Combining the expressions (1) in a matrix equation gives

$$X(p)w(t) + Y(p)w(t) = u(t), \tag{2}$$

with differential operator p, i.e. $p^\ell w(t) = w^{(\ell)}(t)$; with diagonal polynomial matrix $X(p)$, with $x_{jj}(p) = \sum_{\ell=0}^{n_x} \mathsf{x}_{jj,\ell} p^\ell$, containing the components intrinsically related to the nodes (e.g. in the couplings with the zero node); with Laplacian [1] polynomial matrix $Y(p)$, with $y_{jk}(p) = -\sum_{\ell=0}^{n_y} \mathsf{y}_{jk,\ell} p^\ell$ if $k \in \mathcal{N}_j$ and $y_{jk}(p) = 0$ if $k \notin \{\mathcal{N}_j, j\}$, containing the components in the diffusive couplings between the nodes.

Examples of diffusively coupled networks are physical networks, such as electrical circuits, which are characterized by their symmetric components that imply diffusive couplings. For example, a resistor describes the relation between the current and the difference in electric potential on each side of the resistor. A physical network typically exhibits second order dynamics between all node signals. Generalizing to include higher order dynamics is particularly useful for describing a selection of (measured) node signals by removing the other (unmeasured) node signals through a Gaussian elimination procedure (called immersion or Kron reduction (Dankers et al., 2016; Dörfler and Bullo, 2013)).

To exploit the network identification results that have been developed for discrete-time systems, a backward difference method (describing a bijective mapping) is used to approximate (2) bythe equivalent form

$$\bar{X}(q^{-1})w(t) + \bar{Y}(q^{-1})w(t) = u(t), \tag{3}$$

with delay operator q^{-1}, i.e. $q^{-1}w(t) = w(t-1)$, and with $\bar{X}(q^{-1})$ and $\bar{Y}(q^{-1})$ having the same structural properties as $X(p)$ and $Y(p)$, respectively. In the sequel, we will use $A(q^{-1}) = \bar{X}(q^{-1}) + \bar{Y}(q^{-1})$, from which $\bar{X}(q^{-1})$ and $\bar{Y}(q^{-1})$ can uniquely be recovered due to their structure.

2.2 Network model

As explained in Section 2.1, diffusively coupled networks exhibit a symmetric interaction between nodes. We define these networks in line with Kivits and Van den Hof (2023).

Definition 1. A *diffusively coupled linear network model* consists of L internal node signals $w_j(t)$, $j = 1, \dots, L$; $K \leq L$ known excitation signals $r_j(t)$, $j = 1, \dots, K$; L unknown disturbance signals $v_j(t)$, $j = 1, \dots, L$; and $c \leq L$ measured signals $y_j(t)$, $j = 1, \dots, c$ and is defined as

$$A(q^{-1})w(t) = B(q^{-1})r(t) + v(t), \quad y(t) = C(q^{-1})w(t), \tag{4}$$

with $w(t)$, $r(t)$, $v(t)$, and $y(t)$ vectorized versions of $w_j(t)$, $r_j(t)$, $v_j(t)$, and $y_j(t)$, respectively; with $v(t)$ modeled as filtered white noise, i.e. $v(t) = F(q)e(t)$ with $e(t)$ a vector-valued white noise process; and with

(1) $A(q^{-1}) = \sum_{k=0}^{n_a} A_k q^{-k} \in \mathbb{R}^{L\times L}[q^{-1}]$, with $A^{-1}(q^{-1})$ stable; $\text{rank}(A_0) = L$; and $a_{jk}(q^{-1}) = a_{kj}(q^{-1})\ \forall k, j$.
(2) $B(q^{-1}) = \big[\tilde{B}^\top(q^{-1})\ 0\big]^\top \in \mathbb{R}^{L\times K}[q^{-1}]$, with $\tilde{B}(q^{-1}) \in \mathbb{R}^{K\times K}[q^{-1}]$; and $\tilde{b}_{jk}(q^{-1}) = 0$, $\forall j,\ j \neq k$.
(3) $C(q^{-1}) = \big[0\ \bar{C}(q^{-1})\big] \in \mathbb{R}^{c\times L}[q^{-1}]$, with $\bar{C}(q^{-1}) \in \mathbb{R}^{c\times c}[q^{-1}]$; $\text{rank}(\bar{C}_0) = c$; and $\bar{c}_{jk}(q^{-1}) = 0$, $\forall j,\ j \neq k$.
(4) $F(q) \in \mathbb{R}^{L\times L}(q)$, monic, stable, and stably invertible.
(5) $\Lambda \succ 0$ the covariance matrix of the noise $e(t)$.

Assumption 2. It is assumed that the network (4) is:

(1) Connected: Every pair of nodes yields a path [2].
(2) Well-posed: $A^{-1}(q^{-1})$ exists and is proper.

The polynomial matrices $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$ are nonmonic. Stability of the network is induced by stability of $A^{-1}(q^{-1})$. The diffusive character of the model is represented by the symmetry of $A(q^{-1})$. The polynomial $a_{jk}(q^{-1})$ characterizes the dynamics in the link between node signals $w_j(t)$ and $w_k(t)$. Due to the diagonal structure in $B(q^{-1})$ and $C(q^{-1})$, the first K nodes are excited and that the last c node signals are measured. Often, $B(q^{-1})$ and $C(q^{-1})$ are chosen to be binary, implying that each excitation signal directly enters the network at a distinct node and that each measured signal is directly extracted from distinct internal node signals. If $F(q)$ is polynomial or even stronger if $F(q) = I$, the network (4) leads to an ARMAX-like or ARX-like [3] model structure, respectively.

The input-output mapping of (4) is given by

$$y(t) = T_{yr}(q)r(t) + \bar{v}(t), \quad \bar{v}(t) = T_{ye}(q)e(t), \tag{5}$$

[1] A Laplacian matrix is a symmetric matrix with nonpositive off-diagonal elements and with nonnegative diagonal elements that are equal to the negative sum of all other elements in the same row (or column) (Mesbahi and Egerstedt, 2010).

[2] The network is connected if its Laplacian matrix (i.e. the degree matrix minus the adjacency matrix) has a positive second smallest eigenvalue (Fiedler, 1973).

[3] The structure is formally only an ARMAX (autoregressive-moving average with exogenous variables) or ARX (autoregressive with exogenous variables) structure if the $A(q^{-1})$ polynomial is monic (Hannan and Deistler, 2012).

with

$$T_{yr}(q) = C(q^{-1})A^{-1}(q^{-1})B(q^{-1}), \tag{6}$$

$$T_{ye}(q) = C(q^{-1})A^{-1}(q^{-1})F(q), \tag{7}$$

$$\Phi_{\bar{v}}(\omega) = T_{ye}(e^{i\omega})\Lambda T^*_{ye}(e^{i\omega}), \tag{8}$$

with $(\cdot)^*$ the complex conjugate transpose. A standard open-loop identification of (5) can typically lead to consistent estimation of $T_{yr}(q)$ and $\Phi_{\bar{v}}(\omega)$. Observe that for binary $B(q^{-1})$ and $C(q^{-1})$, (6) leads to a subset of rows and columns of $A^{-1}(q^{-1})$ that constitute $T_{yr}(q)$.

3. IDENTIFIABILITY

Identifiability concerns the ability to distinguish between different models in a network model set, given the locations of the external signals in the network. Therefore, identifiability can be analyzed by exploiting the uniqueness of network models.

Definition 3. Network models $M_1 = \big(A_1(q^{-1}), B_1(q^{-1}), C_1(q^{-1}), F_1(q), \Lambda_1\big)$ and $M_2 = \big(A_2(q^{-1}), B_2(q^{-1}), C_2(q^{-1}), F_2(q), \Lambda_2\big)$ are *equivalent* if

$$T_{yr,1}(q) = T_{yr,2}(q) \quad \text{and} \quad \Phi_{\bar{v},1}(\omega) = \Phi_{\bar{v},2}(\omega). \tag{9}$$

This concept of equivalent network models implies that two network models can model the same measured data (y, r), because both models will have the same transfer function $T_{yr}(q)$ and power spectrum $\Phi_{\bar{v}}(\omega)$. Exploiting the spectral factorization of $\Phi_{\bar{v}}(\omega)$ (8) leads to an equivalent network model with a simplified noise model. This result is analogous to Shi et al. (2023, Theorem 1).

Proposition 4. Any network model

$$M = \big(A(q^{-1}), B(q^{-1}), C(q^{-1}), F(q), \Lambda\big) \tag{10}$$

admits an equivalent network model

$$\tilde{M} \triangleq \big(A(q^{-1}), B(q^{-1}), C(q^{-1}), \left[0 \ \tilde{F}^*(q)\right]^*, \tilde{\Lambda}\big), \tag{11}$$

where $\tilde{F}(q) \in \mathbb{R}^{c\times c}(q)$ is monic, stable, and stably invertible and $\tilde{\Lambda} \in \mathbb{R}^{c\times c} \succ 0$.

Proof. Omit the arguments q, q^{-1}, ω, and $e^{i\omega}$ for notational simplicity. The behavior of the measured signals $y(t)$ is described in an immersed network model, which is obtained by eliminating the unmeasured signals (through immersion or Kron reduction (Dankers et al., 2016; Dörfler and Bullo, 2013)). Partition the internal signals as $w(t) = \left[w^\top_{\mathcal{Z}}(t) \ w^\top_{\mathcal{C}}(t)\right]^\top$, such that $y(t) = \bar{C}w_{\mathcal{C}}(t)$. Partition A, B, and F accordingly and define

$$\bar{A} \triangleq d_{\mathcal{Z}\mathcal{Z}}\big(A_{\mathcal{C}\mathcal{C}} - A_{\mathcal{C}\mathcal{Z}}A^{-1}_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{Z}\mathcal{C}}\big),$$

$$\bar{B} \triangleq d_{\mathcal{Z}\mathcal{Z}}\big(B_{\mathcal{C}\bullet} - A_{\mathcal{C}\mathcal{Z}}A^{-1}_{\mathcal{Z}\mathcal{Z}}B_{\mathcal{Z}\bullet}\big),$$

$$\bar{F} \triangleq d_{\mathcal{Z}\mathcal{Z}}\big(F_{\mathcal{C}\bullet} - A_{\mathcal{C}\mathcal{Z}}A^{-1}_{\mathcal{Z}\mathcal{Z}}F_{\mathcal{Z}\bullet}\big),$$

$$d_{\mathcal{Z}\mathcal{Z}} \triangleq \frac{det(A_{\mathcal{Z}\mathcal{Z}})}{gcd\,(det(A_{\mathcal{Z}\mathcal{Z}}), adj(A_{\mathcal{Z}\mathcal{Z}}))},$$

so that $\bar{A}$ and $\bar{B}$ are polynomial (Kivits and Van den Hof, 2022). The immersed network model is now given by

$$\bar{A}w_{\mathcal{C}}(t) = \bar{B}r(t) + \bar{F}e(t), \quad y(t) = \bar{C}w_{\mathcal{C}}(t),$$

which has input-output mapping

$$y(t) = \bar{C}\bar{A}^{-1}\bar{B}r(t) + \bar{C}\bar{A}^{-1}\bar{F}e(t).$$

Together with (5), (7), and (8) this gives

$$\Phi_{\bar{v}} = CA^{-1}F\Lambda F^*A^{-*}C^* = \bar{C}\bar{A}^{-1}\bar{F}\Lambda\bar{F}^*\bar{A}^{-*}\bar{C}^*,$$

where $C = \bar{C}\,[0 \ I]$ and $\bar{A}^{-1} = d^{-1}_{\mathcal{Z}\mathcal{Z}}\,[0 \ I]\,A^{-1}\,[0 \ I]^\top$, i.e.

$$\bar{C}\bar{A}^{-1} = d^{-1}_{\mathcal{Z}\mathcal{Z}}CA^{-1}\begin{bmatrix}0\\I\end{bmatrix}.$$

Further, $\bar{F}\Lambda\bar{F}^*$ can be refactorized as $d_{\mathcal{Z}\mathcal{Z}}\tilde{F}\tilde{\Lambda}\tilde{F}^*d^*_{\mathcal{Z}\mathcal{Z}}$ (Gevers et al., 2019), where $\tilde{F}$ and $\tilde{\Lambda}$ satisfy the properties of this proposition. This leads to

$$\Phi_{\bar{v}} = CA^{-1}\begin{bmatrix}0\\\tilde{F}\end{bmatrix}\tilde{\Lambda}\left[0 \ \tilde{F}^*\right]A^{-*}C^*.$$

Hence, the input-output mapping

$$y(t) = CA^{-1}Br(t) + CA^{-1}\begin{bmatrix}0\\\tilde{F}\end{bmatrix}\tilde{e}(t),$$

where noise signal $\tilde{e}(t)$ has covariance matrix $\tilde{\Lambda}$, leads to the same T_{yr} and $\Phi_{\bar{v}}$ as in (6) and (8), respectively. □

As $T_{yr}(q)$ and $\Phi_{\bar{v}}(\omega)$ only reflect the properties of the measured nodes, there is a freedom in transforming the unmeasured internal signals and in modeling the disturbances affecting the measured signals. In $\tilde{M}$, all unmeasured node signals are disturbance-free. Hence, there are multiple models M (with different noise processes) that admit the same equivalent model $\tilde{M}$. As $\tilde{M}$ admits a simpler noise model, it is more attractive for the identifiability analysis.

Before defining the model set corresponding to the models $\tilde{M}$, let us take a deeper look at the power spectrum $\Phi_{\bar{v}}(\omega)$. As $T_{y\tilde{e}}(q) = C(q^{-1})A^{-1}(q^{-1})\left[0 \ \tilde{F}^*(q)\right]^*$ is not monic, the spectral factorization of $\Phi_{\bar{v}}(\omega)$ into $T_{y\tilde{e}}(e^{i\omega})$ and $\tilde{\Lambda}$ is not unique. However, the spectral factorization of $\Phi_{\bar{v}}(\omega)$ can be made unique by properly scaling $T_{y\tilde{e}}(e^{i\omega})$ and $\tilde{\Lambda}$.

Proposition 5. The power spectrum $\Phi_{\bar{v}}(\omega)$ admits a unique *spectral factorization* into $T_{y\breve{e}}(e^{i\omega})$ and $\breve{\Lambda}$, where $T_{y\breve{e}}(q) = \bar{C}(q^{-1})\bar{A}^{-1}(q^{-1})\breve{F}(q)$ is monic, with $\breve{F}(q) = \tilde{F}(q)\bar{A}_0\bar{C}_0^{-1}$, and $\breve{\Lambda} = \bar{C}_0\bar{A}_0^{-1}\tilde{\Lambda}\bar{A}_0^{-1}\bar{C}_0^\top \succ 0$.

Proof. Redefine the noise model $\bar{v}(t) = \tilde{F}(q)\tilde{e}(t)$ as $\bar{v}(t) = \breve{F}(q)\breve{e}(t)$ with $\breve{F}(q) = \tilde{F}(q)\bar{A}_0\bar{C}_0^{-1}$ and with $\breve{\Lambda} \succ 0$ the covariance of $\breve{e}(t)$. Then $\Phi_{\bar{v}}(\omega) = T_{y\breve{e}}(e^{i\omega})\breve{\Lambda}T^*_{y\breve{e}}(e^{i\omega})$, which admits a unique spectral factorization into $T_{y\breve{e}}(e^{i\omega})$ and $\breve{\Lambda}$ as $T_{y\breve{e}}(q)$ is monic, stable, and stably invertible and $\breve{\Lambda} \succ 0$ (Youla, 1961). □

Now, let us define the model set of the models (11).

Definition 6. The *network model set* $\tilde{\mathcal{M}}$ is defined as a set of parametrized functions as

$$\tilde{\mathcal{M}} := \{\tilde{M}(\theta), \ \theta \in \Theta \subset \mathbb{R}^d\}, \tag{12}$$

with $d \in \mathbb{N}$ and with all particular models

$$\tilde{M}(\theta) := \big(A(q^{-1}, \theta), \ B(q^{-1}, \theta), \ C(q^{-1}, \theta), \ \left[0 \ \tilde{F}^*(q, \theta)\right]^*, \ \tilde{\Lambda}(\theta)\big) \tag{13}$$

satisfying the properties in Definition 1 and Assumption 2, where Property 4 of Definition 1 is replaced by

(4) $\tilde{F}(q) \in \mathbb{R}^{c\times c}(q)$, monic, stable and stably invertible.

Here, θ contains all the unknown coefficients that appear in the entries of the model matrices $A(q^{-1})$, $B(q^{-1})$, $C(q^{-1})$, $\tilde{F}(q)$, and $\tilde{\Lambda}$.

Since the network models that will be considered and the corresponding network model set have been defined, we can now continue with the identifiability analysis. Let us adopt the concept of network identifiability from Weerts et al. (2018).

Definition 7. The network model set $\tilde{\mathcal{M}}$ is *globally network identifiable* from data $z(t) := \{y(t), r(t)\}$ if the parametrized model $\tilde{M}(\theta)$ can uniquely be recovered from $T_{yr}(q,\theta)$ and $\Phi_{\bar{v}}(\omega,\theta)$, that is if for all models $\tilde{M}(\theta_1), \tilde{M}(\theta_2) \in \tilde{\mathcal{M}}$

$$\left.\begin{array}{l} T_{yr}(q,\theta_1) = T_{yr}(q,\theta_2) \\ \Phi_{\bar{v}}(\omega,\theta_1) = \Phi_{\bar{v}}(\omega,\theta_2) \end{array}\right\} \implies \tilde{M}(\theta_1) = \tilde{M}(\theta_2). \quad (14)$$

Using the result of Proposition 5 on the power spectral factorization of $\Phi_{\bar{v}}(\omega)$, we have the following identifiability result.

Proposition 8. For a network model set $\tilde{\mathcal{M}}$, implication (14) can equivalently be formulated as

$$\left.\begin{array}{l} T_{yr}(q,\theta_1) = T_{yr}(q,\theta_2) \\ T_{y\breve{e}}(q,\theta_1) = T_{y\breve{e}}(q,\theta_2) \\ \breve{\Lambda}(\theta_1) = \breve{\Lambda}(\theta_2) \end{array}\right\} \implies \tilde{M}(\theta_1) = \tilde{M}(\theta_2). \quad (15)$$

Proof. From Proposition 5, $T_{y\breve{e}}(q)$ and $\breve{\Lambda}$ are uniquely determined by $\Phi_{\bar{v}}(\omega)$ and therefore, $\Phi_{\bar{v}}(\omega,\theta)$ in (14) can be replaced by $T_{y\breve{e}}(q,\theta)$ and $\breve{\Lambda}(\theta)$. □

4. FULL MEASUREMENT

Consider a network as defined in Definition 1, where all node signals are directly measured. This is the most common instrumentation setting for identification in dynamic networks. Let us recap the corresponding identifiability conditions of Kivits and Van den Hof (2022, 2023).

Assumption 9. Assume $C(q^{-1}) = I$.

Observe that in this case $\tilde{F} = F$ and thus $\tilde{M} = M$. The identifiability analysis is based on the uniqueness of the network model. Therefore, we present a result on the left MFD (LMFD), before formulating the identifiability conditions for our particular network models.

Lemma 10. Consider a network model set $\tilde{\mathcal{M}}$ satisfying Assumption 9. Given the *LMFD* $A(q^{-1})^{-1}B(q^{-1})$, $A(q^{-1})$ and $B(q^{-1})$ are unique within $\tilde{\mathcal{M}}$ up to a scalar factor if the following conditions are satisfied:

(1) $A(q^{-1})$ and $B(q^{-1})$ are left coprime in $\tilde{\mathcal{M}}$.
(2) There exists a permutation matrix P_b such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1 \cdots A_{n_a}\ B_0\ B_1 \cdots B_{n_b}]\,P_b = [D_b\ R_b]$ with D_b square, diagonal, full rank.

Proof. According to Kailath (1980), the LMFD of any two polynomial and left coprime matrices is unique up to a premultiplication with a unimodular matrix. To satisfy Condition 2, the unimodular matrix is restricted to be diagonal. As $A(q^{-1})$ is symmetric, this diagonal matrix is further restricted to have equal elements. □

In general polynomial models, like ARMAX (Deistler, 1983), $A(q^{-1})$ is monic, i.e. $A_0 = I$. Then the LMFD $A(q^{-1})^{-1}B(q^{-1})$ is unique, as the conditions of Lemma 10 are satisfied and scaling with a scalar factor is not possible anymore. Hence, both Condition 2 in Lemma 10 and the scaling factor freedom are a result of the fact that $A(q^{-1})$ is not necessarily monic.

Now the conditions for global network identifiability can be formulated.

Proposition 11. A network model set $\tilde{\mathcal{M}}$ satisfying Assumption 9 is *globally network identifiable* from $z(t)$ if the following conditions are satisfied:

(1) $A(q^{-1})$ and $B(q^{-1})$ are left coprime in $\tilde{\mathcal{M}}$.
(2) There exists a permutation matrix P_b such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1 \cdots A_{n_a}\ B_0\ B_1 \cdots B_{n_b}]\,P_b = [D_b\ R_b]$ with D_b square, diagonal, full rank.
(3) At least one excitation signal $r_j(t)$, $j = 1, \ldots, K$, is present: $K \geq 1$.
(4) There is at least one constraint on the parameters of $A(q^{-1}, \theta_a)$ and $B(q^{-1}, \theta_c)$ of the form $\Gamma\theta_{ab} = \gamma \neq 0$, with Γ full row rank and with $\theta_{ab} := \left[\theta_a^\top\ \theta_b^\top\right]^\top$.

Proof. Condition 3 implies that $T_{yr}(q,\theta)$ is nonzero. According to Lemma 10, Condition 1 and 2 imply that $A(q^{-1},\theta)$ and $B(q^{-1},\theta)$ are unique up to a scalar factor α. According to Proposition 5, $T_{y\breve{e}}(q,\theta)$ and $\breve{\Lambda}(\theta)$ are uniquely recovered from $\Phi_{\bar{v}}(\omega,\theta)$. Together with the fact that $A(q^{-1},\theta)$ is unique up to a scalar factor α, $T_{y\breve{e}}(q,\theta)$ gives a unique $\tilde{F}(q,\theta)$, and $\breve{\Lambda}(\theta)$ gives $\tilde{\Lambda}(\theta)$ up to a scalar factor α^2. Finally, Condition 4 implies that α is unique. □

5. FULL EXCITATION

Consider a network as defined in Definition 1, where now all node signals are directly excited. This is the dual instrumentation setting compared to the full measurement setup in Section 4. In this section, we present the identifiability conditions for networks with full excitation.

Assumption 12. Assume $B(q^{-1}) = I$.

Again, the identifiability analysis is based on the uniqueness of the network model. Here, we present a result on the right MFD (RMFD), before formulating the identifiability conditions for our particular network models.

Lemma 13. Consider a network model set $\tilde{\mathcal{M}}$ satisfying Assumptions 12. Given the *RMFD* $C(q^{-1},)A(q^{-1},)^{-1}$, $C(q^{-1})$ and $A(q^{-1})$ are unique within $\tilde{\mathcal{M}}$ up to a scalar factor if the following conditions are satisfied:

(1) $A(q^{-1})$ and $C(q^{-1})$ are right coprime in $\tilde{\mathcal{M}}$.
(2) There exists a permutation matrix P_c such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1 \cdots A_{n_a}\ C_0\ C_1 \cdots C_{n_c}]\,P_c = [D_c\ R_c]$ with D_c square, diagonal, full rank.

Proof. According to Kailath (1980), the RMFD of any two polynomial and right coprime matrices is unique up to a postmultiplication with a unimodular matrix. To satisfy Condition 2, the unimodular matrix is restricted to be diagonal. As $A(q^{-1})$ is symmetric, this diagonal matrix is further restricted to have equal elements. □

Similar to Section 4, a monic $A(q^{-1})$ implies that the RMFD $C(q^{-1})A(q^{-1})^{-1}$ is unique, as the conditions of Lemma 13 are satisfied and scaling with a scalar factor is not possible anymore. Hence, again Condition 2 in Lemma

13 and the scaling factor freedom are a result of the fact that $A(q^{-1})$ is not necessarily monic.

Now the conditions for global network identifiability can be formulated.

Proposition 14. A network model set $\tilde{\mathcal{M}}$ satisfying Assumption 12 is *globally network identifiable* from $z(t)$ if the following conditions are satisfied:

(1) $A(q^{-1})$ and $C(q^{-1})$ are right coprime in $\tilde{\mathcal{M}}$.
(2) There exists a permutation matrix P_c such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1\ \cdots\ A_{n_a}\ C_0\ C_1\ \cdots\ C_{n_c}]\,P_c = [D_c\ R_c]$ with D_c square, diagonal, full rank.
(3) At least one measured signal $y_j(t)$, $j = 1, \ldots, c$, is present: $c \geq 1$.
(4) There is at least one constraint on the parameters of $A(q^{-1}, \theta_a)$ and $C(q^{-1}, \theta_c)$ of the form $\Gamma\theta_{ac} = \gamma \neq 0$, with Γ full row rank and with $\theta_{ac} := \begin{bmatrix}\theta_a^\top & \theta_c^\top\end{bmatrix}^\top$.

Proof. The proof is fully dual to the proof of Proposition 11. □

6. PARTIAL INSTRUMENTATION

6.1 Network model analysis

This section contains the main results, which are the identifiability conditions for networks with partial instrumentation. Consider a network as defined in Definition 1, where now all node signals are either measured or excited and at least one node signal is both measured and excited:

Assumption 15. Assume $K + c \geq L + 1$.

As before, the analysis is based on the uniqueness of the network model. We present a result on the MFD, before formulating the identifiability conditions for our particular network models.

Lemma 16. For a network model set $\tilde{\mathcal{M}}$ satisfying Assumptions 15, the *MFD* $C(q^{-1})A(q^{-1})^{-1}B(q^{-1},)$, gives unique $C(q^{-1})$, $A(q^{-1})$, and $B(q^{-1})$ within $\tilde{\mathcal{M}}$ if the following conditions are satisfied:

(1) $A(q^{-1})$ and $B(q^{-1})$ are left coprime in $\tilde{\mathcal{M}}$.
(2) $A(q^{-1})$ and $C(q^{-1})$ are right coprime in $\tilde{\mathcal{M}}$.
(3) There exists a permutation matrix P_b such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1\ \cdots\ A_{n_a}\ B_0\ B_1\ \cdots\ B_{n_b}]\,P_b = [D_b\ R_b]$ with D_b square, diagonal, full rank.
(4) There exists a permutation matrix P_c such that within $\tilde{\mathcal{M}}$, $[A_0\ A_1\ \cdots\ A_{n_a}\ C_0\ C_1\ \cdots\ C_{n_c}]\,P_c = [D_c\ R_c]$ with D_c square, diagonal, full rank.
(5) For each $k = 1, 2, \ldots, L$, there is a nonzero linear equality constraint on the parameters related to node $w_k(t)$, i.e. on $a_{\ell k}(q^{-1}, \theta)$, $b_{kk}(q^{-1}, \theta)$, or $c_{kk}(q^{-1}, \theta)$. These L constraints are on L different polynomials.
(6) There is at least one extra constraint on the parameters of $A(q^{-1}, \theta_a)$, $B(q^{-1}, \theta_b)$, and $C(q^{-1}, \theta_c)$ of the form $\Gamma\theta_{abc} = \gamma \neq 0$, with Γ full row rank and with $\theta_{abc} := \begin{bmatrix}\theta_a^\top & \theta_b^\top & \theta_c^\top\end{bmatrix}^\top$.

Proof. According to Kailath (1980), any MFD satisfying Conditions 1 and 2 is unique up to multiplication with unimodular matrices, i.e. $C(q^{-1}, \theta_1)A^{-1}(q^{-1}, \theta_1)B(q^{-1}, \theta_1) = C(q^{-1}, \theta_2)A^{-1}(q^{-1}, \theta_2)B(q^{-1}, \theta_2)$ for all $\theta_1, \theta_2 \in \Theta$, with $C(q^{-1}, \theta_2) \triangleq C(q^{-1}, \theta_1)Z(q^{-1})$, $A(q^{-1}, \theta_2) \triangleq R(q^{-1})A(q^{-1}, \theta_1)Z(q^{-1})$, $B(q^{-1}, \theta_2) \triangleq R(q^{-1})B(q^{-1}, \theta_1)$, and with unimodular matrices $R(q^{-1})$ and $Z(q^{-1})$. Conditions 3 and 4, respectively, imply that $R(q^{-1})$ and $Z(q^{-1})$ are diagonal (and thus static). To satisfy Assumption 2 in $A(q^{-1}, \theta_2) = RA(q^{-1}, \theta_1)Z$, $r_{11}^{-1}z_{11} = r_{22}^{-1}z_{22} = \ldots = r_{LL}^{-1}z_{LL}$. Condition 5 fixes r_{kk} or z_{kk}, $k = 1, 2, \ldots, L$. Finally, Condition 6 fixes the ratios $r_{kk}^{-1}z_{kk}$, resulting in $R = I$ and $Z = I$. □

Condition 5 and 6 of Lemma 16 can for example be satisfied by binary $B(q^{-1})$ and $C(q^{-1})$, implying that nodes are directly excited and measured, respectively. Condition 6 of Lemma 16 is similar to Condition 4 of Proposition 11 and 14.

Now the main results of this paper are formulated, which are the conditions for global network identifiability for diffusively coupled networks with partial instrumentation.

Theorem 17. A network model set $\tilde{\mathcal{M}}$ satisfying Assumption 15 is *globally network identifiable* from $z(t)$ if the conditions in Lemma 16 are satisfied.

Proof. Assumption 15 implies that $K \geq 1$ and $c \geq 1$ and thus $T_{yr}(q, \theta)$ is nonzero. Lemma 16 implies that $A(q^{-1}, \theta)$, $B(q^{-1}, \theta)$, and $C(q^{-1}, \theta)$ are uniquely found from $T_{yr}(q, \theta)$. According to Proposition 5, $T_{y\breve{e}}(q, \theta)$ and $\breve{\Lambda}(\theta)$ are uniquely recovered from $\Phi_{\bar{v}}(\omega, \theta)$. Together with the fact that $A(q^{-1}, \theta)$ and $C(q^{-1}, \theta)$ are unique, $T_{y\breve{e}}(q, \theta)$ gives a unique $\tilde{F}(q, \theta)$, and $\breve{\Lambda}(\theta)$ gives a unique $\tilde{\Lambda}(\theta)$. □

$A(q^{-1}, \theta)$, $B(q^{-1}, \theta)$, and $C(q^{-1}, \theta)$ are uniquely determined from $T_{yr}(q, \theta)$, where the required constraints can be imposed on actuators and sensors locations only. $\Phi_{\bar{v}}(\omega, \theta)$ is used to determine $\tilde{F}(q, \theta)$ and $\tilde{\Lambda}(\theta)$. Extracting information from $\Phi_{\bar{v}}(\omega, \theta)$ on $A(q^{-1}, \theta)$ and $C(q^{-1}, \theta)$ is limited by the non-monicity of these polynomials. In the special case of a known $C(q^{-1})$ and a polynomial $F(q, \theta)$, $T_{y\breve{e}}(q^{-1}, \theta)$ can lead to $A(q^{-1}, \theta)$ under L additional constraints on the network *dynamics* $A(q^{-1}, \theta)$.

6.2 Transfer function analysis

The role of the partial instrumentation condition in Assumption 15, can also be understood from analyzing $T_{yr}(q)$ in (6), which shows that the input dynamics in $B(q^{-1})$ and the output dynamics in $C(q^{-1})$ have an equivalent influence on $T_{yr}(q)$. For simplicity, we restrict to binary B and C in this section. An equivalent analysis is presented by De Angelis et al. (2002) for identifying disturbance-free second-order models from first-order state-space models.

For full instrumentation, $B = I$ and $C = I$. Then $T_{yr}(q^{-1}) = A^{-1}(q^{-1})$ and $A(q^{-1})$ can directly be obtained from $T_{yr}(q^{-1})$. For full measurement, $C = I$ and at least one excitation signal is required, e.g. at $w_i(t)$. Then $T_{yr}(q^{-1}) = (A^{-1})_{\bullet i}(q^{-1})$, i.e. the ith column of $A^{-1}(q^{-1})$. Due to symmetry, the ith row of $A^{-1}(q^{-1})$ is also known. For full excitation, $B = I$ and at least one measured signal is required, e.g. $w_i(t)$. Then $T_{yr}(q^{-1}) = (A^{-1})_{i\bullet}(q^{-1})$, i.e. the ith row of $A^{-1}(q^{-1})$. Due to symmetry, the ith column of $A^{-1}(q^{-1})$ is also known. It might seem surprising that knowing only the ith row and column of

$A^{-1}(q^{-1})$ is sufficient for uniquely determining $A(q^{-1})$, but this is due to the symmetry and the other conditions in Proposition 11 and 14. Observe the equivalent influence of excitations and measurements on identifiability of $A(q^{-1})$.

Partial instrumentation requires at least one node signal to be both excited and measured, e.g. $w_i(t)$. Then $K+c = L+1$, $B = [I_K\ 0]^\top$, and $C = [0\ I_c]$, with I_j the identity matrix of size $j \times j$. Then $T_{yr}(q^{-1}) = [A^{-1}]_{\mathcal{CK}}(q^{-1})$, i.e. all $[a^{-1}]_{jk}(q^{-1})$, with $j \in \mathcal{C} \triangleq \{j \mid L+1-c \leq j \leq L\}$ and $k \in \mathcal{K} \triangleq \{j \mid 1 \leq j \leq K\}$. Due to symmetry, all $[a^{-1}]_{kj}(q^{-1})$, with $j \in \mathcal{C}$ and $k \in \mathcal{K}$ are also known. As $\mathcal{C} \cap \mathcal{K} = \{i\}$, the complete ith row and ith column of $A^{-1}(q^{-1})$ are known, which is sufficient for uniquely determining $A(q^{-1})$. In other words, it is possible to transform the partial instrumentation case (satisfying Assumption 15) to the full measurement or full excitation case if at least one node signal is both excited and measured.

7. CONCLUSION

Identifiability conditions for identifying the complete dynamics of diffusively coupled linear networks have been formulated. Analyzing the uniqueness of the network description lead to more flexible instrumentation requirements than requiring to measure all node signals or to excite all node signals. For identifiability it is sufficient to either measure or excite each node signal and to both measure and excite (at least) one node signal.

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