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Equation (19) in this paper has been corrected.

A dynamic network approach to identification of physical systems

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***Abstract*— System identification problems utilizing a prediction error approach are typically considered in an input/output setting, where a directional cause-effect relationship is presumed and transfer functions are used to estimate the causal relationships. In more complex interconnection structures, as e.g. appearing in dynamic networks, the cause-effect relationships can be encoded by a directed graph. Physical dynamic networks are most commonly described by diffusive couplings between node signals, implying that cause-effect relationships between node signals are symmetric and therefore can be represented by an undirected graph. This paper shows how (prediction error) identification methods developed for linear dynamic networks can be configured to identify components in (undirected) physical networks with known topology.**

I. Introduction

Physical networks are only one example of dynamic networks, which are interconnections of dynamic units. Dynamic networks receive increasing attention from a variety of scientific fields, since systems are growing in complexity and size. Other examples of dynamic networks are biological and chemical processes, neural networks, consensus networks, synchronisation, social interactions, the Internet, the stock market and multi-agent systems [1], [2], [3].

By representing a dynamic network as an interconnection structure of dynamic transfer function modules [4], [5], a framework for system identification in dynamic networks has been developed in [5], by extending classical closed-loop prediction error methods. Other developments focus on topology estimation [6], [7], full network identification [8], local module identification [9], [10], [11] and network identifiability [12], [13], [14]. In this framework, dynamic networks are considered to consist of directed interconnections of dynamic modules that can be of any dynamic order. In contrast, physical systems are typically considered as undirected dynamic interconnections between node signals, where the interconnections represent diffusive couplings [15] and the model is typically described by a vector difference equation of maximum second order. The most well-known example is a mechanical mass-spring-damper system, with positions of masses as node (state) signals and the dynamics being described by a second order vector difference equation. Identification of these physical models can be done by conversion of the model into a state space form, after which matrix transformations [16], [17] or eigenvalue decompositions [18], [19] are being applied to estimate the model parameters. However, during these operations the network structure in the model is generally lost.

The overall objective of this research is to develop a comprehensive theory for the identification of individual interconnections (modules) in physical (undirected) networks, where the order of the individual modules is not restricted and possibly correlated disturbances can be present. The objective includes questions like which nodes to measure (sense) and which nodes to excite (actuate) in order to identify a particular (local) module in the network or to identify the full dynamics and topology of the network. In addition, consistency and minimum variance properties of estimates have to be specified. In this way, the (prediction error) identification theory for directed networks is extended to undirected networks.

This paper includes the first steps towards achieving the above mentioned objective, by addressing the question how current identification methods can be made applicable to undirected networks. This paper includes the first steps towards achieving the above mentioned objective. The physical networks that will be considered in this paper are defined in Section II. Currently, extensive tools are under development for choosing which nodes to measure and / or excite for identifying a local module in directed networks. In order to use the insights of identification in directed networks also for physical / undirected networks, the relationship between physical networks and directed networks needs to be clear. This relationship is described in Section III. Next, insights of identification in directed networks are applied to these networks for estimating the dynamics of the full network (Section IV). Finally, some first results are presented for addressing the local identification problem (Section V), after which Section VI concludes the paper.

II. Physical network

Physical systems are often described by second order differential equations. They can be considered to consist of L interconnected node signals $w_j(t)$, $j = 1, \dots, L$, of which the behaviour is described according to

$$M_j \ddot{w}_j(t) + D_{j0}\dot{w}_j(t) + \sum_{k \in \mathcal{N}_j} D_{jk}\big(\dot{w}_j(t) - \dot{w}_k(t)\big) + K_{j0} w_j(t) + \sum_{k \in \mathcal{N}_j} K_{jk}\big(w_j(t) - w_k(t)\big) = u_j(t), \quad (1)$$

where $M_j \geq 0$, $D_{jk} \geq 0$, $K_{jk} \geq 0$, $D_{jj} = 0$, $K_{jj} = 0$, $\mathcal{N}_j$ is the set of indices of node signals $w_k(t)$ $k \neq j$ with

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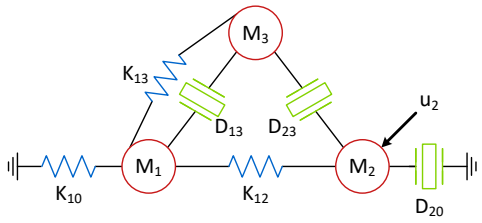

Fig. 1. A network of masses (M_j), dampers (D_{jk}) and springs (K_{jk}).

connections to node signals $w_j(t)$, $u_j(t)$ are the external input signals and $\dot{w}_j(t)$ and $\ddot{w}_j(t)$ are the first and second order derivative of the node signals $w_j(t)$, respectively.

In physical systems, all connections are symmetric, meaning that the strength of the connection from node w_i to node w_k is equal to the strength of the connection (in opposite direction) from node w_k to node w_i. This means that the interconnections of the nodes are diffusive couplings, which emerge in (1) from the symmetric connections: $D_{jk} = D_{kj}$ and $K_{jk} = K_{kj}$ $\forall j, k$.

An example of a physical system with diffusive couplings is the mass-spring-damper system shown in Figure 1, in which masses M_j are interconnected through dampers D_{jk} and springs K_{jk} with $k \neq 0$ and are connected to the earth with dampers D_{j0} and springs K_{j0}. The positions of the masses are the signals of interest and therefore chosen to be the node signals: $w_j(t) := x_j(t)$. The couplings between the masses are diffusive, because springs and dampers are symmetric components. Further, a network as shown in Figure 1 would require at least a two-dimensional position vector $w_j(t)$, but without loss of generality we will restrict our attention to scalar-valued node signals $w_j(t)$.

A. Higher order network

A physical system as the mass-spring-damper system in Section II is typically of second order when all node signals are collected in $w(t)$. The theory can easily be extended to higher order terms, which is useful in for example immersion as explained in Section V-B.

Definition 1 (Physical network): A physical network is a network consisting of L node signals $w_1(t), \ldots, w_L(t)$ interconnected through diffusive couplings and with possibly connections of nodes to a ground node. The behaviour of the node signals $w_j(t)$, $j = 1, \ldots, L$, is described by

$$\sum_{\ell=0}^{n} B_{\ell,j} w_j^{(\ell)}(t) + \sum_{k \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} A_{\ell,jk}[w_j^{(\ell)}(t) - w_k^{(\ell)}(t)] = u_j(t), \tag{2}$$

with $B_{\ell,j} \geq 0$, $A_{\ell,jk} \geq 0$, $A_{\ell,jj} = 0$, $A_{\ell,jk} = A_{\ell,kj}$ and where $w_j^{(\ell)}(t)$ is the ℓ-th derivative of $w_j(t)$. □

The graphical interpretation of the coefficients is as follows: $B_{n,j}$ represent the components intrinsically related to the nodes w_j, $B_{\ell,j}$ with $\ell \neq n$ represent the components connecting the node w_j to the ground node (or earth) and $A_{\ell,jk}$ represent the components in the diffusive couplings between the nodes w_j and w_k. Further, every matrix B_ℓ composed of elements $B_{\ell,j}$ is diagonal and every matrix A_ℓ composed of elements $A_{\ell,jk}$ is Laplacian[1] representing an undirected graph of a specific physical component (i.e. of the diffusive couplings of a specific order).

B. Discretisation

For the purpose of identification in a discrete-time setting, the continuous time network is converted to an equivalent discrete-time network.

Proposition 1 (Discrete time): By using the approximation

$$\frac{dw(t)}{dt} = \frac{w(t_d T_s) - w((t_d - 1)T_s)}{T_s}, \tag{3}$$

the continuous time physical network (2) can be described in discrete time by

$$\sum_{\ell=0}^{n} \bar{B}_{\ell,j} q^{-\ell} w_j(t_d) + \sum_{k \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jk} q^{-\ell}[w_j(t_d) - w_k(t_d)] = u_j(t_d), \tag{4}$$

with q^{-1} the shift operator meaning $q^{-1} w_j(t_d) = w_j(t_d - 1)$ and with matrices

$$\bar{B}_{\ell,j} = (-1)^\ell \textstyle\sum_{i=\ell}^{n} \binom{i}{\ell} T_s^{-i} B_{i,j}, \tag{5}$$

$$\bar{A}_{\ell,jk} = (-1)^\ell \textstyle\sum_{i=\ell}^{n-1} \binom{i}{\ell} T_s^{-i} A_{i,jk}, \tag{6}$$

where T_s is the time interval defined by $t := t_d T_s$.

Proof: Equation (2) is discretised by a similar approach as in [20] by using a backward shift (3). □

In the sequel, t is used for t_d. The expressions for the node signals (4) can be combined in a matrix equation describing the network as

$$\bar{B}(q) w(t) + \bar{A}(q) w(t) = u(t), \tag{7}$$

with $\bar{B}(q)$ and $\bar{A}(q)$ polynomial matrices in the shift operator q^{-1} and composed of elements

$$\bar{B}_{jk}(q) = \begin{cases} \sum_{\ell=0}^{n} \bar{B}_{\ell,j} q^{-\ell}, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

$$\bar{A}_{jk}(q) = \begin{cases} \sum_{m \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jm} q^{-\ell}, & \text{if } k = j \\ -\sum_{\ell=0}^{n-1} \bar{A}_{\ell,jk} q^{-\ell}, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Note that $\bar{B}(q)$ is diagonal and $\bar{A}(q)$ is Laplacian, implying that the structural properties of (2) are maintained in (7)-(9).

C. Identification set-up

In order to connect with the system identification framework formulated for dynamic networks, we will use a slightly different, but equivalent, network description for identification purposes.

Proposition 2 (Physical network): A physical network (7) with $\bar{B}(q)$ diagonal and $\bar{A}(q)$ Laplacian, can uniquely be described by

$$Q(q) w(t) = P(q) w(t) + u(t), \tag{10}$$

[1]A Laplacian matrix is a symmetric matrix with non-positive off-diagonal elements and with non-negative diagonal elements that are equal to the negative sum of all other elements in the same row (or column) [3].

with diagonal polynomial matrix $Q(q) := \bar{B}(q) + \text{diag}\big(\bar{A}(q)\big)$ and hollow and symmetric polynomial matrix $P(q) := -\bar{A}(q) + \text{diag}\big(\bar{A}(q)\big)$, with $\text{diag}\big(\bar{A}(q)\big)$ the diagonal of $\bar{A}(q)$.

Proof: The definitions of $Q(q)$ and $P(q)$ show that $u(t) = \big(Q(q) - P(q)\big)w(t) = \big(\bar{B}(q) + \bar{A}(q)\big)w(t)$. □

Note that there exits a one-to-one relationship between $\big(\bar{A}(q), \bar{B}(q)\big)$ and $\big(P(q), Q(q)\big)$.

In the identification setting as will be considered, the node signals might be affected by a user-applied excitation signal and subject to a disturbance signal. This is achieved by splitting the input signal as $u(t) := Fr(t) + C(q)e(t)$ with F a known binary and diagonal matrix, $C(q)$ a rational matrix, $r(t)$ a known excitation signal and $e(t)$ a stationary white noise process. Applying this partitioning to (10) gives the following identification set-up.

Definition 2 (Identification set-up): The physical network that will be considered during identification is defined as

$$Q(q)w(t) = P(q)w(t) + Fr(t) + C(q)e(t), \tag{11}$$

with

- $Q(q) \in \mathcal{Q} := \{Q \in \mathbb{R}^{L\times L}[q^{-1}] \mid Q_{ij} = 0 \text{ for } i \neq j\}$
- $P(q) \in \mathcal{P} := \{P \in \mathbb{R}^{L\times L}[q^{-1}] \mid P_{ij} = P_{ji},\ P_{ii} = 0\ \forall i, j\}$
- $F \in \mathcal{F} := \{F \in \mathbb{R}^{L\times L} \mid F_{ij} = 0 \text{ for } i \neq j,\ F_{ii} \in \{0, 1\}\ \forall i\}$
- $C(q) \in \mathcal{C} := \{C \in \mathbb{R}^{L\times L}(q) \mid C \text{ monic, stable and stably invertible}\}$

Further, the network is assumed to be well-posed and stable, implying that $(Q(q) - P(q))^{-1}$ exists and is proper and stable. □

In order to address the questions formulated in Section I, we will now show how this network description can be written as a so-called module representation, which is typically used in prediction error identification [5].

III. Module representation

A commonly used description of dynamic networks is the module representation [5], in which a network is considered to be the interconnection of modules through measured node signals. Every node signal $w_j(t)$ is described by

$$w_j(t) = \sum_{k\in\mathcal{N}_j} G_{jk}(q)w_k(t) + R_{jj}(q)r_j(t) + \sum_{p=1}^{L} H_{jp}(q)e_p(t), \tag{12}$$

where $G_{jk}(q)$, $R_{jj}(q)$ and $H_{jp}(q)$ are proper rational transfer functions, $r_j(t)$ are known external excitation signals and $e_p(t)$ are white noises. The module representation does not allow for self-loops, implying that $G_{jj}(q) = 0$. The expressions for the node signals (12) can be combined in a matrix equation describing the network as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t), \tag{13}$$

with matrices $G(q)$, $R(q)$ and $H(q)$ composed of elements $G_{jk}(q)$, $R_{jj}(q)$ and $H_{jp}(q)$, respectively, and where $w(t)$, $r(t)$ and $e(t)$ are vectorised versions of $w_j(t)$, $r_j(t)$ and $e_p(t)$, respectively. Note that $G(q)$ is hollow, $R(q)$ is diagonal and $H(q) \in \mathcal{C}$. In addition, $\big(I - G(q)\big)^{-1}$ must be stable to ensure network stability and $I - G(q)$ must be proper and full rank to ensure well-posedness of the network.

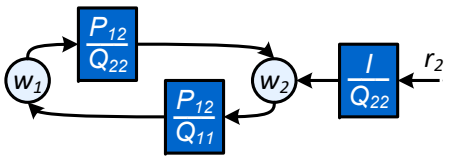

Fig. 2. Module representation of a physical network.

The relationship between the module representation and physical networks is as follows.

Definition 3 (Equivalent network models): A physical network (11) and a module representation (13) are called equivalent if the following equalities hold

$$G(q) = Q^{-1}(q)P(q), \tag{14}$$
$$R(q) = Q^{-1}(q)F, \tag{15}$$
$$H(q) = Q^{-1}(q)Q_0C(q), \tag{16}$$

with $Q_0 = \lim_{q\to\infty} Q(q)$. □

As a result, physical networks lead to module representations that satisfy particular symmetric properties: in the factorisations (14)-(16), $G_{jk}(q)$ and $G_{kj}(q)$ have the same numerator for all j, k; $G_{jk}(q)$ and $R_{jj}(q)$ have the same denominator for all k; $G_{jk}(q)$ and $H_{jj}(q)$ have the same denominator for all k if $C(q)$ is polynomial.

The structure of $G(q)$ and $R(q)$ for a physical network with two nodes is illustrated by Figure 2. It shows that the modules $G_{12}(q) = \frac{P_{12}(q)}{Q_{11}(q)}$ and $G_{21}(q) = \frac{P_{12}(q)}{Q_{22}(q)}$ between w_1 and w_2 have the same numerator related to their interconnection and a different denominator related to the node they enter. It can also be seen that both paths entering node w_2 indeed have the same denominator. Since $G_{12}(q)$ and $G_{21}(q)$ have the same numerator, they will either be both present or both absent, which is in accordance with the fact that they represent a single physical interconnection.

Furthermore, the connections to the earth are only present in the denominators, because they are only present in $Q(q)$. This means that they do not have an effect on the topology in the module representation, although they are part of the topology in the physical network.

Next, the relationship between the module representation and physical networks is specified further, leading to a unique mapping between the models.

Lemma 1 (Left matrix fraction description (LMFD)): For two left co-prime matrices $Q(q) \in \mathcal{Q}$ and $P(q) \in \mathcal{P}$, the LMFD $Q(q)^{-1}P(q)$ is unique up to a scalar factor.

Proof: According to [21], the LMFD of any two polynomial and left co-prime matrices is unique up to a unimodular matrix multiplication. In order to preserve diagonality of $Q(q)$ and symmetry of $P(q)$, the unimodular matrix is restricted to be diagonal with equal elements. □

Proposition 3 (Unique equivalent network models): Given a module representation (13) with $G(q) \in \mathcal{G} := \{G \in$

$\mathbb{R}^{L\times L}(q) \mid \exists Q \in \mathcal{Q}, P \in \mathcal{P}$ that satisfy $G = Q^{-1}P$ with Q, P left co-prime$\}$, $R(q) \in \mathcal{R} := \{R \in \mathbb{R}^{L\times L}(q) \mid \exists Q \in \mathcal{Q}, F \in \mathcal{F}$ that satisfy $R = Q^{-1}F\}$ and $H(q) \in \mathcal{C}$, there exists a unique equivalent network model (11) with $\big(Q(q), P(q), F, C(q)\big) \in \mathcal{Q}\times\mathcal{P}\times\mathcal{F}\times\mathcal{C}$ if

- $Q(q)$ and $P(q)$ are left co-prime, and
- F is non-zero.

Proof: According to Lemma 1, the LMFD (14) is unique up to a scalar factor if $Q(q)$ and $P(q)$ are left co-prime. If in addition F is non-zero, this scalar factor is fixed to 1 in order to preserve binarity in F. $C(q) \in \mathcal{C}$ is uniquely obtained from (16). □

IV. Full network identification

The module representation of a physical network can now be used to identify a dynamic network on the basis of measured data. The main difference with a general prediction error network identification problem [8], is that the symmetric structure of the interconnections has to be accommodated. This symmetry can simply be encoded in the parameterised model set that will be used for identification. This identification can be directed towards identifying particular dynamic modules while the topology of the network is given (it is known which nodes are interconnected) or for identifying a full network in which all interconnections are being identified.

Definition 4 (Data generating network): Consider a data generating network $\mathcal{S}$ (11), defined according to

$$Q^0(q)w(t) = P^0(q)w(t) + F^0r(t) + C^0(q)e(t), \quad (17)$$

with $Q^0(q) \in \mathcal{Q}$, $P^0(q) \in \mathcal{P}$, $F^0 \in \mathcal{F}$, $C^0(q) \in \mathcal{C}$, external excitations $r(t)$ being uncorrelated with white noise process $e(t)$ with bounded moments of order higher than 4[2]. □

Definition 5 (Network model structure): A network model structure used for identifying (17) is defined as a set of parameterised matrices

$$\mathcal{M}(\theta) := \{Q(q,\theta), P(q,\theta), F, C(q,\theta), \theta \in \Theta\}, \quad (18)$$

with $Q(q,\theta) \in \mathcal{Q}$, $P(q,\theta) \in \mathcal{P}$, $C(q,\theta) \in \mathcal{C}$ and with known $F = F^0$. □

Since $Q(q,\theta)$ is not monic, special attention is required for the identification set-up. We denote the parameterised residual[3] $\varepsilon(t,\theta)$ as

$$\varepsilon(t,\theta) := C^{-1}(q,\theta)\left[Q(q,\theta) - P(q,\theta)\right]w(t) - C^{-1}(q,\theta)Fr(t). \quad (19)$$

[Q_0(\theta)-P_0(\theta)]^{-1}

[Q_0(\theta)-P^0(\theta)]^{-1}

Theorem 1 (Joint-direct method): Consider a network that has generated data according to (17) with $C^0(q)$ polynomial and an "ARMAX" network model structure according to (18) with $C(q,\theta)$ being polynomial. The joint-direct method with identification criterion

$$\hat{\theta}_N = \arg\min_\theta \frac{1}{N}\sum_{t=1}^{N} \varepsilon^\top(t,\theta)\varepsilon(t,\theta), \quad (20)$$

with $\varepsilon(t,\theta)$ given by (19), results in consistent estimates of the transfer functions $G(q,\hat{\theta}_N)$, $R(q,\hat{\theta}_N)$ and $H(q,\hat{\theta}_N)$ determined by

$$G(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)P(q,\hat{\theta}_N), \quad (21a)$$
$$R(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)F, \quad (21b)$$
$$H(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)Q_0(\hat{\theta}_N)C(q,\hat{\theta}_N), \quad (21c)$$

provided that

- The data generating network $\mathcal{S}$ is in the model set $\mathcal{M}$,
- The data is sufficiently informative [23], and
- The model set $\mathcal{M}(\theta)$ is globally network identifiable in $\mathcal{S}$ [12].

If in addition, there exists at least one external excitation signal $r(t)$ (i.e. $F^0 \neq 0$), and Q^0, P^0 are left co-prime, then the consistency result also applies to the polynomials $Q(q,\hat{\theta}_N)$, $P(q,\hat{\theta}_N)$, $C(q,\hat{\theta}_N)$.

Proof: If $C^0(q)$ and $C(q,\theta)$ are polynomial, the network model structure is "ARMAX" and the identification problem is similar to the joint-direct identification method in [23], for the particular situation that P^0 and $P(q,\theta)$ are symmetric. Consistency of the polynomial terms follows from Proposition 3. □

Remark 1: The estimated module dynamics are proper but not necessarily strictly proper. This has consequences for the conditions under which the network is identifiable. In the presence of algebraic loops, additional conditions on the presence of excitation signals need to be satisfied for achieving network identifiability [12], [23]. □

Proposition 4 (Linear regression): Consider a network that has generated data according to (17) with $C^0(q) = I$, $F^0 \neq 0$, and an "ARX" network model structure according to (18), that is $C(q,\theta) = I$, with $\mathcal{S} \in \mathcal{M}$. Under the conditions of Theorem 1, the physical components represented by the polynomials $P(q)$ and $Q(q)$ are consistently estimated through a linear regression according to

Q_0^0=I, P_0^0=0,

$$\hat{\theta}_N = \left[\frac{1}{N}\sum_t \varphi(t)\varphi^\top(t)\right]^{-1}\left[\frac{1}{N}\sum_t \varphi(t)Fr(t)\right], \quad (22)$$

with $\varphi(t)$ defined as

$$\varphi^\top(t) = \big(\varphi^\top_{Q_0}(t)\ \cdots\ \varphi^\top_{Q_n}(t)\ \ \varphi^\top_{P_0}(t)\ \cdots\ \varphi^\top_{P_{n-1}}(t)\big), \quad (23)$$

with $\varphi^\top_{Q_i}(t) = \mathrm{diag}\big(q^{-i}w_1(t)\ \cdots\ q^{-i}w_L(t)\big)$ and

$$\varphi^\top_{P_i}(t) = -\left(\begin{bmatrix} Z_{0,L-1} \\ q^{-i}W_{2,L}(t) \\ q^{-i}d_{L-1}(w_1(t)) \end{bmatrix} \begin{bmatrix} Z_{1,L-2} \\ q^{-i}W_{3,L}(t) \\ q^{-i}d_{L-2}(w_2(t)) \end{bmatrix} \cdots \begin{bmatrix} Z_{L-2,1} \\ q^{-i}W_{L,L}(t) \\ q^{-i}d_1(w_{L-1}(t)) \end{bmatrix}\right),$$

where $Z_{j,k}$ is a matrix of size $j\times k$ with all elements equal to 0, $W_{j,L}(t) = \big(w_j(t)\ \cdots\ w_L(t)\big)$ and $d_i\big(w_j(t)\big)$ is a square and diagonal matrix of size $i\times i$ (with $i = L-j$) with all elements on the diagonal equal to $w_j(t)$.

[Q_0(\theta)-P^0(\theta)]

Proof: If $C^0(q) = I$ and $C(q,\theta) = I$, the network model structure is "ARX" and the residual $\varepsilon(t,\theta)$ is affine in the parameters θ, meaning that it can be written as

$$\varphi^\top(t)\theta - Fr(t) = \left[Q(q,\theta) - P(q,\theta)\right]w(t) - Fr(t), \quad (24)$$
$$= \left[\sum_{i=0}^{n} Q_i(\theta)q^{-i} - \sum_{i=0}^{n-1} P_i(\theta)q^{-i}\right]w(t) - Fr(t), \quad (25)$$

[2]This is the typical assumption for consistency of prediction error estimation [22].

[3]~~The parameterised residual $\varepsilon(t,\theta)$ is equal to the prediction error of $Q_0(\theta)w(t)$, that is $\varepsilon(t,\theta) = Q_0(\theta)w(t) - Q_0(\theta)\hat{w}(t|t-1;\theta)$.~~

where the structure of $Q(q,\theta)$ and $P(q,\theta)$ is retained in $Q_i(\theta)$ and $P_i(\theta)$, respectively, and with parameter vector

$$\theta = \begin{pmatrix} \theta_{Q_0}^\top & \cdots & \theta_{Q_n}^\top & \theta_{P_0}^\top & \cdots & \theta_{P_{n-1}}^\top \end{pmatrix}^\top, \tag{26}$$

with $\theta_{Q_i} = \begin{pmatrix} Q_{i,1} \cdots Q_{i,L} \end{pmatrix}^\top$, $\theta_{P_i} = \begin{pmatrix} P_{i,1} \cdots P_{i,L(L-1)/2} \end{pmatrix}^\top$, where these parameter vectors parameterise the matrices according to $Q_i(\theta) = \text{diag}(\theta_{Q_i})$ and

$$P_i(\theta) = \begin{pmatrix} 0 & P_{i,1} & P_{i,2} & \cdots & P_{i,L-1} \\ \star & 0 & P_{i,L} & \cdots & P_{i,2L-3} \\ \star & \star & 0 & \ddots & \vdots \\ \star & \star & \star & 0 & P_{i,L(L-1)/2} \\ \star & \star & \star & \star & 0 \end{pmatrix}, \tag{27}$$

where the elements $\star$ follow from the symmetry. □

The symmetry in $P(q,\theta)$ is included in the parametrisation and therefore, the resulting optimisation problem is unconstrained. That is, the identification procedure of the network results in an unconstrained least squares optimisation problem in which the structure of $P(q,\theta)$ is taken into account.

V. Local network identification

A. Problem definition

The local identification problem in networks is in general formulated as the objective to identify a single module in the network [5], [9], [10], [11]. However, due to the symmetry in the symmetric couplings in the networks considered in this paper, it is attractive to formulate the local identification problem slightly different.

Definition 6 (Local identification problem): The local identification problem concerns the identification of a single coupling between two nodes in the physical network. □

A single coupling in the physical network is described by two modules in the module representation, meaning that the objective is to identify two modules simultaneously. For the nodes w_j and w_k, these modules are $G_{jk}(q) = Q_{jj}^{-1}(q)P_{jk}(q)$ and $G_{kj}(q) = Q_{kk}^{-1}(q)P_{kj}(q)$, which contain the full information on how the nodes w_j and w_k interact with each other. Due to the symmetry in $P(q)$, $P_{jk}(q) = P_{kj}(q)$ and hence, this identification problem concerns the identification of three polynomials: $Q_{jj}(q)$, $Q_{kk}(q)$, and $P_{jk}(q)$. In order to take account of the symmetric properties in physical network interconnections, the currently available methods for local module identification need to be reconsidered.

B. Immersion

In order to decide which of the node signals need to be taken into account for the identification of a local module, the procedure introduced in [10] suggests to remove (immerse) node signals from the network, while adapting the dynamic modules such that the retained node signals are kept invariant. If node signals can be removed (immersed) while the target module remains invariant, the immersed node signals can be discarded in the identification. The results for target module invariance under immersion are described in [10]. Applying these results to the two modules $G_{jk}(q)$ and $G_{kj}(q)$ simultaneously, leads to the following conditions on the graph of the network:

- Every loop around w_j and every loop around w_k needs to pass through a retained node.
- Every parallel path[4] from w_j to w_k and every parallel path from w_k to w_j needs to pass through a retained node.

Because of the symmetric properties of a physical network, these conditions lead to the following result.

Proposition 5 (Immersion in physical network): Immersion in module representations of physical networks keeps two modules $G_{jk}(q)$ and $G_{kj}(q)$ invariant if w_j and w_k and all their neighbour nodes are retained.

Proof: Since $P(q)$ is symmetric, all nodes are bilaterally connected. Therefore, all loops around w_j and all loops around w_k contain a retained node if and only if all neighbour nodes of w_j and all neighbour nodes of w_k are retained, respectively. As a consequence, all parallel paths from w_j to w_k and from w_k to w_j contain a retained node as well. □

This proposition shows that for identification of a single coupling between two nodes w_j and w_k, all nodes that are not neighbours of w_j and w_k can be immersed from the network.

Remark 2: Note that immersion of nodes can lead to higher order dynamics in the modules after immersion. This order increase can simply be accommodated for in the module representation of physical networks, i.e. the modules do not need to be restricted to second order dynamics. □

Remark 3: By using immersion, nodes are removed from the network and the identification problem can be solved locally, meaning that not all nodes are needed and not all dynamics in the network need to be modelled in order to identify the dynamics of a specific coupling in the network. □

C. Identification set-up

After immersion, the system representation is as follows:

$$\begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{jk}(q) & G_{j\mathcal{L}}(q) \\ G_{kj}(q) & 0 & G_{k\mathcal{L}}(q) \\ \breve{G}_{\mathcal{L}j}(q) & \breve{G}_{\mathcal{L}k}(q) & \breve{G}_{\mathcal{L}\mathcal{L}}(q) \end{bmatrix} \begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \end{bmatrix} + \breve{R}(q)r(t) + \breve{H}(q)e(t), \tag{28}$$

where $w_{\mathcal{L}}(t)$ is the set of node signals that are being retained, i.e. the neighbour node signals of $w_j(t)$ and $w_k(t)$. Following the local identification approaches in [11] and [23], (28) is used for locally identifying the two modules $G_{jk}(q)$ and $G_{kj}(q)$. The output signals of this local identification problem are $w_y(t) = \begin{bmatrix} w_j(t) & w_k(t) \end{bmatrix}^\top$ and the input signals are $w_m(t) = \begin{bmatrix} w_y^\top(t) & w_{\mathcal{L}}^\top(t) \end{bmatrix}^\top$ and $\tilde{r}(t) = \begin{bmatrix} r_j(t) & r_k(t) \end{bmatrix}^\top$. This means that the first two rows of (28) will be estimated. In order to do so, the disturbances affecting $w_{\mathcal{L}}(t)$ have to be uncorrelated with the disturbances affecting $w_j(t)$ and $w_k(t)$, but the disturbances affecting $w_j(t)$ and $w_k(t)$ can be mutually correlated, as well as the disturbances affecting $w_{\mathcal{L}}(t)$.

[4] A parallel path from w_j to w_k is a path from w_j to w_k that does not pass through $G_{kj}(q)$.

This identification set-up has the parameterised residual

$$\varepsilon(t,\theta) = \tilde{C}^{-1}(q,\theta)\left[\tilde{Q}(q,\theta)w_y(t) - \tilde{P}(q,\theta)w_m(t)\right] - \tilde{C}^{-1}(q,\theta)\tilde{F}\tilde{r}(t), \quad (29)$$

with $\tilde{Q}(q,\theta) = \mathrm{diag}\big(Q_{jj}(q,\theta), Q_{kk}(q,\theta)\big)$,

$$\tilde{P}(q,\theta) = \begin{bmatrix} 0 & P_{jk}(q,\theta) & P_{j\mathcal{L}}(q,\theta) \\ P_{kj}(q,\theta) & 0 & P_{k\mathcal{L}}(q,\theta) \end{bmatrix}, \quad (30)$$

$$\tilde{C}(q,\theta) = \begin{bmatrix} C_{jj}(q,\theta) & C_{jk}(q,\theta) \\ C_{kj}(q,\theta) & C_{kk}(q,\theta) \end{bmatrix}, \quad (31)$$

$\tilde{F} = \mathrm{diag}\big(F_{jj}, F_{kk}\big)$.

The results in [11] and [23] for single module identification can now be used to formulate conditions for consistent estimation of the two modules $G_{jk}(q)$ and $G_{kj}(q)$ simultaneously. If in the original network, the disturbances affecting $w_{\mathcal{L}}(t)$ are uncorrelated to the disturbances affecting $w_j(t)$ and $w_k(t)$, then the identification set-up described above will lead to consistent estimates $G_{jk}(q,\hat{\theta}_N)$ and $G_{kj}(q,\hat{\theta}_N)$ under the usual conditions on data informativity (sufficient excitation), as formalised in [11].

Once $G_{jk}(q)$ and $G_{kj}(q)$ have been identified consistently, the physical components in the network model (2) can be retrieved from the estimated model $\big(\tilde{Q}(q), \tilde{P}(q), \tilde{F}, \tilde{C}(q)\big)$.

Remark 4: The presented results for both full and local network identification are based on network description (11). Instead, (7) can also be used, leading to similar results. □

VI. Conclusion

The undirected network description of physical networks has been extended by allowing for higher order diffusive couplings. Undirected network descriptions of physical systems with diffusive couplings can be represented as directed dynamic networks with particular structural properties. This allows for effective identification of the global and local properties of the physical network.

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