Local identification in diffusively coupled linear networks

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***Abstract*— Physical dynamic networks most commonly consist of interconnections of physical components that can be described by diffusive couplings. Diffusive couplings imply symmetric cause-effect relationships in the interconnections and therefore diffusively coupled networks can be represented by undirected graphs. This paper shows how local dynamics of (undirected) diffusively coupled networks can be identified on the bases of local signals only. Sensors and actuators are allocated to guarantee consistent identification. An algorithm is developed for identifying the local dynamics.**

I. Introduction

Physical networks can describe a diversity of physical processes from various domains, such as electrical, mechanical, hydraulic, thermal, and chemical processes. Their dynamic behavior is typically described by undirected dynamic interconnections between node signals, where the interconnections represent diffusive couplings [1], [2]. The network is typically described by a vector differential equation of maximum second order. Some famous examples of physical networks are electrical resistor-inductor-capacitor circuits and mechanical mass-spring-damper systems.

In literature, there are several methods available for identifying the physical components in the network on the basis of measured signals. Black-box prediction error identification methods [3] can model the transfer functions from measured excitation signals to node signals. These models need to be converted to the structure of the physical network for estimating the component values, which is nontrivial. Moreover, this modeling procedure depends on the particular location of the external signals. Second, black-box state-space models can be estimated from which the model parameters can be derived by applying matrix transformations [4], [5], [6] or eigenvalue decompositions [7], [8]. However, these methods typically do not have any guarantees on the statistical accuracy of the estimates. Third, physical networks can be considered to be directed dynamic networks with specific structural properties [9]. Dynamic networks can be modeled as directed interconnections of transfer function modules [10], [11] for which an identification framework has been developed in [11]. However, the network structure in the model is generally lost.

Instead of identifying the full network dynamics, one can also aim for identifying only a part of the network, such that more simple experiments can be used to obtain a particular component. This is often referred to as 'local', 'single module', or 'subnetwork' identification, for which several methods have been developed for dynamic networks [12], [13], [14], see [15] for an overview. Again, the structural properties of undirected network models cannot easily be accounted for in these identification procedures for directed dynamic networks.

This paper builds further on the preliminary work presented in [16], in which the identification of the full diffusively coupled network dynamics is discussed, including detailed identifiability and consistency results as well as the implementation into a convex multi-step algorithm. This paper addresses the problem of identifying a particular (local) dynamics in the diffusively coupled network. The order of the dynamics is not restricted and possibly correlated disturbances can be present. The question that is addressed is: Which nodes to measure (sense) and which nodes to excite (actuate) in order to identify the dynamics of a local interconnection in the network? An identification procedure is developed that is shown to lead to consistent estimates thereof.

The networks that will be considered in this paper are defined in Section II. The identification problem is specified in Section III. Section IV and Section V describe how to remove unmeasured nodes from the network, without affecting the target component. Section VI describes the identification procedure, including experiment design and conditions for consistent estimates. Section VII discusses some algorithmic aspects. Section VIII shows a simulation example of local identification. Finally, Section IX concludes the paper. For simplicity, we restrict to representations in the discrete-time domain.

We consider the following notation throughout the paper. A polynomial matrix $A(q^{-1})$ consists of matrices A_ℓ and (j,k)-th polynomial elements $a_{jk}(q^{-1})$ such that $A(q^{-1}) = \sum_{\ell=0}^{n_a} A_\ell q^{-\ell}$ and $a_{jk}(q^{-1}) = \sum_{\ell=0}^{n_a} a_{jk,\ell} q^{-\ell}$. Hence, the (j,k)-th element of the matrix A_ℓ is denoted by $a_{jk,\ell}$. Further, let $A_{\mathcal{J}\bullet}(q^{-1})$ indicate all j-th rows of $A(q^{-1})$ for which $j \in \mathcal{J}$.

II. Diffusively coupled networks

Diffusively coupled networks are linear dynamic networks in which the interaction between the nodes depends on the difference between the node signals. Such an interaction implies a symmetric coupling between the nodes. The nodes can also be a coupled with the zero node, referred to as the ground node. The networks that will be considered in this paper are defined in accordance with [16] as follows.

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Definition 1 (Network model): The network that will be considered consists of L node signals $w(t)$, K known excitation signals $r(t)$, and L unknown disturbance signals $v(t)$ and is defined as

$$A(q^{-1})w(t) = B(q^{-1})r(t) + v(t), \tag{1}$$

with q^{-1} the delay operator, i.e. $q^{-1}w(t) = w(t-1)$; with $v(t)$ modeled as filtered white noise, i.e. $v(t) = F(q)e(t)$ with $e(t)$ a vector-valued wide-sense stationary white noise process, i.e. $\mathbb{E}[e(t)e^T(t-\tau)] = 0$ for $\tau \neq 0$; and with

- $A(q^{-1}) = \sum_{k=0}^{n_a} A_k q^{-k} \in \mathbb{R}^{L\times L}[q^{-1}]$, with $A^{-1}(q^{-1})$ stable; rank$(A_0) = L$; and $a_{jk}(q^{-1}) = a_{kj}(q^{-1})\ \forall k, j$.
- $B(q^{-1}) \in \mathbb{R}^{L\times K}[q^{-1}]$.
- $F(q) \in \mathbb{R}^{L\times L}(q)$, monic, stable, and stably invertible.
- $\Lambda \succ 0$ the covariance matrix of the noise $e(t)$. □

The diffusive character of the model is represented by the symmetry property of $A(q^{-1})$. It is assumed that the network is connected, which means that there is a path between every pair of nodes[1]. If the network has at least one connection to the ground node, then the network is well-posed, which means that $A^{-1}(q^{-1})$ exists and is proper. Stability of the network is induced by stability of $A^{-1}(q^{-1})$.

Both $A(q^{-1})$ and $B(q^{-1})$ are nonmonic polynomial matrices. In the symmetric $A(q^{-1})$, the polynomial $a_{ij}(q^{-1})$ characterizes the dynamics in the link between node signals $w_i(t)$ and $w_j(t)$. Often, $B(q^{-1})$ is chosen to be a submatrix of the identity matrix, implying that each external excitation signal directly enters the network at a distinct node. If $F(q)$ is polynomial or even stronger if $F(q) = I$, the network (1) leads to an ARMAX-like or ARX-like[2] model structure, respectively.

A diffusively coupled network induces an undirected graph, where the vertices (nodes) represent the node signals and the links (interconnections) represent the symmetric couplings. Figure 1 shows a diffusively coupled network with the dynamics captured by the boxes containing the polynomials $a_{ij}(q^{-1})$ and $b_{ij}(q^{-1})$ and with the nodes represented by the circles, which sum the diffusive couplings and the external signals.

III. Identification problem

In view of the symmetric couplings in the considered networks, the local identification problem is formulated as follows.

Definition 2 (Local identification problem): The local identification problem concerns the identification of a single *coupling* between two nodes in the network on the bases of selected measured signals $w(t)$ and $r(t)$. □

A single coupling in the network contains the full information on how two nodes interact with each other. For the nodes w_i and w_j, this coupling is described by the polynomials $a_{ii}(q^{-1})$, $a_{ij}(q^{-1}) = a_{ji}(q^{-1})$, and $a_{jj}(q^{-1})$. One could interpret this identification problem as the identification of the *subnetwork* described by the nodes w_i and w_j. For solving this identification problem, it is assumed to be known which nodes are the neighbor nodes of the subnetwork.

[1]The network is connected if its Laplacian matrix (i.e. the degree matrix minus the adjacency matrix) has a positive second smallest eigenvalue [2].

[2]The structure is formally only an ARMAX (autoregressive-moving average with exogenous variables) or ARX (autoregressive with exogenous variables) structure if the $A(q^{-1})$ polynomial is monic [17].

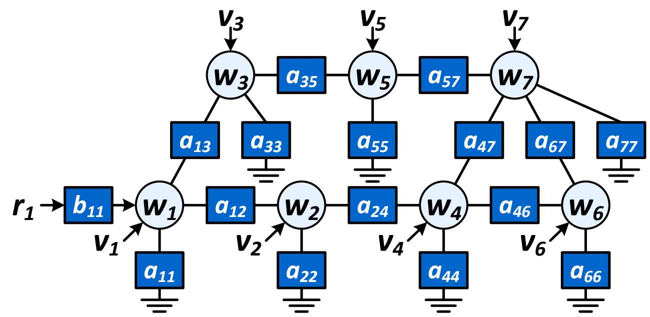

Fig. 1. A diffusively coupled network as defined in Definition 1, with nodes w_j, excitations r_j, disturbances v_j, network dynamics a_{jk}, and input dynamics b_{jk}.

IV. Immersion

The identification of a subnetwork is preferably based on partial measurement of the network. This means that only a selected set of node signals is measured. One way to deal with unmeasured node signals is by eliminating them from the representation. In literature, this Gaussian elimination is referred to as Kron reduction [2] or immersion [18]. In this section, this reduction procedure is adapted to polynomial representations in order to preserve the network model structure.

For the purpose of immersion, consider a network as defined in Definition 1, with the node signals partitioned into two groups: the signals that will be immersed $w_\mathcal{Z}(t)$ and the signals that will be preserved $w_\mathcal{Y}(t)$. Define the sets $\mathcal{Z} := \{\ell \mid w_\ell \in w_\mathcal{Z}\}$ and $\mathcal{Y} := \{\ell \mid w_\ell \notin w_\mathcal{Z}\}$. The external signal $v(t)$ is partitioned accordingly, as well as the network matrices $A(q^{-1})$ and $B(q^{-1})$. This partitioning leads to the equivalent network description

$$\begin{bmatrix} A_{\mathcal{Y}\mathcal{Y}}(q^{-1}) & A_{\mathcal{Y}\mathcal{Z}}(q^{-1}) \\ A_{\mathcal{Z}\mathcal{Y}}(q^{-1}) & A_{\mathcal{Z}\mathcal{Z}}(q^{-1}) \end{bmatrix} \begin{bmatrix} w_\mathcal{Y}(t) \\ w_\mathcal{Z}(t) \end{bmatrix} = \begin{bmatrix} B_{\mathcal{Y}\bullet}(q^{-1}) \\ B_{\mathcal{Z}\bullet}(q^{-1}) \end{bmatrix} r(t) + \begin{bmatrix} v_\mathcal{Y}(t) \\ v_\mathcal{Z}(t) \end{bmatrix}. \tag{2}$$

Proposition 1 (Immersion in diffusively coupled networks): Consider the network in (2). Removing the nodes $w_\mathcal{Z}(t)$ through a Gaussian elimination procedure results in the *immersed* network representation

$$\breve{A}(q^{-1})\breve{w}(t) = \breve{B}(q^{-1})r(t) + \breve{v}(t), \tag{3}$$

with $\breve{w}(t) = w_\mathcal{Y}(t)$, $\breve{A}(q^{-1})$ symmetric, and (omitting arguments q^{-1}, t)

$$\breve{A}(q^{-1}) = d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{Y}\mathcal{Y}} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{Y}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}A_{\mathcal{Z}\mathcal{Y}}, \tag{4a}$$

$$\breve{B}(q^{-1}) = d_{\mathcal{Z}\mathcal{Z}}B_{\mathcal{Y}\bullet} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{Y}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}B_{\mathcal{Z}\bullet}, \tag{4b}$$

$$\breve{v}(t) = d_{\mathcal{Z}\mathcal{Z}}v_\mathcal{Y} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{Y}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}v_\mathcal{Z}, \tag{4c}$$

$$d_{\mathcal{Z}\mathcal{Z}}(q^{-1}) := \frac{det(A_{\mathcal{Z}\mathcal{Z}})}{gcd\left(det(A_{\mathcal{Z}\mathcal{Z}}), adj(A_{\mathcal{Z}\mathcal{Z}})\right)}, \tag{4d}$$

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where $det(A_{\mathcal{Z}\mathcal{Z}})$ and $adj(A_{\mathcal{Z}\mathcal{Z}})$ are the determinant and the adjugate of the polynomial matrix $A_{\mathcal{Z}\mathcal{Z}}(q^{-1})$, respectively, and $gcd(X,Y)$ is the greatest common divisor of X and Y.

Proof: This follows from Gaussian elimination of $w_{\mathcal{Z}}(t)$ and the fact that $\breve{A}(q^{-1})$ is a symmetric polynomial matrix. As $A_{\mathcal{Z}\mathcal{Z}}^{-1}$ is rational, an additional scaling with the monic scalar polynomial $d_{\mathcal{Z}\mathcal{Z}}(q^{-1})$ is needed in order to make the representation polynomial again. □

The immersed network represents the dynamical relations between a selected subset of nodes in the network. It plays a crucial role in the identification of local network properties that is based on a selected set of (local) node measurements.

V. Invariant local dynamics

As mentioned in Section III, the objective is to identify a subnetwork described by the nodes w_i and w_j. Let this subnetwork be described by node signals $w_{\mathcal{J}}(t)$ and dynamics $A_{\mathcal{J}\mathcal{J}}(q^{-1})$, where we define the set $\mathcal{J} := \{\ell \mid w_\ell \in w_{\mathcal{J}}\}$. From (4a), it follows that the dynamics of $A_{\mathcal{Y}\mathcal{Y}}(q^{-1})$ is preserved after immersion, up to the scalar polynomial factor $d_{\mathcal{Z}\mathcal{Z}}(q^{-1})$, if the signals $w_{\mathcal{Y}}(t)$ are preserved such that $A_{\mathcal{Y}\mathcal{Z}}(q^{-1}) = 0$. This can simply be done by preserving the nodes $w_{\mathcal{J}}(t)$ and all their neighbor nodes and immersing all remaining nodes.

In line with this reasoning, we partition the node signals into three groups: the signals of interest $w_{\mathcal{J}}(t)$, their neighbor signals $w_{\mathcal{D}}(t)$, and the remaining signals $w_{\mathcal{Z}}(t)$. Define the set $\mathcal{D} := \{\ell \mid w_\ell \in w_{\mathcal{D}}\}$. The external signal $v(t)$ is partitioned accordingly, as well as the network matrices $A(q^{-1})$, $B(q^{-1})$, and $F(q)$. We assume that the disturbance signals $v_{\mathcal{J}}(t)$ are uncorrelated to the other disturbances in the network ($v_{\mathcal{D}}(t)$ and $v_{\mathcal{Z}}(t)$). This partitioning leads to the network description

$$\begin{bmatrix} A_{\mathcal{J}\mathcal{J}}(q^{-1}) & A_{\mathcal{J}\mathcal{D}}(q^{-1}) & 0 \\ A_{\mathcal{D}\mathcal{J}}(q^{-1}) & A_{\mathcal{D}\mathcal{D}}(q^{-1}) & A_{\mathcal{D}\mathcal{Z}}(q^{-1}) \\ 0 & A_{\mathcal{Z}\mathcal{D}}(q^{-1}) & A_{\mathcal{Z}\mathcal{Z}}(q^{-1}) \end{bmatrix} \begin{bmatrix} w_{\mathcal{J}}(t) \\ w_{\mathcal{D}}(t) \\ w_{\mathcal{Z}}(t) \end{bmatrix} = \begin{bmatrix} B_{\mathcal{J}\bullet}(q^{-1}) \\ B_{\mathcal{D}\bullet}(q^{-1}) \\ B_{\mathcal{Z}\bullet}(q^{-1}) \end{bmatrix} r(t) + \begin{bmatrix} v_{\mathcal{J}}(t) \\ v_{\mathcal{D}}(t) \\ v_{\mathcal{Z}}(t) \end{bmatrix}. \tag{5}$$

Immersing the node signals $w_{\mathcal{Z}}(t)$ leads to

$$\begin{bmatrix} \breve{A}_{\mathcal{J}\mathcal{J}}(q^{-1}) & \breve{A}_{\mathcal{J}\mathcal{D}}(q^{-1}) \\ \breve{A}_{\mathcal{D}\mathcal{J}}(q^{-1}) & \breve{A}_{\mathcal{D}\mathcal{D}}(q^{-1}) \end{bmatrix} \begin{bmatrix} w_{\mathcal{J}}(t) \\ w_{\mathcal{D}}(t) \end{bmatrix} = \begin{bmatrix} \breve{B}_{\mathcal{J}\bullet}(q^{-1}) \\ \breve{B}_{\mathcal{D}\bullet}(q^{-1}) \end{bmatrix} r(t) + \begin{bmatrix} \breve{F}_{\mathcal{J}\mathcal{J}}(q) & 0 \\ 0 & \breve{F}_{\mathcal{D}\mathcal{D}}(q) \end{bmatrix} \begin{bmatrix} e_{\mathcal{J}}(t) \\ \breve{e}_{\mathcal{D}}(t) \end{bmatrix}, \tag{6}$$

that is, $\breve{A}(q^{-1})\breve{w}(t) = \breve{B}(q^{-1})r(t) + \breve{F}(q)\breve{e}(t)$, with (omitting arguments q^{-1}, q, t)

$$\breve{A}_{\mathcal{J}\mathcal{J}} = d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{J}\mathcal{J}}, \qquad \breve{A}_{\mathcal{J}\mathcal{D}} = d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{J}\mathcal{D}}, \tag{7a}$$
$$\breve{A}_{\mathcal{D}\mathcal{J}} = d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{J}}, \qquad \breve{B}_{\mathcal{J}\bullet} = d_{\mathcal{Z}\mathcal{Z}}B_{\mathcal{J}\bullet}, \tag{7b}$$
$$\breve{A}_{\mathcal{D}\mathcal{D}} = d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{D}} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}A_{\mathcal{Z}\mathcal{D}}, \tag{7c}$$
$$\breve{B}_{\mathcal{D}\bullet} = d_{\mathcal{Z}\mathcal{Z}}B_{\mathcal{D}\bullet} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}B_{\mathcal{Z}\bullet}, \tag{7d}$$
$$\breve{F}_{\mathcal{J}\mathcal{J}} = d_{\mathcal{Z}\mathcal{Z}}F_{\mathcal{J}\mathcal{J}}, \tag{7e}$$

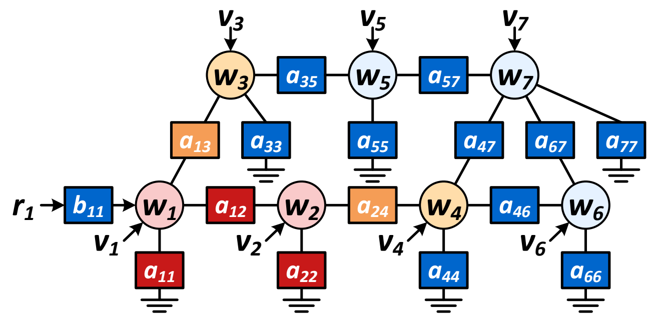

Fig. 2. A diffusively coupled network with a target subnetwork indicated in red, with its neighbor dynamics and nodes indicated in orange.

$$\breve{F}_{\mathcal{D}\mathcal{D}}\breve{e}_D = \left(d_{\mathcal{Z}\mathcal{Z}}F_{\mathcal{D}\mathcal{D}} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}F_{\mathcal{Z}\mathcal{D}}\right)e_{\mathcal{D}}(t) + \left(d_{\mathcal{Z}\mathcal{Z}}F_{\mathcal{D}\mathcal{Z}} - d_{\mathcal{Z}\mathcal{Z}}A_{\mathcal{D}\mathcal{Z}}A_{\mathcal{Z}\mathcal{Z}}^{-1}F_{\mathcal{Z}\mathcal{Z}}\right)e_{\mathcal{Z}}(t), \tag{7f}$$

with $\breve{F}_{\mathcal{D}\mathcal{D}}(q)$ a monic, stable, and stably invertible transfer function matrix and with $\breve{e}_{\mathcal{D}}(t)$ white noise.

Proposition 2 (Invariant local dynamics): Immersion for diffusively coupled networks, as described in Proposition 1, applied to the network (5) resulting in (6), gives for $\ell \in \mathcal{J}$:

$$\breve{a}_{\ell\ell}^{-1}(q^{-1})\breve{A}_{\ell\bullet}(q^{-1}) = a_{\ell\ell}^{-1}(q^{-1})A_{\ell\bullet}(q^{-1}), \tag{8a}$$
$$\breve{a}_{\ell\ell}^{-1}(q^{-1})\breve{B}_{\ell\bullet}(q^{-1}) = a_{\ell\ell}^{-1}(q^{-1})B_{\ell\bullet}(q^{-1}), \tag{8b}$$
$$\breve{a}_{\ell\ell}^{-1}(q^{-1})\breve{F}_{\ell\bullet}(q) = a_{\ell\ell}^{-1}(q^{-1})F_{\ell\bullet}(q). \tag{8c}$$

Proof: For $\ell \in \mathcal{J}$ (omitting arguments q^{-1}, q):

$$\breve{a}_{\ell\ell}^{-1}\breve{A}_{\ell\bullet} = a_{\ell\ell}^{-1}d_{\mathcal{Z}\mathcal{Z}}^{-1}d_{\mathcal{Z}\mathcal{Z}}A_{\ell\bullet} = a_{\ell\ell}^{-1}A_{\ell\bullet},$$
$$\breve{a}_{\ell\ell}^{-1}\breve{B}_{\ell\bullet} = a_{\ell\ell}^{-1}d_{\mathcal{Z}\mathcal{Z}}^{-1}d_{\mathcal{Z}\mathcal{Z}}B_{\ell\bullet} = a_{\ell\ell}^{-1}B_{\ell\bullet},$$
$$\breve{a}_{\ell\ell}^{-1}\breve{F}_{\ell\bullet} = a_{\ell\ell}^{-1}d_{\mathcal{Z}\mathcal{Z}}^{-1}d_{\mathcal{Z}\mathcal{Z}}F_{\ell\bullet} = a_{\ell\ell}^{-1}F_{\ell\bullet}.$$

□

The result of Proposition 2 is that the local identification problem of identifying a subnetwork really becomes a local problem in the sense that (6) can be used to identify $A_{\mathcal{J}\mathcal{J}}(q^{-1})$ on the basis of the signals of interest $w_{\mathcal{J}}(t)$ and their neighbor node signals $w_{\mathcal{D}}(t)$ only and that all other node signals $w_{\mathcal{Z}}(t)$ can be discarded.

Figure 2 shows a diffusively coupled network with the subnetwork described by the node signals $w_{\mathcal{J}}(t) = \begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix}^\top$ indicated in red. The neighbor node signals $w_{\mathcal{D}}(t) = \begin{bmatrix} w_3(t) & w_4(t) \end{bmatrix}^\top$ are indicated in orange. Immersing the remaining node signals from the network, results in the immersed network representation shown in Figure 3, where $\breve{a}_{ij}$ and $\breve{b}_{11}$ are related to a_{ik} and b_{11} according to the relations in (7).

Remark 1 (Module representation): The result of Proposition 2 is a specific version of the condition on parallel paths and loops around the output as defined in [18]. To see this, observe that all loops around $w_j(t)$ contain a measured node signal if and only if all neighbor nodes of $w_j(t)$ are measured and consequently, all parallel paths from $w_i(t)$ to $w_j(t)$ contain a measured node signal as well [9]. □

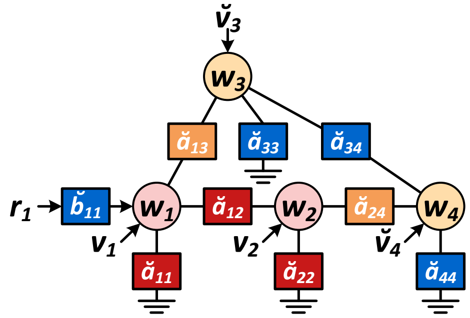

Fig. 3. The immersed network representation corresponding to the diffusively coupled network in Figure 2 with the target subnetwork indicated in red.

VI. Identification procedure

A. Identifying the immersed network

For identifying the complete immersed network, a predictor model is set up based on the parametrized model set

$$\breve{\mathcal{M}} := \big(\breve{A}(q^{-1},\eta), \breve{B}(q^{-1},\eta), \breve{F}(q,\eta), \breve{\Lambda}(\eta), \eta \in \Pi\big), \quad (9)$$

while the corresponding data generating network is denoted by $\breve{\mathcal{S}} := \{\breve{A}^0(q^{-1}), \breve{B}^0(q^{-1}), \breve{F}^0(q), \breve{\Lambda}^0\}$. Considering the one-step-ahead predictor of $\breve{w}(t)$ in line with [19] as

$$\hat{\breve{w}}(t|t-1) = \mathbb{E}\{\breve{w}(t) \mid \breve{w}^{t-1},\ r^t\}, \quad (10)$$

where $\breve{w}^\ell$ and r^ℓ refer to signal samples $\breve{w}(\tau)$ and $r(\tau)$, respectively, for all $\tau \leq \ell$. The resulting prediction error becomes [16] (omitting arguments q^{-1}, q)

$$\breve{\varepsilon}(t,\eta) = \breve{w}(t) - \hat{\breve{w}}(t|t-1;\eta), \quad (11)$$

$$= \breve{A}_0^{-1}(\eta)\breve{F}^{-1}(\eta)\left[\breve{A}(\eta)\breve{w}(t) - \breve{B}(\eta)r(t)\right]. \quad (12)$$

The parameters of the immersed network are estimated through the least squares identification criterion $\hat{\eta}_N = \arg\min_\eta \frac{1}{N}\sum_{t=1}^{N} \breve{\varepsilon}^\top(t,\eta)\Lambda\breve{\varepsilon}(t,\eta)$, $\Lambda \succ 0$. Under some mild conditions[3] this criterion converges with probability 1 to $\eta^* := \arg\min_\eta \lim_{N\to\infty}\sum_{t=1}^{N}\mathbb{E}\left\{\breve{\varepsilon}^\top(t,\eta)\Lambda\breve{\varepsilon}(t,\eta)\right\}$.

Proposition 3 (Consistent full identification): The parameter estimate $\hat{\eta}_N$ provides a consistent estimate of the system $\breve{\mathcal{S}}$ if the following conditions hold:

1) The true system is in the model set: $\breve{\mathcal{S}} \subset \breve{\mathcal{M}}$.
2) At least one excitation signal is present: $K \geq 1$.
3) $\Phi_r(\omega) \succ 0$ for a sufficiently high number of frequencies.
4) The polynomials $\breve{A}(q^{-1},\eta)$ and $\breve{B}(q^{-1},\eta)$ are left coprime.
5) There exists a permutation matrix P such that $\big[\breve{A}_0(\eta)\ \breve{A}_1(\eta)\ \cdots\ \breve{A}_{n_a}(\eta)\ \breve{B}_0(\eta)\ \breve{B}_1(\eta)\ \cdots\ \breve{B}_{n_b}(\eta)\big]P = \big[D(\eta)\ R(\eta)\big]$ with $D(\eta)$ square, diagonal, full rank.
6) There is at least one parameter constraint on the parameters of $\breve{A}(q^{-1},\eta_A)$ and $\breve{B}(q^{-1},\eta_B)$ of the form $\Gamma\bar{\eta} = \gamma \neq 0$, with $\bar{\eta} := \big[\eta_A^\top \quad \eta_B^\top\big]^\top$.

[3]The standard conditions for convergence of predictor error estimates include the condition that the white noise process e has bounded moments of an order larger than 4, see [3].

Proof: A consistent estimate is obtained if the model is uniquely recovered from the data. Condition 1 is necessary for this. Condition 3 ensures that the transfer functions from $r(t)$ and $\bar{e}(t) := A_0^{-1}e(t)$ to $\breve{w}(t)$ (i.e. $T_{wr}(q,\eta)$ and $T_{w\bar{e}}(q,\eta)$) can uniquely be recovered from data [16]. Condition 2 implies that $T_{wr}(q,\eta)$ is nonzero. From $T_{wr}(q,\eta)$, Condition 4 ensures that $\breve{A}(q^{-1},\eta)$ and $\breve{B}(q^{-1},\eta)$ are found up to a premultiplication with a unimodular matrix. To satisfy Condition 5, this unimodular matrix is restricted to be diagonal. To preserve symmetry of $\breve{A}(q^{-1},\eta)$, this diagonal matrix is further restricted to have equal elements. Condition 6 fixes the remaining scaling factor. As $\breve{A}(q^{-1},\eta)$ is uniquely found, $T_{w\bar{e}}(q,\eta)$ gives unique $\breve{F}(q,\eta)$ and $\breve{\Lambda}(\eta)$ [16]. ☐

For Condition 6 it is possible to choose a custom constraint, leading to a scaled immersed network representation.

B. Identifying the target subnetwork

Once the complete immersed network representation (6) is identified, the target subnetwork can be estimated. The correct scaling is obtained through a parameter constraint on the target subnetwork. An additional identification step is needed for this, because this dynamics is only present in the identified immersed network with a scaled polynomial factor that needs to be removed. The relations in (7) lead to

$$\tilde{A}^0_{\mathcal{J}\bullet}(q^{-1}) = \alpha d^0_{\mathcal{Z}\mathcal{Z}}(q^{-1})A^0_{\mathcal{J}\bullet}(q^{-1}), \quad (13a)$$

$$\tilde{B}^0_{\mathcal{J}\bullet}(q^{-1}) = \alpha d^0_{\mathcal{Z}\mathcal{Z}}(q^{-1})B^0_{\mathcal{J}\bullet}(q^{-1}), \quad (13b)$$

with $\tilde{A}^0_{\mathcal{J}\bullet}(q^{-1}) := \alpha\breve{A}^0_{\mathcal{J}\bullet}(q^{-1})$, $\tilde{B}^0_{\mathcal{J}\bullet}(q^{-1}) := \alpha\breve{B}^0_{\mathcal{J}\bullet}(q^{-1})$, and unknown scaling factor $\alpha \in \mathbb{R}_+$.

Proposition 4 (Consistent local identification): If a nonzero polynomial element $a_{ij}(q^{-1})$ or $b_{ij}(q^{-1})$ of $A_{\mathcal{J}\bullet}(q^{-1})$ or $B_{\mathcal{J}\bullet}(q^{-1})$, respectively, is known, a consistent estimate of $A^0_{\mathcal{J}\bullet}(q^{-1})$ and $B^0_{\mathcal{J}\bullet}(q^{-1})$ is obtained through (13).

Proof: From Proposition 3, $\tilde{A}^0_{\mathcal{J}\bullet}(q^{-1})$ and $\tilde{B}^0_{\mathcal{J}\bullet}(q^{-1})$ have been estimated consistently with a custom parameter constraint on η. Using known $a_{ij}(q^{-1})$ or $b_{ij}(q^{-1})$, polynomial factor $\alpha d_{\mathcal{Z}\mathcal{Z}}(q^{-1})$ can be extracted from (13). Then, (13) leads to consistent estimates of $A^0_{\mathcal{J}\bullet}(q^{-1})$ and $B^0_{\mathcal{J}\bullet}(q^{-1})$. ☐

The constraint that a nonzero polynomial element $a_{ij}(q^{-1})$ or $b_{ij}(q^{-1})$ should be known, means that a single interconnection in the network is known or that an excitation signal enters the network through known dynamics (e.g. $b_{ij}(q^{-1}) = 1$), respectively. If only one of the parameters of $A_{\mathcal{J}\bullet}(q^{-1})$ or $B_{\mathcal{J}\bullet}(q^{-1})$ is constraint (similar to the constraint in Condition 6 of Proposition 3), a consistent estimate of the target subnetwork is obtained through a null-space fitting, see Section VII.

VII. Algorithmic aspects

For performing the identification we adopt the multi-step algorithm for full network identification presented in [16] for systems with a polynomial noise model, i.e. $F(q) = C(q^{-1})$ polynomial. The prime steps of this algorithm are: 1) estimate a nonstructured high order ARX model; 2) reduce this model to a structured network model through a weighted null-space fitting (WNSF); 3) improve the structured network model through a WNSF; 4) obtain the noise model.

While in (4) the matrix expressions are forced to become polynomial by premultiplying with the common polynomial factor $d_{\mathcal{ZZ}}(q^{-1})$, this causes many polynomial terms in (4) to have common factors. As this can lead to undesired effects in our identification algorithm because of canceling terms, we adopt a different route for arriving at a polynomial model. We remove $d_{\mathcal{ZZ}}(q^{-1})$ from (4) and approximate the rational term $A^{-1}_{\mathcal{ZZ}}(q^{-1})$ by a symmetric polynomial matrix. If the order of this polynomial matrix is chosen sufficiently high, $A^{-1}_{\mathcal{ZZ}}(q^{-1})$ is approximated sufficiently well. The order of $\breve{A}(q^{-1})$ can be controlled and no terms will cancel out. In addition, the target subnetwork appears directly in the immersed network representation. Because of these advantages, we continue with this alternative approach. Observe that if $B_{\mathcal{Z}\bullet}(q^{-1}) = 0$, then $A^{-1}_{\mathcal{ZZ}}(q^{-1})$ only appears in $\breve{A}_{\mathcal{DD}}(q^{-1})$ and $\breve{A}_{\mathcal{DD}}(q^{-1})$ can be approximated by a symmetric polynomial matrix instead. The identification procedure simplifies in the sense that Condition 6 in Proposition 3 directly applies to the target subnetwork and that the subnetwork can be extracted from the immersed network representation, without an additional identification step. This means that Proposition 3 guarantees a consistent estimate of the target subnetwork.

VIII. Simulation Example

This simulation example serves to illustrate that indeed a subnetwork can be identified from a single excitation signal and by measuring the nodes of interest and their neighbor nodes only.

A. Experimental set-up

Consider the network (1) consisting of seven scalar nodes, with a single excitation signal directly entering the network at node w_1 and with a polynomial noise model $F(q) = C(q^{-1})$. This network is shown in Figure 1, where $b_{11} = 1$. The objective is to identify the coupling between the nodes $w_1(t)$ and $w_2(t)$ indicated in red in Figure 2. Hence, $w_{\mathcal{J}}(t) = \begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix}^\top$ and thus $w_{\mathcal{D}}(t) = \begin{bmatrix} w_3(t) & w_4(t) \end{bmatrix}^\top$. The corresponding immersed network representation is shown in Figure 3, where $\breve{b}_{11} = 1$. The exact parameter values are

$$A_0 = \begin{bmatrix} 80 & -40 & -20 & 0 & 0 & 0 & 0 \\ -40 & 80 & 0 & -10 & 0 & 0 & 0 \\ -20 & 0 & 50 & 0 & -5 & 0 & 0 \\ 0 & -10 & 0 & 35 & 0 & -5 & -5 \\ 0 & 0 & -5 & 0 & 15 & 0 & -5 \\ 0 & 0 & 0 & -5 & 0 & 25 & -20 \\ 0 & 0 & 0 & -5 & -5 & -20 & 30 \end{bmatrix}, \quad (14a)$$

$$A_1 = \begin{bmatrix} -60 & 30 & 0 & 0 & 0 & 0 & 0 \\ 30 & -60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -40 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -40 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 20 \\ 0 & 0 & 0 & 0 & 0 & 20 & -20 \end{bmatrix}, \quad (14b)$$

$$A_2 = \text{diag}(\begin{bmatrix} 20 & 20 & 20 & 20 & 0 & 0 & 0 \end{bmatrix}), \quad (14c)$$

TABLE I

The true parameters values of $A_{\mathcal{J}\bullet}(q^{-1}, \theta)$ and the mean and standard deviation (SD) of their estimates.

Parameter	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
True value	80	-60	20	-40	30	0
Mean	79.4219	-59.7437	20.1294	-39.0483	29.3419	0
SD	0.6564	0.3255	0.1626	1.1800	0.7409	0
Parameter	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}
True value	-20	0	0	0	0	0
Mean	-19.7780	-0.0534	0	-0.1235	0.0339	0
SD	0.3821	0.1623	0	0.2261	0.1078	0
Parameter	θ_{13}	θ_{14}	θ_{15}	θ_{16}	θ_{17}	θ_{18}
True value	80	-60	20	0	0	0
Mean	78.0318	-58.6126	19.6240	-0.0104	-0.0957	0
SD	2.1832	1.6962	0.4578	0.6798	0.4375	0
Parameter	θ_{19}	θ_{20}	θ_{21}			
True value	-10	0	0			
Mean	-9.6627	-0.0820	0			
SD	0.3790	0.1841	0			

$$B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = 10^{-2} \begin{bmatrix} 2 & 9 & 6 & 3 & 8 & 4 & 1 \\ 3 & 6 & 1 & 7 & 1 & 6 & 7 \\ 9 & 6 & 5 & 7 & 6 & 3 & 6 \\ 7 & 9 & 4 & 6 & 6 & 3 & 7 \\ 3 & 6 & 4 & 9 & 9 & 5 & 8 \\ 7 & 8 & 6 & 6 & 5 & 6 & 5 \\ 8 & 2 & 9 & 5 & 2 & 4 & 4 \end{bmatrix}. \quad (14d)$$

The external excitation signal $r_1(t)$ is an independent white noise process with mean 0 and variance $\sigma_r^2 = 1$. All nodes are subject to disturbances $e_\ell(t)$, which are independent white noise processes (uncorrelated with $r_1(t)$) with mean 0 and variance $\sigma_e^2 = 10^{-2}$. The experiments consists of 100 Monte-Carlo simulations, where in each run new excitation and noise signals are generated. The number of samples generated for each data set is $N = 10,000$.

In the immersed network representation, $\breve{A}_{\mathcal{DD}}(q^{-1})$ is approximated by a second order polynomial matrix. The full immersed network is identified through the algorithm in [16], where in Step 1, the order of the ARX model approximation is chosen to be 10. The network topology of the immersed network is assumed to be unknown, meaning that all connections between nodes are parametrized. However, it is assumed to be known that $\breve{A}_2(\theta)$ is diagonal and that $\breve{A}_k(\theta) = 0,\ \forall k \geq 3$, such that Condition 5 in Proposition 3 is satisfied. The knowledge that the excitation signal enters the network directly at node w_1 induces that $\breve{B}$ is the 4×1 unit vector, such that Condition 6 in Proposition 3 is satisfied.

The target subnetwork is extracted from the immersed network, where the symmetric structure of $A(q^{-1})$ is incorporated in the parametrization:

$$A_{\mathcal{J}\bullet,0}(\theta) = \begin{bmatrix} \theta_1 & \theta_4 & \theta_7 & \theta_{10} & 0 & 0 & 0 \\ \theta_4 & \theta_{13} & \theta_{16} & \theta_{19} & 0 & 0 & 0 \end{bmatrix}, \quad (15a)$$

$$A_{\mathcal{J}\bullet,1}(\theta) = \begin{bmatrix} \theta_2 & \theta_5 & \theta_8 & \theta_{11} & 0 & 0 & 0 \\ \theta_5 & \theta_{14} & \theta_{17} & \theta_{20} & 0 & 0 & 0 \end{bmatrix}, \quad (15b)$$

$$A_{\mathcal{J}\bullet,2}(\theta) = \begin{bmatrix} \theta_3 & \theta_6 & \theta_9 & \theta_{12} & 0 & 0 & 0 \\ \theta_6 & \theta_{15} & \theta_{18} & \theta_{21} & 0 & 0 & 0 \end{bmatrix}. \quad (15c)$$

Table I shows the true parameter values of $A_{\mathcal{J}\bullet}(q^{-1}, \theta)$. The assumption that $\breve{A}_2$ is diagonal implies the constraints $\theta_6 =$

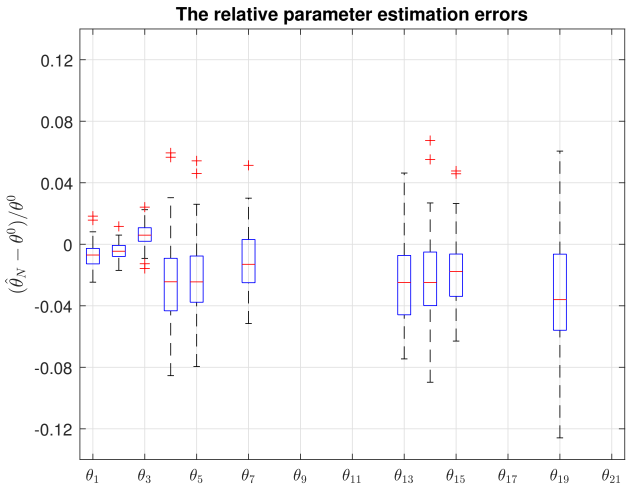

Fig. 4. Boxplot of the relative parameter estimation errors of the parameters of $A_{\mathcal{J}\bullet}(q^{-1}, \theta)$ (15), for parameters with a nonzero true value.

$\theta_9 = \theta_{12} = \theta_{18} = \theta_{21} = 0$.

B. Simulation results

The simulation results are shown in Table I and Figure 4. Table I shows the mean and standard deviation of the estimated parameters of $A_{\mathcal{J}\bullet}(q^{-1}, \theta)$. It can be seen that the constraints $\theta_6 = \theta_9 = \theta_{12} = \theta_{18} = \theta_{21} = 0$ are incorporated well, as these parameters are estimated without bias and variance. The other parameters are estimated with a bias that is within a bound of 1 standard deviation. This bias is less than 3.5% deviation for θ_{19} and within 2.5% deviation for all other (nonzero) parameters. The bias is caused by the limited order of the ARX approximation in Step 1 of the algorithm. Figure 4 shows the relative estimation errors of the parameters of $A_{\mathcal{J}\bullet}(q^{-1}, \theta)$ that have a nonzero true value. Here, the bias is also visible, as the median deviates from 0. It can be seen that all parameters are estimated within 13% error of their true values.

To conclude, the subnetwork described by the nodes w_1 and w_2 has been identified by measuring only four node signals ($w_1(t)$, $w_2(t)$, $w_3(t)$, and $w_4(t)$) and with a single excitation signal ($r_1(t)$) only. The dynamics between the target subnetwork and its neighbor nodes has been identified as well. The variance can be reduced further by adding external excitation signals $r(t)$ to the experiment.

IX. Conclusions

A method and an algorithm for identifying a subnetwork in a diffusively coupled linear network have been presented. For this local identification problem, it is sufficient to measure only the node signals of interest and their neighbor node signals, while all other node signals can be discarded. In addition, only a single excitation signal is required. The identification is performed by identifying the complete immersed network representation from which the target subnetwork is identified.

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