INPUT DESIGN FOR FREQUENCY-FUNCTION VARIANCE REDUCTION IN PREDICTION ERROR IDENTIFICATION

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Abstract

A new approach is proposed for reducing and influencing the variance distribution of finite order frequency-function estimates. The relation between the Hessian of the identification criterion and the variance of the estimates if $\mathcal{S} \in \mathcal{M}$ and $N \to \infty$ is exploited for input design. Maximizing functions of the Hessian yields the optimal input signal. Input design procedures are developed for both open-loop and closed-loop identification. An analytic and a numerical example illustrate the developed method.

1 Introduction

In system identification, not only the estimated model, but also the accuracy is of importance to evaluate the quality of a model. In Prediction Error identification [6, 11], the accuracy is measured in terms of the *bias* and the *variance*.
It is well-known that both bias and variance depend on, among other things, the input signal, and this fact has been used over the years as a basis for input design.

Existing results on optimal input design can be divided into three categories.
"Classical" methods are based on the evaluation of the parameter variance. The input signal is designed such that the estimated parameters have a minimum variance, measured in some sense. Scalar functions of the inverse of the Fisher Information Matrix, such as the trace, the determinant or the largest eigenvalue, are minimized with respect to the input signal [3, 4, 7, 17].
This approach implicitly assumes that the system can be modeled exactly ($\mathcal{S} \in \mathcal{M}$, with $\mathcal{S}$ the system and $\mathcal{M}$ the model set), and hence the parameters are estimated consistently (no bias error).

The second class of input design methods aims at obtaining suitable approximate models, in terms of influencing the bias distribution over the frequency domain. These methods have been analyzed in, e.g., [6, 15]. The bias of the parameter estimates is converted into a corresponding effect on the estimated frequency function.

Optimal input design based on the variance of estimated models, evaluated in terms of the frequency function, has only received minor attention. One of the rare contributions in this area is [1], where an optimal experiment is designed for application of the model in minium variance control. Again, the situation $\mathcal{S} \in \mathcal{M}$ is considered, and consistent estimation is assumed (no bias). This is also the case in [16], where variance aspects have been studied in the situation that the model order tends to infinity.

In this paper we introduce an input design methodology, that is based on a criterion that considers the variance of a finite order estimated model, in the situation $\mathcal{S} \in \mathcal{M}$. However, in contrast with the classical approach, in which the parameter variance is considered, we reparametrize the model parameters to a set of points of the frequency response. This enables us to optimize the input signal to specifically desired variance effects on the frequency function of the estimated model.

The motivation for this approach is given by the fact that for many model applications the frequency function is a "design" variable that is better suited than the parameters of a difference equation. Especially for model-based control design, the variance of the estimated frequency function in the control-relevant frequency band (e.g., near the closed-loop bandwidth) determines the achievable performance of the closed-loop system.

The proposed input design methodology can be incorporated in the iterative schemes of identification and controller design, that have been developed recently [2, 10, 13]. These schemes attempt to exploit the connection between identification and controller design, to arrive at a controller with higher performance. The ana-

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lyses of the iterative schemes have focused mainly on the bias effects, whereas the variance effects play an important role as well.

In Section 2 the problem setting is provided. A brief on system identification is followed by the statement of the input design problem considered. In Section 3 we present the solution to the posed problem. In Section 4 the results are shown to be extendible to closed-loop identification as well.

An analytic example in Section 5 shows how to apply the proposed input design procedure. The numerical example in Section 6 illustrates the effect of different user's choices in the procedure, such as the input design criterion and the weighting matrix. Some conclusions are drawn in Section 7.

2 Problem setting

We consider the prediction error identification framework according to [6], for SISO systems.

Consider a data generating process

$$y_t = G_0(q)u_t + H_0(q)e_t \tag{1}$$

with u_t and y_t the scalar-valued input and output of the system, $G_0(z)$ a rational transfer function, analytic in $|z| \geq 1$, $H_0(z)$ a monic stable and stably invertible transfer function, and $\{e_t\}$ a zero mean white noise process. The collection $(G_0(z), H_0(z))$ is denoted as the *system* $\mathcal{S}$.

A parametrized set of models is considered, denoted as $\mathcal{M} : \{(G(z,\theta), H(z,\theta)), \theta \in \Theta\}$, where $\Theta \subset \mathbb{R}^d$ is an appropriate parameter space. In this paper we assume that $\mathcal{S} \in \mathcal{M}$, implying that there exists a $\theta_0 \in \Theta$ such that $(G(z,\theta_0), H(z,\theta_0)) = (G_0(z), H_0(z))$ for almost all z.

Given input and output data of the process (1), a parameter estimate $\hat{\theta}_N$ is obtained according to

$$\hat{\theta}_N = \arg\min_{\theta\in\Theta} V_N(\theta) \tag{2a}$$

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon_{L,t}^2(\theta) \tag{2b}$$

with $\varepsilon_{L,t}(\theta)$ the filtered prediction error of the corresponding model, defined as

$$\varepsilon_{L,t}(\theta) = L(q)\left\{y_t - \hat{y}_t(\theta)\right\} = \frac{L(q)}{H(q,\theta)}\left\{y_t - G(q,\theta)u_t\right\} \tag{3}$$

where $L(z)$ is a stable filter.

Under weak conditions [6] the parameter estimate $\hat{\theta}_N$ satisfies

$$\hat{\theta}_N \overset{N\to\infty}{\longrightarrow} \arg\min_{\theta\in\Theta} \overline{V}(\theta) \qquad \text{w.p. } 1 \tag{4}$$

with $\overline{V}(\theta) = \overline{\mathrm{E}}\varepsilon_{L,t}^2(\theta)$ ($\overline{\mathrm{E}}$ is the generalized expectation operator) and *w.p.* 1 meaning *with probability one.*

Assuming that $\mathcal{S} \in \mathcal{M}$, $\hat{\theta}_N$ will also satisfy

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in As\mathcal{N}(0, P_\theta) \tag{5}$$

with P_θ the asymptotic covariance matrix of the estimated parameter, given by

$$P_\theta = \sigma_e^2 \overline{V}_{\theta\theta}^{-1}(\theta_0) \tag{6}$$

where σ_e^2 is the variance of the white noise process $\{e_t\}$, and $\overline{V}_{\theta\theta}(\theta)$ is the Hessian (second derivative) of the asymptotic criterion $\overline{V}(\theta)$ with respect to the parameter vector θ.

To determine an input signal u_t, the dependence of P_θ and $\overline{V}_{\theta\theta}(\theta_0)$ on u_t is exploited. These define an uncertainty ellipsoid around $\hat{\theta}_N$. The size and shape of this ellipsoid can be characterized by the determinant, the largest eigenvalue and the trace of the inverse of the Hessian, and the smaller these scalar functions, the smaller is the uncertainty ellipsoid. A smaller largest eigenvalue also implies that the uncertainty ellipsoid becomes more spherical, which results in a Hessian with a smaller condition number.

Input constraints, such as a limited input power, are always present, and the input design problem can be written as a constrained optimization problem. To make this problem solvable, in general the input signal is parametrized by the *input protocol* ξ, such that $u_t = u_t(\xi)$. The input protocol contains all parameters that uniquely define an input signal, such as the amplitudes and frequencies if u_t is a multisine, or the filter parameters if u_t is generated as filtered white noise. The input constraints can in general be expressed by a vector function h as $h(\xi) \leq 0$, and the optimal input protocol is obtained as

$$\xi^* = \arg \min_{\substack{\xi \\ h(\xi)\leq 0}} J\left(\overline{V}_{\theta\theta}^{-1}(\theta_0,\xi)\right) \tag{7}$$

where J is the determinant, the largest eigenvalue or the trace.

To avoid the inversion of the Hessian, it can be decided to maximize the determinant or the smallest eigenvalue of $\overline{V}_{\theta\theta}(\theta_0,\xi)$ instead.

The classical input design methodology (7) is directed toward the parameter variance. As stated before, we are interested in the effect of the variance on the frequency function $G(e^{i\omega})$ of the model. To this end we define a bijective mapping (parametrization) from the parameter vector θ to a vector γ, which uniquely defines the frequency function of the model. Calculating the Hessian $\overline{V}_{\gamma\gamma}(\theta_0,\xi)$ then provides us with a basis for input design. This is discussed in the next section.

3 Input design methodology

Let $\gamma(\theta)$ represent, in some way, the frequency function $G(e^{i\omega},\theta)$. The gradient and Hessian of $\overline{V}(\theta,\xi)$ with respect to γ are, respectively,

$$\overline{V}_\gamma(\theta,\xi) \triangleq \frac{\partial \overline{V}(\theta,\xi)}{\partial \gamma(\theta)} \overline{V}_{\gamma\gamma}(\theta,\xi) \triangleq \frac{\partial^2 \overline{V}(\theta,\xi)}{\partial \gamma^2(\theta)} \qquad (8)$$

It can be shown [8, 14] that $\overline{V}_{\gamma\gamma}(\theta_0,\xi)$ can be calculated as

$$\overline{V}_{\gamma\gamma}(\theta_0,\xi) = \left[\gamma_\theta^{-1}(\theta_0)\right]' \overline{V}_{\theta\theta}(\theta_0)\gamma_\theta^{-1}(\theta_0) \qquad (9)$$

irrespective of $\gamma(\theta)$, as long as $\gamma_\theta(\theta)$ is invertible (which is the case if the mapping from θ to γ is bijective).
Hence the Hessian with respect to $\gamma(\theta)$ is obtained from the Hessian with respect to θ by pre- and postmultiplication with the inverse Jacobian of γ with respect to θ.

To derive a specific input design method, we proceed as follows.
The vector $\gamma(\theta)$ should represent the frequency function $G(e^{i\omega},\theta)$. A possible choice is

$$\Gamma(\omega,\theta) = \left(\ \mathrm{Re}\{G(e^{i\omega},\theta)\} \quad \mathrm{Im}\{G(e^{i\omega},\theta)\}\ \right)' \quad (10a)$$

$$\gamma(\theta) = \left(\ \Gamma'(\omega_1,\theta) \quad \Gamma'(\omega_2,\theta)\cdots\Gamma'(\omega_m,\theta)\ \right)' \quad (10b)$$

To have a bijective relationship between θ and γ, the dimension of γ is d and the m frequencies ω_j are all different with $0 \le \omega_j < \pi$, and $m = \mathrm{ent}(d/2)$, which is the largest integer such that $m \le d/2$.
If d is even, ω_j is nonzero for all j. If d is odd, the zero frequency is added, and $G(e^{i0},\theta) = G(1,\theta)$ (which is real) is added.

Remark: In general it is very dangerous to restrict the design to the minimum number of frequencies [9]. However, we explicitly assume that $\mathcal{S} \in \mathcal{M}$ and therefore it is not a problem in our approach.

We assume that the process model and the noise model are independently parametrized as $(G(z,\theta), H(z,\beta))$. The asymptotic criterion $\overline{V}(\theta,\xi)$ can then be written as

$$\overline{V}(\theta,\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |L(e^{i\omega})H^{-1}(e^{i\omega},\beta)|^2 \cdot \cdot\left[|G_0(e^{i\omega}) - G(e^{i\omega},\theta)|^2\Phi_u(\omega,\xi) + \Phi_v(\omega)\right]\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |LH^{-1}(\beta)|^2 \left[|\Gamma_0 - \Gamma(\theta)|^2\Phi_u(\xi) + \Phi_v\right]\, d\omega$$

An expression for $\overline{V}_{\theta\theta}$ can be derived by twice differentiating $\overline{V}(\theta,\xi)$ with respect to θ to obtain

$$\overline{V}_{\theta\theta}(\theta,\xi) = \frac{1}{\pi}\int_{-\pi}^{\pi} |LH^{-1}(\beta)|^2 \left[\Gamma'_\theta(\theta)\Gamma_\theta(\theta) - (\Gamma_0 - \Gamma(\theta))'\Gamma_{\theta\theta}(\theta)\right]\Phi_u(\xi)\, d\omega$$

Evaluating $\overline{V}_{\theta\theta}(\theta,\xi)$ in $\theta = \theta_0$, and noting that, since it is assumed that $\mathcal{S} \in \mathcal{M}$, $G(e^{i\omega},\theta_0) = G_0(e^{i\omega})$ and therefore $\Gamma_0(\omega) = \Gamma(\omega,\theta_0)$, (11) becomes

$$\overline{V}_{\theta\theta}(\theta_0,\xi) = \frac{1}{\pi}\int_{-\pi}^{\pi} |LH^{-1}(\beta)|^2\Gamma'_\theta(\theta_0)\Gamma_\theta(\theta_0)\Phi_u(\xi)\, d\omega \qquad (11)$$

Note that for linear parametrizations $\Gamma_{\theta\theta} = 0$, and (11) holds even if $\mathcal{S} \notin \mathcal{M}$.

Given $\overline{V}_{\gamma\gamma}(\theta_0,\xi)$, the input design problem can be formulated as (7). To emphasize certain points of the frequency function in the optimization, a weighting matrix W can be added, that is given by

$$W = \mathrm{diag}\left(w(\omega_1), w(\omega_1), \cdots, w(\omega_m), w(\omega_m)\right) \qquad (12)$$

where $w(\omega_j)$ is the weight for frequency ω_j.
The input design problem then becomes

$$\xi^* = \arg \min_{\substack{\xi \\ h(\xi)\le 0}} J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right) \qquad (13)$$

where the input design criteria J are either of

$$J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right) = -\det\left\{W'\overline{V}_{\gamma\gamma}(\theta_0,\xi)W\right\} \quad (14a)$$

$$J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right) = -\lambda_{\min}\left\{W'\overline{V}_{\gamma\gamma}(\theta_0,\xi)W\right\} (14b)$$

$$J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right) = \mathrm{trace}\left\{W'\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi)W\right\} \quad (14c)$$

Note that the minus-signs are used in (14a) and (14b) to avoid the inversion of the Hessian and still have a minimization problem (13). Generally this problem will not be convex, and therefore global convergence cannot be guaranteed.

The optimal protocol ξ^* that is obtained by minimizing the determinant criterion (14a) is independent of the weighting matrix W and the Jacobian γ_θ. Minimizing (14a) is therefore equivalent to maximizing the determinant of $\overline{V}_{\theta\theta}(\theta_0,\xi)$, and results in the same protocol ξ^*.

In practice, θ_0 is of course not known. In (13) θ_0 is therefore replaced by θ_{init}, which is an initial estimate of θ, obtained by applying an (arbitrary) initial protocol ξ_{init}.

4 Extension to closed loop

In many situations an identification experiment must be carried out under closed-loop conditions. For example, to guarantee the safety of operators and environment, the controllers of chemical plants must stay active during the experiment. This implies that the feedback must be taken into account in the identification procedure. In this section it is shown that the proposed input design method is easily extendible toward closed-loop identification problems.

A typical closed-loop configuration is depicted in Figure 1, where (C_b, C_f) is a two-degree of freedom controller, and G_0 is again the process. The signals u_t and y_t are assumed to be measurable, and either one of the external excitation signals $r_{1,t}$ or $r_{2,t}$ is assumed to be present, such that r_t is measurable and *persistently exciting* of sufficient order.

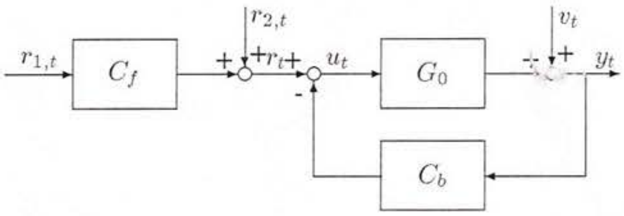

Figure 1: Closed-loop configuration

The closed-loop identification problem is to estimate a model $\hat{G}$ of G_0 from the data set $\{y_t, u_t, r_t\}_N$. In [14] an overview is given of different closed-loop identification methods, and their cost functions are unified in the *Generalized Identification* (GI) criterion

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left\{(L_y[y_t - \hat{y}_t(\theta)])^2 + \lambda^2(L_u[u_t - \hat{u}_t(\theta)])^2\right\} \tag{15}$$

where L_y and L_u are monic stable and inversely stable filters, and λ is a constant weighting parameter.
The parametrizations of $\hat{u}_t$ and $\hat{y}_t$ are

$$\hat{u}_t(\theta) = S(q,\theta)r_t \tag{16a}$$

$$\hat{y}_t(\theta) = G(q,\theta)S(q,\theta)r_t = G(\theta)\hat{u}_t \tag{16b}$$

For $N \to \infty$, $V_N(\theta)$ (15) tends to $\overline{V}(\theta)$ under weak conditions

$$\begin{aligned}\overline{V}(\theta,\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} & (|L_y|^2|G_0S_0 - G(\theta)S(\theta)|^2 + \\ & +|L_u|^2|S_0 - S(\theta)|^2)\Phi_r(\xi) \\ & +(|L_y|^2 + |L_u|^2|C_b|^2)|S_0|^2\Phi_v \, d\omega \end{aligned} \tag{17}$$

where the dependence on $e^{i\omega}$ has been omitted for readability.

Different choices of L_y, L_u, λ and the parametrizations of $y_t(\theta)$ and $u_t(\theta)$ result in different closed-loop identification methods. For example, choosing $\hat{u}_t(\beta) = S(\beta)r_t$ and $\hat{y}_t = G(\theta)S(\beta)r_t$, and minimizing V_N (15) in two steps, results in the *Two-Step* method (TS) [12].
Choosing $\hat{u}_t = u_t$ and $\hat{y}_t(\theta) = G(\theta)u_t$ results in the *Direct Identification* method (DI).

The relation (9) between $\overline{V}_{\theta\theta}$ and $\overline{V}_{\gamma\gamma}$ is valid for any identification criterion, both open- and closed-loop. For closed-loop identification, however, a different $\overline{V}_{\theta\theta}$ results.
In view of (17), we define $\Gamma(\omega,\theta)$ as in (10a), and

$$\Psi(\omega,\theta) = \begin{pmatrix} \mathrm{Re}\{S(e^{i\omega},\theta)\} & \mathrm{Im}\{S(e^{i\omega},\theta)\}\end{pmatrix}' \tag{18}$$

$$\Xi(\omega,\theta) = \begin{pmatrix} \mathrm{Re}\{GS(e^{i\omega},\theta)\} & \mathrm{Im}\{GS(e^{i\omega},\theta)\}\end{pmatrix}' \tag{19}$$

The vector $\gamma(\theta)$ is now defined as

$$\gamma(\theta) = \begin{pmatrix} \Gamma'(\omega_1,\theta)\cdots\Gamma'(\omega_m,\theta) & \Psi'(\omega_1,\theta)\cdots\Psi'(\omega_m,\theta)\end{pmatrix}' \tag{20}$$

and we derive [14]

$$\begin{aligned}\overline{V}_{\theta\theta}(\theta_0,\xi) = \frac{1}{\pi}\int_{-\pi}^{\pi} & [|L_y|^2\Xi'_\theta(\theta_0)\Xi_\theta(\theta_0) + \\ & +|L_u|^2\Psi'_\theta(\theta_0)\Psi_\theta(\theta_0)]\,\Phi_r(\xi)\,d\omega\end{aligned}$$

where $\Xi_\theta(\theta) = \Xi_\Psi(\theta)\Psi_\theta(\theta) + \Xi_\Gamma(\theta)\Gamma_\theta(\theta)$.
The Hessian $\overline{V}_{\theta\theta}(\theta_0,\xi)$ in (21) can be used in (9) to calculate $\overline{V}_{\gamma\gamma}(\theta_0,\xi)$, which in turn can be used in the input design criteria (14).

5 Analytic example

To illustrate the proposed input design procedure we present an analytic open-loop example, in which a second order FIR model $G(z,\theta) = b_1z^{-1} + b_2z^{-2}$ with $\theta = \begin{pmatrix} b_1 & b_2\end{pmatrix}'$ is estimated by applying a filtered white noise input $u_t(\xi)$

$$u_t(\xi) = K(1+d_1q^{-1}+d_2q^{-2})e_t \qquad \xi = \begin{pmatrix} K & d_1 & d_2\end{pmatrix}' \tag{21}$$

with e_t Zero Mean White Noise with unit variance.
There is a constraint on the input power, formulated as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(\omega,\xi)\,d\omega \le P_m \tag{22}$$

and hence the optimal K is obtained when the input power is maximal:

$$K^* = \sqrt{\frac{P_m}{1+d_1^2+d_2^2}} \tag{23}$$

The input design criterion is (14a), the maximization of the determinant, and hence there is no influence from W or γ_θ on ξ^*.

It can be shown [14] that in the case of an FIR parametrization, $\overline{V}_{\theta\theta}(\theta_0,\xi)$ is in fact the covariance matrix of $u_t(\xi)$, and hence we maximize the determinant of the input signal covariance matrix. It is then to be expected that a white noise signal $u_t(\xi)$ is optimal.
This is confirmed by calculating the criterion J (14a):

$$J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right) = -4K^4\frac{(1+d_1^2+d_2^2)^2 - d_1^2(1+4d_2)^2}{(1+d_1^2+d_2^2)^4} \tag{24}$$

The function $-J\left(\overline{V}_{\gamma\gamma}^{-1}(\theta_0,\xi), W\right)$ is plotted in Figure 2 for several combinations of d_1 and d_2. As expected, it is maximal for $d_1 = d_2 = 0$, for which u_t is a white noise signal. This is confirmed by analytically calculating the minimizing argument of (24) as

$$\xi^* = \begin{pmatrix} \sqrt{P_m} & 0 & 0\end{pmatrix}' \tag{25}$$

Although this example is rather trivial, it shows that the proposed input design procedure easily results in the expected optimal input protocol.
In the next section we present an example with an OE parametrization, that cannot be solved analytically.

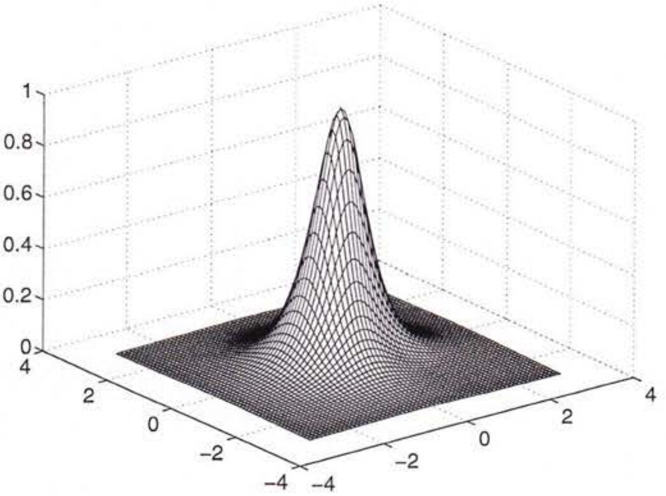

Figure 2: Input design criterion (24) as a function of d_1 and d_2

6 Numerical example

The proposed input design procedure has been implemented in Matlab. In this example we show how the different choices of input design criterion J (14), weighting matrix W (12) and points of the frequency function ω_j influence the optimal protocol ξ^*. The optimization is done using a constrained Gauss-Newton method.

A second order OE model $G(z,\theta)$ is identified

$$G(z,\theta) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + f_1 z^{-1} + f_2 z^{-2}} \qquad \theta = (\ b_1 \quad b_2 \quad f_1 \quad f_2\ )' \tag{26}$$

with $\theta_0 = (\ 1 \quad 0.5 \quad -1.5 \quad 0.7\ )'$ and a multisine input signal $u_t(\xi)$:

$$u_t(\xi) = A_1 \sin \omega_{u,1} t + A_2 \sin \omega_{u,2} t$$
$$\xi = (\ A_1 \quad \omega_{u,1} \quad A_2 \quad \omega_{u,2}\ )'$$

The input power is limited as in (22), with $P_m = 1$.

A data set is generated with the initial protocol $\xi_{\text{init}} = (\ 0.4 \quad 0.5 \quad 1.3565 \quad 2.5\ )'$, which is chosen arbitrarily. With this data set an initial model θ_{init} is estimated, with which we can design an input protocol.

Table 1 shows the results of the different experiments. In the first column the input design criterion is given. The second column shows the set of ω_j's, selected in $\gamma(\theta)$, and the third column gives the corresponding weights. In the fourth column the values of the different criteria are calculated for the initial protocol ξ_{init}. The fifth column gives the values for the optimal protocol ξ^*, and the sixth column gives ξ^*.

The first three experiments have equal ω_j's and W, and different criteria J. The second, fourth and fifth experiment have equal criterion $\lambda_{\min}$, but differ in ω_j's and W. For each experiment, different protocols ξ^* result.

Two specific observations are made. First, the determinant criterion (first row) results in a protocol for which the amplitude is equally distributed over each frequency in ξ^*. This is explained by the assumption that $\mathcal{S} \in \mathcal{M}$, which implies white output noise. The variance of $G(e^{i\omega}, \hat{\theta})$ is reduced by choosing the amplitude spectrum of the input signal proportional to Φ_v, which in this case is achieved by choosing equal amplitude fo r each frequency.

Second, the frequencies for the optimal protocols of the fourth and fifth experiment are equal. The same effect can thus be achieved by varying ω_j's or W. The amplitudes, however, are different.

Altough we have considered the frequency function variance at only two frequencies, in this example these frequencies are representative for the behavior over the complete frequnecy range [8]. This implies that the variance at lower frequencies is significantly reduced at the cost of a slight increase in variance for higher frequencies. All three criteria, (determinant, smallest eigenvalue, trace) minimize the volume of the uncertainty region, which is defined by the weighted Hessian $W'\overline{V}_{\gamma\gamma}W$. The smallest-eigenvalue criterion also takes effort in making the uncertainty region spherical. The trace criterion tries to obtain a decoupling with respect to the parameters.

7 Conclusions

In this paper a new input design methodology has been developed for reducing and influencing the variance of finite order frequency-function estimates. It is based on the relation between the Hessian $\overline{V}_{\gamma\gamma}$ of the identification criterion with respect to γ, which represents the frequency function, and the variance of the frequency function, under the assumption that $\mathcal{S} \in \mathcal{M}$ and $N \to \infty$.

The Hessian $\overline{V}_{\gamma\gamma}$ is derived for open-loop identification and the results are extended toward closed-loop identification. The input design problem is then posed as a constrained optimization problem (13).

The analytic example shows how the input design procedure calculates an optimal input protocol. The numerical example (Table 1) shows how to use the input design procedure in practice. It illustrates the effect of the different user's choices, such as the input design criterion J, the frequencies ω_j that determine γ, and the weighting matrix W.

Of course it is necessary to compare the resulting transfer function uncertainties not only at the selected frequencies but over the complete frequency band of interest. If $\mathcal{S} \in \mathcal{M}$ it is sufficient to examine the transfer function variance only at a finite number of frequencies, but it is not clear yet if this is true in the general case. Further study will be necessary.

crit	$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$	$\begin{pmatrix}w(\omega_1)\\ w(\omega_2)\end{pmatrix}$	$\text{crit}(\xi_{\text{init}})$	$\text{crit}(\xi^*)$	ξ^*
det	$\begin{pmatrix}0.3\\ 2.6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{aligned}\det &= 0.461\\ \lambda_{\min} &= 6.95\cdot 10^{-4}\\ \text{tr} &= 1446\end{aligned}$	$\begin{aligned}\det &= 1.486\cdot 10^{5}\\ \lambda_{\min} &= 0.89\\ \text{tr} &= 1.963\end{aligned}$	$\begin{pmatrix}1\\ 0.3609\\ 1\\ 0.5660\end{pmatrix}$
$\lambda_{\min}$	$\begin{pmatrix}0.3\\ 2.6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{aligned}\det &= 0.461\\ \lambda_{\min} &= 6.95\cdot 10^{-4}\\ \text{tr} &= 1446\end{aligned}$	$\begin{aligned}\det &= 5027\\ \lambda_{\min} &= 1.82\\ \text{tr} &= 1.36\end{aligned}$	$\begin{pmatrix}1.3503\\ 0.3105\\ 0.4203\\ 0.6733\end{pmatrix}$
trace	$\begin{pmatrix}0.3\\ 2.6\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{aligned}\det &= 0.461\\ \lambda_{\min} &= 6.95\cdot 10^{-4}\\ \text{tr} &= 1446\end{aligned}$	$\begin{aligned}\det &= 2.44\cdot 10^{4}\\ \lambda_{\min} &= 1.66\\ \text{tr} &= 1.27\end{aligned}$	$\begin{pmatrix}1.3027\\ 0.3226\\ 0.5505\\ 0.6214\end{pmatrix}$
$\lambda_{\min}$	$\begin{pmatrix}0.3\\ 2.6\end{pmatrix}$	$\begin{pmatrix}10\\ 1\end{pmatrix}$	$\begin{aligned}\det &= 4610\\ \lambda_{\min} &= 0.0678\\ \text{tr} &= 15.34\end{aligned}$	$\begin{aligned}\det &= 4\cdot 10^{8}\\ \lambda_{\min} &= 13.60\\ \text{tr} &= 0.0935\end{aligned}$	$\begin{pmatrix}0.8279\\ 0.3468\\ 1.1466\\ 0.7075\end{pmatrix}$
$\lambda_{\min}$	$\begin{pmatrix}0.7\\ 1.2\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{aligned}\det &= 0.137\\ \lambda_{\min} &= 0.0036\\ \text{tr} &= 285\end{aligned}$	$\begin{aligned}\det &= 955\\ \lambda_{\min} &= 1.39\\ \text{tr} &= 1.51\end{aligned}$	$\begin{pmatrix}0.3694\\ 0.3468\\ 1.3651\\ 0.7075\end{pmatrix}$

Table 1: Input design results

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