Prediction error identification in physical networks

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***Abstract*— System identification problems utilizing a prediction error approach are typically considered in an input/output setting, where a directional cause-effect relationship is presumed and transfer functions are used to estimate the causal relationships. In more complex interconnection structures, as e.g. appearing in dynamic networks, the cause-effect relationships can be encoded by a directed graph. Physical dynamic networks are most commonly described by diffusive couplings between node signals, implying that cause-effect relationships between node signals are symmetric and therefore can be represented by an undirected graph. This paper shows how prediction error methods developed for linear dynamic networks can be configured to identify components in (undirected) physical networks with known topology.**

I. Introduction

Physical networks are only one example of dynamic networks, which are interconnections of dynamic units. Dynamic networks receive increasing attention from a variety of scientific fields, since systems are growing in complexity and size. Other examples of dynamic networks are biological and chemical processes, neural networks, consensus networks, synchronisation, social interactions, the Internet, the stock market and multi-agent systems [1], [2], [3].

A framework for system identification in dynamic networks has been introduced in [4], by extending classical closed-loop prediction error methods. Other developments focus on topology estimation [5], [6], full network identification [7], local module identification [8], [9], [10] and network identifiability [11], [12], [13]. In this framework, dynamic networks are considered to consist of directed interconnections of dynamic modules that can be of any dynamic order. In contrast, physical networks are typically considered as undirected dynamic interconnections between node signals, where the interconnections represent diffusive couplings [14] and the model is typically described by a vector difference equation of maximum second order. The most well-known example is a mechanical mass-spring-damper system, with positions of masses as node (state) signals and the dynamics being described by a second order vector difference equation. Identification of these physical models can be done by conversion of the model into a state space form, after which matrix transformations [15], [16] or eigenvalue decompositions [17], [18] are being applied to estimate the dynamics of the model. However, during these operations the network structure in the model is generally lost.

With the objective to preserve the network structure in the models, we choose to extend the prediction error identification framework to the situation of undirected network interconnections. We will do so, by rewriting vector difference equation models in terms of module representations, without restricting the modules to be limited to second order. By exploiting the symmetrical nature of the diffusive couplings, we can build a framework for prediction error identification in physical networks and use all the insights from dynamic network identification for the physical network models too.

The concept of a diffusively coupled physical network is defined in Section II. Subsequently, the network identification setting is presented in Section III. In Section IV identification of the full network is discussed, while in Section V attention is given to the local identification results. Finally, Section VI concludes the paper.

II. Physical network

Physical networks are often described by second order differential equations. They can be considered to consists of L interconnected scalar node signals $w_j(t)$, $j = 1, \dots, L$, of which the behaviour is described according to

$$M_j\ddot{w}_j(t) + D_{j0}\dot{w}_j(t) + \sum_{k\in\mathcal{N}_j} D_{jk}\big(\dot{w}_j(t) - \dot{w}_k(t)\big) + K_{j0}w_j(t) + \sum_{k\in\mathcal{N}_j} K_{jk}\big(w_j(t) - w_k(t)\big) = u_j(t), \quad (1)$$

with $M_j > 0$, $D_{jk} \geq 0$, $K_{jk} \geq 0$, $D_{jj} = 0$, $K_{jj} = 0$, $\mathcal{N}_j$ is the set of indices of node signals $w_k(t)$ $k \neq j$ with connections to node signals $w_j(t)$, $u_j(t)$ are the external input signals and $\dot{w}_j(t)$ and $\ddot{w}_j(t)$ are the first and second order derivative of the node signals $w_j(t)$, respectively.

In a physical network, all connections are symmetric, meaning that the strength of the connection from node $w_i(t)$ to node $w_k(t)$ is equal to the strength of the connection (in opposite direction) from node $w_k(t)$ to node $w_i(t)$. More precise, the interconnections of the node signals are diffusive couplings, which emerge in (1) from the symmetric connections: $D_{jk} = D_{kj}$ and $K_{jk} = K_{kj}$.

Proposition 1 (Second order physical network): A second order physical network is described by

$$\bar{M}\ddot{w}(t) + \bar{D}\dot{w}(t) + \bar{K}w(t) = u(t), \quad (2)$$

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with matrices $\bar{M}$, $\bar{D}$ and $\bar{K}$ composed of elements containing the physical parameters in the network as

$$\bar{M}_{jk} = \begin{cases} M_j, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

$$\bar{D}_{jk} = \begin{cases} D_{j0} + \sum_{m \in \mathcal{N}_j} D_{jm}, & \text{if } k = j \\ -D_{jk}, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

$$\bar{K}_{jk} = \begin{cases} K_{j0} + \sum_{m \in \mathcal{N}_j} K_{jm}, & \text{if } k = j \\ -K_{jk}, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

and $w(t)$ and $u(t)$ are vectorised versions of $w_j(t)$ and $u_j(t)$, respectively.

Proof: The expressions for the node signals (1) can be stacked for $w_1(t), \dots, w_L(t)$ and combined in a matrix equation. ■

Hence, $\bar{M}$ is a diagonal matrix, $\bar{D}$ and $\bar{K}$ are a diagonal matrix plus a Laplacian matrix, which is a symmetric matrix with non-positive off-diagonal elements and with non-negative diagonal elements that are equal to the negative sum of all other elements in the same row (or column) [3].

A. *Mass-spring-damper network*

An example of a physical network with diffusive couplings is a mass-spring-damper system, in which the nodes are masses interconnected through springs and dampers. The couplings between the masses are diffusive, because springs and dampers are symmetric components. To be more precise, for two masses m_1 and m_2 with position $x_1(t)$ and $x_2(t)$, respectively, the strength of a connection through a spring K is equal to $K\big(x_1(t) - x_2(t)\big)$ seen from m_1 and equal to $-K\big(x_1(t) - x_2(t)\big)$ seen from m_2.

Figure 1 shows an example of a mass-spring-damper system. The positions of the masses are the signals of interest and therefore chosen to be the node signals: $w_j(t) := x_j(t)$. For $k \neq 0$, D_{jk} and K_{jk} represent the dampers and springs interconnecting the masses M_j and M_k, while D_{j0} and K_{j0} represent the dampers and springs connecting to the earth.

B. *Higher order network*

A physical network is typically of second order, as the mass-spring-damper system in Section II-A, but the theory can easily be extended to higher order terms.

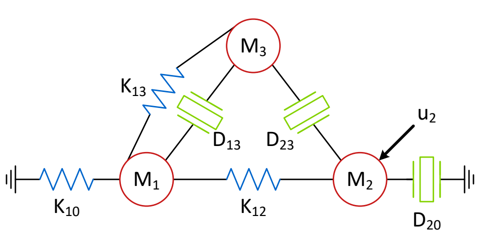

Fig. 1. A network of masses (M), dampers (D) and springs (K).

Definition 1 (Physical network): A physical network is a network consisting of L node signals $w_1(t), \dots, w_L(t)$ interconnected through diffusive couplings and with possibly connections of node signals to a ground node. The behaviour of the node signals $w_j(t)$, $j = 1, \dots, L$, is described by

$$\sum_{\ell=0}^{n} B_{\ell,j} w_j^{(\ell)}(t) + \sum_{k \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} A_{\ell,jk} [w_j^{(\ell)}(t) - w_k^{(\ell)}(t)] = u_j(t), \quad (6)$$

with $B_{n,j} > 0$, $B_{\ell,j} \geq 0$, $A_{\ell,jk} \geq 0$, $A_{\ell,jj} = 0$, $A_{\ell,jk} = A_{\ell,kj}$, and where $w_j^{(\ell)}(t)$ is the ℓ-th derivative of $w_j(t)$. ■

The graphical interpretation of the coefficients is as follows: $B_{n,j}$ represent the components intrinsically related to the nodes $w_j(t)$, $B_{\ell,j}$ with $\ell \neq n$ represent the components connecting the node $w_j(t)$ to the ground node (or earth) and $A_{\ell,jk}$ represent the components in the diffusive couplings between the node signals $w_j(t)$ and $w_k(t)$. In addition, every matrix A_ℓ composed of elements $A_{\ell,jk}$ is a Laplacian matrix representing an undirected graph of a specific physical component (i.e. of the diffusive couplings of a specific order).

Proposition 2 (Physical network): A physical network (6) can be described by

$$B(p)w(t) + A(p)w(t) = u(t), \quad (7)$$

with $B(p)$ and $A(p)$ polynomial matrices in the difference operator $p = \frac{d}{dt}$ and composed of elements

$$B_{jk}(p) = \begin{cases} \sum_{\ell=0}^{n} B_{\ell,j} p^\ell, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

$$A_{jk}(p) = \begin{cases} \sum_{m \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} A_{\ell,jm} p^\ell, & \text{if } k = j \\ -\sum_{\ell=0}^{n-1} A_{\ell,jk} p^\ell, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

Proof: The expressions for the node signals (6) can be stacked for $w_1(t), \dots, w_L(t)$ and combined in a matrix equation. ■

Note that $B(p)$ is diagonal and $A(p)$ is Laplacian.

C. *Discretisation*

There are several options for identifying a physical network from experimental data. One option would be to identify the network in continuous time. Here, the approach is to identify the network in discrete time in order to connect with the system identification framework formulated for dynamic networks. For this purpose, the continuous time network is converted to an equivalent discrete-time network.

Proposition 3 (Discrete time): By using the approximation

$$\frac{dw(t)}{dt} = \frac{w(t_d T_s) - w((t_d - 1)T_s)}{T_s}, \quad (10)$$

the continuous time physical network (6) can be described in discrete time by

$$\sum_{\ell=0}^{n} \bar{B}_{\ell,j} q^{-\ell} w_j(t_d) + \sum_{k \in \mathcal{N}_j} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jk} q^{-\ell} [w_j(t_d) - w_k(t_d)] = u_j(t_d), \quad (11)$$

with q^{-1} the shift operator meaning $q^{-1}w_j(t_d) = w_j(t_d - 1)$ and with matrices

$$\bar{B}_{\ell,j} = (-1)^\ell \sum_{i=\ell}^{n} \binom{i}{\ell} T_s^{-i} B_{i,j}, \tag{12}$$

$$\bar{A}_{\ell,jk} = (-1)^\ell \sum_{i=\ell}^{n-1} \binom{i}{\ell} T_s^{-i} A_{i,jk}, \tag{13}$$

where T_s is the time interval defined by $t := t_d T_s$.

Proof: Equation (6) is discretised by a similar approach as in [19] by using a backward shift (10). ■

In the sequel, t is used for t_d. The expressions for the node signals (11) can be combined in a matrix equation describing the network as

$$\bar{B}(q)w(t) + \bar{A}(q)w(t) = u(t), \tag{14}$$

with $\bar{B}(q)$ and $\bar{A}(q)$ polynomial matrices in the shift operator q and composed of elements

$$\bar{B}_{jk}(q) = \begin{cases} \sum_{\ell=0}^{n} \bar{B}_{\ell,j} q^{-\ell}, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

$$\bar{A}_{jk}(q) = \begin{cases} \sum_{m\in\mathcal{N}_j} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jm} q^{-\ell}, & \text{if } k = j \\ -\sum_{\ell=0}^{n-1} \bar{A}_{\ell,jk} q^{-\ell}, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Note that $\bar{B}(q)$ is diagonal and $\bar{A}(q)$ is Laplacian, i.e. the structure of $B(p)$ and $A(p)$ is preserved in $\bar{B}(q)$ and $\bar{A}(q)$, respectively, while there exits a one-to-one relationship between $\big(A(p), B(p)\big)$ and $\big(\bar{A}(q), \bar{B}(q)\big)$.

D. Identification setup

In the identification setting as will be considered, the nodes signals might be affected by a user-applied excitation signal, while all measured node signals are subject to disturbance signals that are mutually uncorrelated. This is represented by splitting the input signal $u(t)$ into two contributions: the known excitation signal and the unknown disturbance signal. Applying this partitioning to (14), the network description becomes

$$\bar{B}(q)w(t) + \bar{A}(q)w(t) = Fr(t) + C(q)e(t), \tag{17}$$

where F is a binary diagonal matrix, i.e. a sub-matrix of the identity matrix, and $C(q)$ is a diagonal rational matrix restricted to be monic. The disturbance $C(q)e(t)$ is a realisation of a stationary stochastic process with diagonal spectral density. In the next section it will be shown how this network representation can be written in a so-called module representation, which is typically used in prediction error identification [4].

III. Module representation

A standard description of dynamic networks is the module representation [4], in which a network is considered to be the interconnection of modules through measured node signals. Every node signal $w_j(t)$ is described by

$$w_j(t) = \sum_{k\in\mathcal{N}_j} G_{jk}(q)w_k(t) + R_{jj}(q)r_j(t) + H_{jj}(q)e_j(t), \tag{18}$$

where $G_{jk}(q)$, $R_{jj}(q)$ and $H_{jj}(q)$ are proper rational transfer functions, $r_j(t)$ are known external excitation signals and $e_j(t)$ are white noises. The module representation does not allow for self-loops, implying that $G_{jj}(q) = 0$.

The expressions for the node signals (18) can be combined in a matrix equation describing the network as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t), \tag{19}$$

with matrices $G(q)$, $R(q)$ and $H(q)$ composed of elements $G_{jk}(q)$, $R_{jj}(q)$ and $H_{jj}(q)$, respectively, and where $w(t)$, $r(t)$ and $e(t)$ are vectorised versions of $w_j(t)$, $r_j(t)$ and $e_j(t)$, respectively. Note that $G(q)$ is hollow and $R(q)$ and $H(q)$ are diagonal. In addition, $H(q)$ is restricted to be monic, stable and stably invertible.

Proposition 4 (Module representation): A physical network (17) with (15) and (16) can be described in the module representation

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)\tilde{e}(t), \tag{20}$$

with $\tilde{e}(t) = Q_0^{-1}e(t)$, where Q_0 is a diagonal matrix composed of the constant elements of $Q(q)$ (the elements of $Q(q)$ related to q^0) and with

$$G(q) = Q^{-1}(q)P(q), \tag{21a}$$

$$R(q) = Q^{-1}(q)F, \tag{21b}$$

$$H(q) = Q^{-1}(q)Q_0C(q), \tag{21c}$$

with diagonal $Q(q)$ and hollow and symmetric $P(q)$ being polynomial matrices composed of elements

$$Q_{jk}(q) = \begin{cases} \displaystyle\sum_{\ell=0}^{n} \bar{B}_{\ell,j} q^{-\ell} + \sum_{m\in\mathcal{N}_j} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jm} q^{-\ell}, & \text{if } k = j \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

$$P_{jk}(q) = \begin{cases} \sum_{\ell=0}^{n-1} \bar{A}_{\ell,jk} q^{-\ell}, & \text{if } k \in \mathcal{N}_j \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

and with F and $C(q)$ as in (17).

Proof: The diagonal elements of $\bar{B}(q)$ and $\bar{A}(q)$ are collected in $Q(p)$ and the remaining elements of $\bar{A}(q)$ yield $P(q)$, that is $Q_{jj}(q) = \bar{B}_{jj}(q) + \bar{A}_{jj}(q)$ and $P_{jk}(q) = \bar{A}_{jk}(q)$ for $k \neq j$. All entries of $G(q)$, $R(q)$ and $H(q)$ are proper rational transfer functions. $G(q)$ is hollow and $R(q)$ and $H(q)$ are diagonal, because $P(q)$ is hollow and $Q(q)$, F, and $C(q)$ are diagonal. $H(q)$ is monic due to the scaling with Q_0. ■

Note that there exits a one-to-one relationship between $\big(\bar{A}(q), \bar{B}(q)\big)$ and $\big(P(q), Q(q)\big)$. As a result of Proposition 4, physical networks lead to module representations that satisfy particular symmetric properties. This is specified next.

Proposition 5 (Symmetric properties): The module representation (20) of a physical network has the following symmetric properties

1) $G_{jk}(q)$ and $G_{kj}(q)$ have the same numerator.
2) $G_{jk}(q)$ and $R_{jj}(q)$ have the same denominator for all k.

3) $G_{jk}(q)$ and $H_{jj}(q)$ have the same denominator for all k if $C(q)$ is polynomial.

Proof:

1) Because $Q(q)$ and $P(q)$ are polynomial in (21a), the numerators of $G_{jk}(q)$ and $G_{kj}(q)$ are completely determined by $P_{jk}(q)$ and $P_{kj}(q)$, respectively, which are the same because $P(q)$ is symmetric.
2) Because $Q(q)$ and $P(q)$ are polynomial and F is constant in (21a) and (21b) and $Q(q)$ is diagonal, the denominators of $G_{jk}(q)$ and $R_{jj}(q)$ are completely determined by $Q_{jj}(q)$ for all k.
3) Because $Q(q)$, $P(q)$ and $C(q)$ are polynomial in (21a) and (21c) and $Q(q)$ is diagonal, the denominators of $G_{jk}(q)$ and $H_{jj}(q)$ are completely determined by $Q_{jj}(q)$ for all k.

■

The structure of $G(q)$ and $R(q)$ for a physical network (20) with two nodes is illustrated by Figure 2. It shows that the modules $G_{12}(q) = \frac{P_{12}(q)}{Q_{11}(q)}$ and $G_{21}(q) = \frac{P_{12}(q)}{Q_{22}(q)}$ related to the interconnection between $w_1(t)$ and $w_2(t)$ have the same numerator related to their interconnection and a different denominator related to the node signal they enter. It can also be seen that both paths entering node signal $w_2(t)$ indeed have the same denominator. Since $G_{12}(q)$ and $G_{21}(q)$ have the same numerator, they will either be both present or both absent, which is in accordance with the fact that they represent one physical interconnection.

In addition, the connections to the earth are only present in the denominators, because they are only present in $Q(q)$. This means that they do not have an effect on the topology in the module representation, although they are part of the topology in the physical network.

IV. Full network identification

The module representation of a physical network (20) can now be used to identify a dynamic network on the basis of measured data. The main difference with a general prediction error network identification problem [7], is that the symmetric structure of the interconnections has to be accommodated. This symmetry can simply be encoded in the parameterised model set that will be used for identification. This identification can be directed towards identifying the dynamic modules while the topology of the network is given (it is known which nodes are interconnected) or for a full network in which all interconnections are being identified.

Either of the two network descriptions (17) or (20)-(21) can be used for identification. Here, (20)-(21) is chosen, because it is more closely related to the module representation of

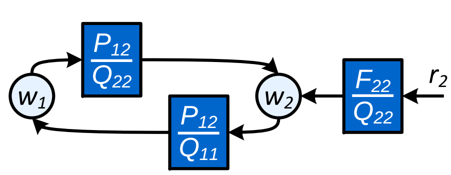

Fig. 2. Module representation of a physical network.

dynamic networks. Adopting the approach of [20], the one-step-ahead predictor of the node signals $w_j(t)$ is defined as

$$\hat{w}_j(t|t-1) := \mathbb{E}\{w_j(t)|w_j(t-1)^-; w_k(t)^-, k \neq j; r(t)^-\}, \quad (24)$$

where $w_j(t-1)^-$ reflects the past of $w_j(t)$. All node signals are considered to be output signals and at the same time input signals to other node signals. Following the approach of [20] further, results in the one-step-ahead predictor

$$\hat{w}_j(t|t-1) = \left[I - C^{-1}(q)\big(Q(q) - P(q)\big)\right] w(t) + C^{-1}(q)Fr(t). \quad (25)$$

Definition 2 (Data generating network): The data generating network of (20)-(21) is defined as

$$Q^0(q)w(t) = P^0(q)w(t) + F^0 r(t) + C^0(q)e(t), \quad (26)$$

with $Q^0(q)$, $P^0(q)$, F^0 and $C^0(q)$ satisfying the properties of $Q(q)$, $P(q)$, F and $C(q)$ in Proposition 4 and with $e(t)$ a white noise process with bounded moments of order higher than 4. ■

Definition 3 (Network model structure): A network model structure used for identifying (26) is defined as the set of parameterised polynomial matrices

$$\mathcal{M}(\theta) := \{Q(q,\theta), P(q,\theta), F, C(q,\theta), \theta \in \Theta\}, \quad (27)$$

with $Q(q,\theta)$, $P(q,\theta)$ and $C(q,\theta)$ being polynomials in the delay operator q^{-1} with the first coefficient of $Q(q,\theta)$ and $C(q,\theta)$ equal to I, F is given and equal to F^0, and $Q^{-1}(q,\theta)C(q,\theta)$ being monic, stable and stably invertible. The network model structure determines a one-step-ahead predictor $\hat{w}_j(t|t-1;\theta)$ similar to (25), which leads to the prediction error $\varepsilon(t,\theta) = w(t) - \hat{w}_j(t|t-1;\theta)$ as

$$\varepsilon(t,\theta) = C^{-1}(q,\theta)\left[Q(q,\theta) - P(q,\theta)\right] w(t) - C^{-1}(q,\theta)Fr(t). \quad (28)$$

Proposition 6 (Joint-direct method): Consider a network that has generated data according to (26) with $C^0(q)$ polynomial and an ARMAX network model structure according to (27), that is $C(q,\theta)$ is polynomial. Then the transfer functions $G(q,\hat{\theta}_N)$, $R(q,\hat{\theta}_N)$ and $H(q,\hat{\theta}_N)$ determined by

$$G(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)P(q,\hat{\theta}_N), \quad (29a)$$

$$R(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)F, \quad (29b)$$

$$H(q,\hat{\theta}_N) = Q^{-1}(q,\hat{\theta}_N)Q_0 C(q,\hat{\theta}_N), \quad (29c)$$

are consistent estimates of the transfer functions $(Q^0)^{-1}P^0, (Q^0)^{-1}F^0$, and $(Q^0)^{-1}Q_0 C^0$ respectively, through the joint-direct method according to

$$\hat{\theta}_N = \arg\min_{\theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^\top(t,\theta)\varepsilon(t,\theta), \quad (30)$$

where $\varepsilon(t,\theta)$ is given by (28), and under the conditions that the network is identifiable [12] and the external excitations $r(t)$ are persistently exciting of sufficiently high order while being uncorrelated with $e(t)$.

Proof: If $C^0(q)$ and $C(q,\theta)$ are polynomial, the network model structure is ARMAX and the identification problem is similar to the joint-direct identification method in Proposition 3 of [7], for the particular situation that P^0 and $P(q,\theta)$ are symmetric. ■

Remark 1: Note that the module dynamics that are estimated are proper but not necessarily strictly proper. This has consequences for the conditions under which the network is identifiable. In the presence of algebraic loops in the network, additional conditions on the presence of excitation signals need to be satisfied for achieving network identifiability [12], and [21][Chapter 5]. ■

Corollary 1 (Linear regression): Consider a network that has generated data according to (26) with $C^0(q) = I$ and an ARX network model structure according to (27), that is $C(q,\theta) = I$. Based on Proposition 6, the corresponding transfer functions are then consistently estimated through a linear regression according to

$$\hat{\theta}_N = \left[\frac{1}{N}\sum \varphi(t)\varphi^\top(t)\right]^{-1}\left[\frac{1}{N}\sum \varphi(t)[w(t) - Fr(t)]\right], \tag{31}$$

with $\varphi(t)$ defined as

$$\varphi^\top(t) = \begin{pmatrix}\varphi^\top_{Q_1}(t) & \cdots & \varphi^\top_{Q_n}(t) & \varphi^\top_{P_0}(t) & \cdots & \varphi^\top_{P_{n-1}}(t)\end{pmatrix}, \tag{32}$$

with

$$\varphi^\top_{Q_i}(t) = \text{diag}\begin{pmatrix}q^{-i}w_1(t) & \cdots & q^{-i}w_L(t)\end{pmatrix}, \tag{33}$$

$$\varphi^\top_{P_i}(t) = \left(\begin{bmatrix} Z_{0,L-1} \\ q^{-i}W_{2,L}(t) \\ q^{-i}d_{L-1}(w_1(t))\end{bmatrix} \quad \begin{bmatrix} Z_{1,L-2} \\ q^{-i}W_{3,L}(t) \\ q^{-i}d_{L-2}(w_2(t))\end{bmatrix} \quad \cdots \quad \begin{bmatrix} Z_{L-2,1} \\ q^{-i}W_{L,L}(t) \\ q^{-i}d_1(w_L(t))\end{bmatrix}\right), \tag{34}$$

where $Z_{j,k}$ is a matrix of size $j \times k$ with all elements equal to 0, $W_{j,L}(t) = \begin{pmatrix}w_j(t) & \cdots & w_L(t)\end{pmatrix}$ and $d_i(w_j(t))$ is a square and diagonal matrix of size $i \times i$ with all elements on the diagonal equal to $w_j(t)$.

Proof: If $C^0(q) = I$ and $C(q,\theta) = I$, the network model structure is ARX and the one-step-ahead predictor $\hat{w}_j(t|t-1;\theta)$ is affine in the parameters θ, meaning that it can be written as a linear regression

$$\varphi^\top(t)\theta - Fr(t) = \left[I - \left(Q(q,\theta) - P(q,\theta)\right)\right] w(t), \tag{35}$$

$$= \left[\sum_{i=0}^{n-1} P_i(\theta)q^{-i} - \sum_{i=1}^{n} Q_i(\theta)q^{-i}\right] w(t), \tag{36}$$

where the structure of $Q(q,\theta)$ and $P(q,\theta)$ is retained in $Q_i(\theta)$ and $P_i(\theta)$, respectively, and with parameter vector

$$\theta = \begin{pmatrix}\theta^\top_{Q_1} & \cdots & \theta^\top_{Q_n} & \theta^\top_{P_0} & \cdots & \theta^\top_{P_{n-1}}\end{pmatrix}^\top, \tag{37}$$

with

$$\theta_{Q_i} = \begin{pmatrix}Q_{i,1} & Q_{i,2} & \cdots & Q_{i,L}\end{pmatrix}^\top, \tag{38}$$

$$\theta_{P_i} = \begin{pmatrix}P_{i,1} & P_{i,2} & \cdots & P_{i,L(L-1)/2}\end{pmatrix}^\top, \tag{39}$$

where these parameter vectors parameterise the matrices according to $Q_i(\theta) = \text{diag}(\theta_{Q_i})$ and

$$P_i(\theta) = \begin{pmatrix} 0 & P_{i,1} & P_{i,2} & \cdots & P_{i,L-1} \\ \star & 0 & P_{i,L} & \cdots & P_{i,2L-3} \\ \star & \star & 0 & \ddots & \vdots \\ \star & \star & \star & 0 & P_{i,L(L-1)/2} \\ \star & \star & \star & \star & 0 \end{pmatrix}, \tag{40}$$

where the elements $\star$ follow from the symmetry. ■

The symmetry in $P(q)$ is included in the parametrisation and therefore, the resulting optimisation problem is unconstrained. That is, the identification procedure of the network results in an unconstrained least squares optimisation problem in which the structure of $P(q)$ is taken into account.

The physical components in the network, represented by $(Q^0(q), P^0(q))$ or equivalently by $(\bar{A}(q), \bar{B}(q))$ can now be determined from the estimated transfer functions (29a)-29c). However, since $Q^0(q)$ is not necessarily scaled to be monic, a consistent estimate of $(Q^0)^{-1}P_0$ will determine the polynomial numerator and denominator only up to a constant scaling factor. For components in row j of these polynomial matrices, the scaling factor is fixed to 1 if node j has an external excitation signal r_j, as in this situation the consistent estimation of $(Q^0)^{-1}F$ will uniquely fix the scaling factor.

V. Local network identification

A. Problem definition

The local identification problem in networks is in general formulated as the objective to identifying a single module in the network [4], [8], [9], [10]. However, due to the symmetry in the networks considered in this paper, it is attractive to formulate the local identification problem slightly different.

Definition 4 (Local identification problem): The local identification problem concerns the identification of an interconnection between two nodes in the physical network. ■

A single interconnection in the physical network is described by two modules in the module representation, meaning that the objective is to identify two modules simultaneously. For the interconnection between the node signals $w_j(t)$ and $w_k(t)$, these modules are $G_{jk}(q) = Q_{jj}^{-1}(q)P_{jk}(q)$ and $G_{kj}(q) = Q_{kk}^{-1}(q)P_{kj}(q)$, which contain the full information on how the node signals $w_j(t)$ and $w_k(t)$ interact with each other. Due to the symmetry in $P(q)$, $P_{jk}(q) = P_{kj}(q)$ and hence, this identification problem concerns the identification of three polynomials: $Q_{jj}(q)$, $Q_{kk}(q)$, and $P_{jk}(q)$. In order to take account of the symmetric properties in physical network interconnections, the currently available methods for local module identification need to be reconsidered.

B. Immersion

In order to decide which of the node signals to take into account for the identification of a local module, a standard procedure is to remove (immerse) node signals from the network, while guaranteeing that the target module remains invariant [9]. The standard rules for this immersion are described in [9]. Applying these rules to the two modules $G_{jk}(q)$ and $G_{kj}(q)$ simultaneously, leads to the conditions

- Every loop around $w_j(t)$ and every loop around $w_k(t)$ needs to be blocked by a retained node signal.
- Every parallel path from $w_j(t)$ to $w_k(t)$ and every parallel path from $w_k(t)$ to $w_j(t)$ needs to be blocked by a retained node signal.

Because of the symmetric properties of a physical network, these conditions lead to the following result.

Proposition 7 (Immersion in physical network): Immersion in physical networks (18) keeps two modules $G_{jk}(q)$ and $G_{kj}(q)$ invariant if all neighbour node signals of $w_j(t)$ and $w_k(t)$ are retained.

Proof: Since $P(q)$ is symmetric, all nodes are bilaterally connected. Therefore, all loops around $w_j(t)$ and all loops around $w_k(t)$ contain a retained node signal if and only if all neighbour nodes of $w_j(t)$ and all neighbour nodes of $w_k(t)$ are retained, respectively. As a consequence, all parallel paths from $w_j(t)$ to $w_k(t)$ and from $w_k(t)$ to $w_j(t)$ contain a retained node signal as well. ■

This proposition shows that for identification of a single / local interconnection, all nodes that are not neighbours of $w_j(t)$ and $w_k(t)$ can be immersed from the network. The system equations can now be rewritten as

$$\begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \\ w_{\mathcal{Z}}(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{jk}(q) & G_{j\mathcal{L}}(q) & 0 \\ G_{kj}(q) & 0 & G_{k\mathcal{L}}(q) & 0 \\ G_{\mathcal{L}j}(q) & G_{\mathcal{L}k}(q) & G_{\mathcal{L}\mathcal{L}}(q) & G_{\mathcal{L}\mathcal{Z}}(q) \\ 0 & 0 & G_{\mathcal{Z}\mathcal{L}}(q) & G_{\mathcal{Z}\mathcal{Z}}(q) \end{bmatrix} \begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \\ w_{\mathcal{Z}}(t) \end{bmatrix} + R(q)r(t) + H(q)e(t), \quad (41)$$

where $w_{\mathcal{L}}(t)$ is the set of node signals that are being retained, i.e. the neighbour node signals of $w_j(t)$ and $w_k(t)$, and $w_{\mathcal{Z}}(t)$ is the set of node signals that will be discarded, i.e. all other node signals. Since all neighbour nodes of $w_j(t)$ and $w_k(t)$ are retained, there are no unmeasurable signals affecting both $w_j(t)$ and $w_k(t)$, i.e. there are no confounding variables related to $w_j(t)$ and $w_k(t)$.

Remark 2: If the model set is restricted to at most second order dynamics, the network resulting from immersion contains higher order dynamics and therefore, the network is not in the model set anymore. That is, allowing for higher order dynamics in the network model set allows for immersing nodes from the network. ■

Remark 3: By using immersion, nodes are removed from the network and the identification problem can be solved locally, meaning that not all nodes are needed and not all dynamics in the network need exactly be recovered in order to identify the dynamics of a specific interconnection in the network. ■

C. Identification setup

After immersion, the system representation is as follows:

$$\begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \end{bmatrix} = \begin{bmatrix} 0 & G_{jk}(q) & G_{j\mathcal{L}}(q) \\ G_{kj}(q) & 0 & G_{k\mathcal{L}}(q) \\ G_{\mathcal{L}j}(q) & G_{\mathcal{L}k}(q) & \breve{G}_{\mathcal{L}\mathcal{L}}(q) \end{bmatrix} \begin{bmatrix} w_j(t) \\ w_k(t) \\ w_{\mathcal{L}}(t) \end{bmatrix} + \breve{R}(q)r(t) + \breve{H}(q)e(t). \quad (42)$$

Following the local identification approach in [10], (42) is used for locally identifying the two modules $G_{jk}(q)$ and $G_{kj}(q)$. The input signals of this local identification problem will be $w_j(t)$, $w_k(t)$ and $w_{\mathcal{L}}(t)$ and the output signals will be $w_j(t)$ and $w_k(t)$. Hence, the first two rows of (42) will be estimated consistently through the joint-direct method according to Proposition 6, where the symmetry of $P(q)$ is taken into account in the parametrisation similar as in the full network case and where the prediction error is

$$\varepsilon(t,\theta) = C_{j-k}^{-1}(q,\theta)\left[Q_{j-k}(q,\theta) - P_{j-k}(q,\theta)\right] w_{j-\mathcal{L}}(t) - C_{j-k}^{-1}(q,\theta) F_{j-k} r_{j-k}(t), \quad (43)$$

with

$$Q_{j-k}(q,\theta) = \begin{bmatrix} Q_{jj}(q,\theta) & 0 & 0 \\ 0 & Q_{kk}(q,\theta) & 0 \end{bmatrix}, \quad (44)$$

$$P_{j-k}(q,\theta) = \begin{bmatrix} 0 & P_{jk}(q,\theta) & P_{j\mathcal{L}}(q,\theta) \\ P_{kj}(q,\theta) & 0 & P_{k\mathcal{L}}(q,\theta) \end{bmatrix}, \quad (45)$$

with $C_{j-k}(q,\theta) = \mathrm{diag}\big(C_{jj}(q,\theta), C_{kk}(q,\theta)\big)$, $F_{j-k}(\theta) = \mathrm{diag}\big(F_{jj}, F_{kk}\big)$, $w_{j-\mathcal{L}}(t) = [w_j(t), w_k(t), w_{\mathcal{L}}^\top(t)]^\top$ and $r_{j-k}(t) = [r_j(t), r_k(t)]^\top$.

Once $G_{jk}(q)$ and $G_{kj}(q)$ have been identified, the physical components in the network model (7) can be retrieved from the estimated model $(Q_{j-k}(q), P_{j-k}(q), F_{j-k}, C_{j-k}(q))$. Like in the full network identification case they are determined up to a scaling factor, which is fixed to be 1 in the case that both node signals w_j and w_k are excited by an external excitation signal.

VI. Conclusion

The undirected network description of physical networks has been extended by allowing for higher order diffusive couplings. The resulting undirected network has been reformulated into a directed module representation with specific structural properties. This representation allows for effective identification of the global and local properties of the physical network.

Appendix

A. Full network identification: alternative linear regression

Instead of using the network equations in terms of $Q(q)$, $P(q)$, F, and $C(q)$ (20)-(21), it is also possible to use the network equations in terms of $\bar{B}(q)$, $\bar{A}(q)$, F, and $C(q)$ (17) for identifying the full network. The linear regression result in Corollary 1 changes to the following result.

Corollary 2 (Alternative linear regression): Consider a network that has generated data according to

$$\bar{B}^0(q)w(t) = \bar{A}^0(q)w(t) + F^0 r(t) + e(t). \quad (46)$$

and an ARX network model structure according to

$$\mathcal{M}(\theta) := \{\bar{B}(q,\theta), \bar{A}(q,\theta), F, C(q,\theta), \theta \in \Theta\} \quad (47)$$

that is, $C(q,\theta) = I$, with $\bar{B}(q,\theta)$ and $\bar{A}(q,\theta)$ containing polynomials in the delay operator q^{-1} and scaled such that the diagonals of their constant terms (the terms related to q^0) sum up to I and F being constant and known to be equal to F^0. Then the transfer functions $(Q^0)^{-1}P^0, (Q^0)^{-1}F^0,$

and $(Q^0)^{-1}Q_0C^0$ are estimated consistently through a linear regression according to (31) with $\varphi(t)$ defined as

$$\varphi^\top(t) = \begin{pmatrix} \varphi^\top_{\bar{B}_1}(t) & \cdots & \varphi^\top_{\bar{B}_n}(t) & \varphi^\top_{\bar{A}_0}(t) & \cdots & \varphi^\top_{\bar{A}_{n-1}}(t) \end{pmatrix}, \tag{48}$$

with $\varphi^\top_{\bar{B}_i}(t) = \varphi^\top_{Q_i}(t)$ (33), $\varphi^\top_{\bar{A}_0}(t) = \varphi^\top_{P_0}(t)$ (34) and with

$$\varphi^\top_{\bar{A}_i}(t) = \left(\begin{bmatrix} Z_{0,L-1} \\ q^{-i}V_{2,L}(t) \\ -q^{-i}dV_{2,L}(t) \end{bmatrix} \begin{bmatrix} Z_{1,L-2} \\ q^{-i}V_{3,L}(t) \\ -q^{-i}dV_{3,L}(t) \end{bmatrix} \cdots \begin{bmatrix} Z_{L-2,1} \\ q^{-i}V_{L,L}(t) \\ -q^{-i}dV_{L,L}(t) \end{bmatrix} \right), \tag{49}$$

for $i = 1, \ldots, n-1$, where $Z_{j,k}$ is a matrix of size $j \times k$ with all elements equal to 0, $V_{j,L}(t) = [w_j(t) - w_{j-1}(t), \cdots, w_L(t) - w_{j-1}(t)]$ and $dV_{j,L}(t) = \text{diag}\big(V_{j,L}(t)\big)$ is a square and diagonal matrix of size $(L - j + 1) \times (L - j + 1)$.

Proof: The one-step-ahead predictor $\hat{w}_j(t|t-1;\theta)$ is affine in the parameters θ, meaning that it can be written as a linear regression

$$\varphi^\top(t)\theta - Fr(t) = \left[I - \big(\bar{B}(q,\theta) + \bar{A}(q,\theta)\big)\right] w(t), \tag{50}$$

$$= \left[-\sum_{i=0}^{n-1} \bar{A}_i(\theta)q^{-i} - \sum_{i=1}^{n} \bar{B}_i(\theta)q^{-i} \right] w(t), \tag{51}$$

where the structure of $\bar{B}(q,\theta)$ and $\bar{A}(q,\theta)$ is retained in $\bar{B}_i(\theta)$ and $\bar{A}_i(\theta)$, respectively, and with parameter vector

$$\theta = \begin{pmatrix} \theta^\top_{\bar{B}_1} & \cdots & \theta^\top_{\bar{B}_n} & \theta^\top_{\bar{A}_0} & \cdots & \theta^\top_{\bar{A}_{n-1}} \end{pmatrix}^\top, \tag{52}$$

with

$$\theta_{\bar{B}_i} = \begin{pmatrix} \bar{B}_{i,1} & \bar{B}_{i,2} & \cdots & \bar{B}_{i,L} \end{pmatrix}^\top, \tag{53}$$

$$\theta_{\bar{A}_i} = \begin{pmatrix} \bar{A}_{i,1} & \bar{A}_{i,2} & \cdots & \bar{A}_{i,L(L-1)/2} \end{pmatrix}^\top, \tag{54}$$

where these parameter vectors parameterise the matrices according to $\bar{B}_i(\theta) = \text{diag}(\theta_{\bar{B}_i})$ and

$$\bar{A}_i(\theta) = \begin{pmatrix} \star & \bar{A}_{i,1} & \bar{A}_{i,2} & \cdots & \bar{A}_{i,L-1} \\ \star & \star & \bar{A}_{i,L} & \cdots & \bar{A}_{i,2L-3} \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \bar{A}_{i,L(L-1)/2} \\ \star & \star & \star & \star & \star \end{pmatrix}, \tag{55}$$

where elements $\star$ follow from the Laplacian structure. In addition, note that $\bar{A}_0(\theta)$ is hollow, since the diagonal elements of $\bar{A}_0(\theta)$ are nullified by $\bar{B}_0(\theta)$ and I in (50). ■

Note that $\bar{B}(q) + \bar{A}(q) = Q(q) - P(q)$ in (17) and (20)-(21). As a result, the scaling of $\bar{B}(q)$ and $\bar{A}(q)$ such that their diagonals sum up to I is the same as the scaling of $Q(q)$ such that it is monic (remember that $P(q)$ is hollow). In other words, $\bar{B}(q)$ and $\bar{A}(q)$ are scaled with Q_0.

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