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Detecting Nonlinear Modules in a Dynamic Network: A Step-by-Step Procedure ⋆

M. Schoukens * **P.M.J. Van den Hof** *

* *Control Systems, Eindhoven University of Technology, Eindhoven, The Netherlands (e-mail: m.schoukens@tue.nl).*

Abstract: Adopting a dynamic network viewpoint allows one to analyze and identify subsystems of a complex interconnected system. When studying a network of dynamic systems, it is important to know if significant nonlinear behavior is present in a dynamic network under study, and where the nonlinearity is located in the network.
This work extends the Best Linear Approximation framework from the closed-loop to the networked setting. The framework is illustrated using a practical step-by-step estimation and analysis procedure. It is shown how nonlinear behavior can be quantified and located in a dynamic network using this framework.

Keywords: Nonlinear Systems, Dynamic Networks, System Identification, Linear Approximation

1. INTRODUCTION

Many systems can be viewed as an interconnection of multiple subsystems, e.g. large scale mechanical systems consisting of many components, the electrical grid, biological systems or industrial plants. Adopting the viewpoint of interconnected subsystems in a systems and control framework results in the so-called dynamic networks setting.

Many aspects of the identification of a linear dynamic network have received quite some attention over the last years, e.g. network structure detection (Goncalves and Warnick, 2008; Materassi and Innocenti, 2010; Chiuso and Pillonetto, 2012), identification of one or more subsystems in the network (Chiuso and Pillonetto, 2012; Van den Hof et al., 2013; Linder and Enqvist, 2017; Van den Hof et al., 2018), input selection (Dankers et al., 2016), multiple noise frameworks (Weerts et al., 2017; Van den Hof et al., 2018). However, the identification of systems operating in a nonlinear dynamic network has received consideringly less attention (Yuan et al., 2011; Cooman et al., 2016; Pan et al., 2017).

This papers focuses on the detection and quantification of nonlinearity in dynamic networks using the Best Linear Approximation (BLA) framework (Pintelon and Schoukens, 2012). The results presented in the present work are an extension of the work by Cooman et al. (2016). A noiseless setting is adopted in (Cooman et al., 2016), and also full access to all signals in the dynamic network is required. The current work adopts the framework developed in (Van den Hof et al., 2013; Dankers, 2014): noise is present in the network, and only the node signals (sum of the noisy subsystem outputs) are accessible.

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A short introduction to nonlinear dynamic networks is given in Section 2. The BLA framework is discussed next in Section 3. A step-by-step procedure for the estimation of the BLA of the subsystems in a dynamic network is presented in Section 4. Finally, the developed nonlinearity detection and quantification approach is illustrated on a simulation example in Section 5.

2. NONLINEAR DYNAMIC NETWORKS

The dynamic networks considered here follow the same definitions and visualization as in (Van den Hof et al., 2013; Dankers, 2014). A dynamic network (see Figure 1) consists of a total of L nodes, representing internal variables of the network, which are interconnected with other nodes by (nonlinear) dynamic systems. A node signal, denoted $w_i(t)$, is obtained as the sum of the outputs of the incoming (nonlinear) dynamic subsystems ($y_{ij}(t)$ denotes the output of the subsystem connecting node j to node i), an external reference signal $r_i(t)$, and a noise signal $v_i(t)$:

$$w_i(t) = \sum_{j=1,\, j\neq i}^{L} y_{ij}(t) + r_i(t) + v_i(t). \tag{1}$$

Only the node signals $w_i(t)$ and the reference signals $r_i(t)$ are known.

The node noise signal $v_i(t)$ is assumed to be zero-mean and to have a finite variance $\sigma^2_{v_i}$. Note that only noise at the network nodes are considered. No measurement noise is present in the networked system.

3. BEST LINEAR APPROXIMATION

The BLA approximates the behavior of a nonlinear system in least-squares sense using a linear time-invariant model. This results in the following definition in the single-input-single-output open-loop case in the presence of noise at

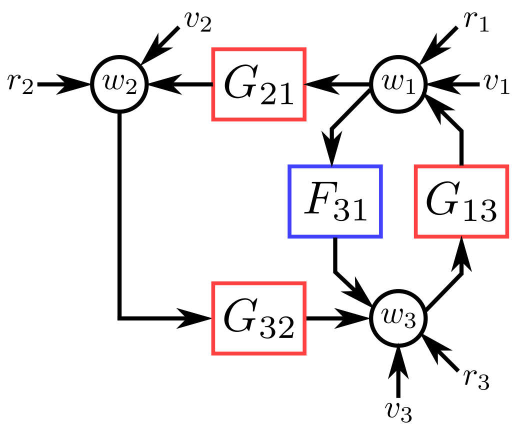

Fig. 1. An example of a nonlinear dynamic network with 3 nodes. The node signal $w_i(t)$ is obtained as the sum of the outputs y_{ij} of the linear G_{ij} and nonlinear F_{ij} subsystems connecting to it, the noise signal $v_i(t)$ and the known reference signal $r_i(t)$.

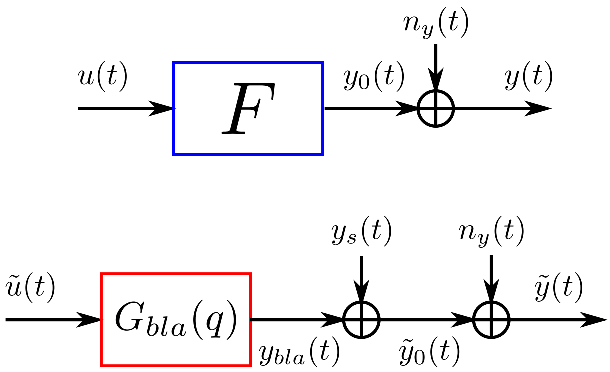

Fig. 2. A dynamic nonlinear system (F) with zero-mean (colored) additive noise $n_y(t)$ at the output only (top) is represented as the Best Linear Approximation of the system behavior $G_{bla}(q)$ and a stochastic nonlinear distortion source $y_s(t)$ (bottom).

the output of the nonlinear system only (Pintelon and Schoukens, 2012):

$$G_{bla}(q) = \arg\min_{G(q)} E_{u,n_y}\left\{|\tilde{y}(t) - G(q)\tilde{u}(t)|^2\right\}, \quad (2)$$
$$\tilde{u}(t) = u(t) - E_u\{u(t)\}, \quad (3)$$
$$\tilde{y}(t) = y(t) - E_{u,n_y}\{y(t)\}, \quad (4)$$

where $u(t)$ is the system input and $y(t)$ is the noisy system output. $E_{u,n_y}\{.\}$ denotes the expected value operator taken w.r.t. the random variations due to the input $u(t)$ and the output noise $n_y(t)$. This definition of the BLA is equivalent to the definition of the linear time-invariant second-order equivalent model defined in (Enqvist and Ljung, 2005) when the stability and causality restrictions imposed there are omitted.

Throughout this paper it is assumed that the system under study belongs to the periodic in same period out (PISPO) class of systems (Pintelon and Schoukens, 2012). This excludes nonlinear systems that present biffurcating or chaotic behavior.

As depicted in Figure 2 the output of a nonlinear system in the BLA framework is given by:

$$y(t) = G_{bla}(q)u(t) + y_s(t) + n_y(t). \quad (5)$$

The stochastic nonlinear distortion $y_s(t)$ captures the nonlinear behavior of the system that is not explained by the BLA. The stochastic distortion is shown to be zero-mean and uncorrelated with $u(t)$, but not independent of $u(t)$. Note that the BLA and the stochastic nonlinear distortion depend on the selected input excitation class. A different linear approximation can be obtained for an input excitation class with a different power spectrum or probability density function.

The nonlinearity of a system, for a given input excitation class, can be quantified by the variance of the stochastic nonlinear distortion. This variance can be estimated using, for instance, the robust BLA estimation approach (Schoukens et al., 2012).

4. ESTIMATING THE BLA IN DYNAMIC NETWORKS

Instead of estimating the BLA of an open loop system or a system operating in closed loop, this paper aims to identify the BLA of the subsystems present in a nonlinear dynamic network. This is obtained by combining the BLA framework for systems operating in closed loop (Pintelon and Schoukens, 2013) and the BLA framework for MIMO systems (Dobrowiecki and Schoukens, 2007b,a).

The closed-loop BLA framework is based upon a two-stage identification approach. Firstly, the BLA is estimated from the references to the nodes, resulting in the estimate $\mathbf{\hat{S}_{bla}}$. The disturbance-free node signal estimates $\tilde{\hat{w}}_i(t)$ are constructed next, based on the BLA estimates $\mathbf{\hat{S}_{bla}}$. Finally, BLA of the network modules $\mathbf{G_{bla,w}}$ is estimated from node to node, using the node signal estimates $\tilde{\hat{w}}_i(t)$.

The two-stage approach avoids the introduction of a bias on the estimates caused by correlated disturbance contributions due to the presence of feedback loops in the dynamic network. One could also use direct identification approaches (see Van den Hof et al. (2013)) in a dynamic network to obtain the BLA of a nonlinear module. However, the latter relies on the identification of consistent disturbance models, which can be a challenging task due to the presence of the extra nonlinear disturbance sources y_s.

Step 1: BLA from Reference to Nodes

The MIMO BLA from the references to the network nodes is defined as:

$$\mathbf{S_{bla}}(q) = \arg\min_{\mathbf{S}(q)} E_{r,v}\left\{\sum_{i=1}^{L}\left|\tilde{w}_i(t) - \sum_{j=1}^{L} S_{i,r_j}(q)\tilde{r}_j(t)\right|^2\right\}, \quad (6)$$

The expectation $E_{r,v}\{.\}$ is taken with respect to all possible realizations of the reference signals r_i and the noise signals v_i within the considered signal class. The zero-mean signals $\tilde{r}(t)$ and $\tilde{w}(t)$ are defined in the time domain as:

$$\tilde{r}_i(t) = r_i(t) - E_{r_i}\{r_i(t)\}, \quad (7)$$
$$\tilde{w}_i(t) = w_i(t) - E_{r,v}\{w_i(t)\}. \quad (8)$$

The MIMO BLA can be estimated using the MIMO Robust BLA approach from all references to all nodes (Dobrowiecki and Schoukens, 2007b,a). This results in the estimates $\hat{S}_{bla,i,r_j}(q)$ representing the BLA from reference j to node i.

Step 2: Simulate the Nodes from the References Only

The noise-free node estimates are obtained by simulating the network from the reference signals to the network nodes:

$$\tilde{\bar{w}}_i(t) = \sum_{j=1}^{L} S_{bla,i,r_j}(q)\tilde{r}_j(t). \tag{9}$$

These signals are used in a next step to obtain an estimate of the BLA in between the nodes. In practice $\tilde{\bar{w}}_i(t)$ is obtained by using the estimates $\hat{S}_{bla,i,r_j}(q)$ instead of the unknown true underlying BLA $S_{bla,i,r_j}(q)$.

Step 3: BLA from Node to Node

The BLA of the modules that are present in the nonlinear dynamic network is defined as:

$$\mathbf{G_{bla}}(q) = \tag{10}$$
$$\arg\min_{\mathbf{G}(q)} E_{r,v}\left\{\sum_{i=1}^{L}\left|\tilde{\bar{w}}_i(t)-\tilde{r}_i(t)-\sum_{j=1,\,j\neq i}^{L} G_{i,j}(q)\tilde{\bar{w}}_j(t)\right|^2\right\}.$$

The BLA is estimated using the MIMO Robust BLA approach from node to node (Dobrowiecki and Schoukens, 2007b,a), taking into account the direct contribution coming from the reference signals. The signals $\tilde{\bar{w}}_j(t)$ $(j \neq i)$ act as the input signals during the estimation, while the signal $\tilde{\bar{w}}_i(t) - \tilde{r}_i(t)$ acts as the output signal. This results in the estimate $\hat{G}_{bla,i,j}(q)$ representing the BLA from node j to node i.

Step 4: Simulate the Nodes using the Networked Framewok

The simulated node signals are obtained by:

$$\tilde{\bar{\bar{w}}}_i(t) = \tilde{r}_i(t) + \sum_{j=1,\,j\neq i}^{L} G_{bla,i,j}(q)\tilde{w}_j(t). \tag{11}$$

Note that the estimated node signal $\tilde{\bar{\bar{w}}}_i(t)$ is obtained using the noisy node signals $\tilde{w}_j(t)$ and the reference signal $\tilde{r}_i(t)$. In practice $\tilde{\bar{\bar{w}}}_i(t)$ is obtained by using the estimates $\hat{G}_{bla,i,j}(q)$ instead of the unknown true underlying BLA $G_{bla,i,j}(q)$.

In the case of a purely linear dynamic network the difference between $\tilde{\bar{\bar{w}}}_i(t)$ and $\tilde{w}_i(t)$ is the node noise $v_i(t)$. In the case of a nonlinear dynamic network the difference will also include the combined stochastic nonlinear distortion generated by all nonlinear subsystems connecting to node i.

Step 5: Determine Total, Nonlinear and Noise Distortion

Altough the robust BLA identification approach was used in this paper, Step 1 to Step 4 can be performed using multiple BLA identification approaches (Pintelon and Schoukens, 2012; Enqvist and Ljung, 2005). In this step, it is assumed that the system was excited by M random realizations and P periods of a chosen periodic signal class, as is also detailed in the so-called robust BLA esitmation method (Schoukens et al., 2012; Pintelon and Schoukens, 2012).

The measured and simulated node signals w_i are split over the multiple periods and realizations, resulting in the signals $\tilde{w}_i^{[m,p]}$ and $\tilde{\bar{\bar{w}}}_i(t)_i^{[m,p]}$, where p denotes the period and m the realization. One can observe that when a periodic signal is applied to a PISPO nonlinear system a periodic response is observed. In other words, the nonlinear response, and by extension the stochastic nonlinear distortion, does not change from one period to another. However, when the input signal changes a different nonlinear response is obtained. The stochastic nonlinear distortion varies over the signal realization, but not over the signal periods. Observe that the noise contributions vary over both the signal periods and the signal realizations.

Define the residuals $e_i^{[m,p]}(j\omega)$ as:

$$e_i^{[m,p]}(j\omega) = \tilde{\bar{\bar{w}}}_i^{[m,p]}(j\omega) - \tilde{w}_i^{[m,p]}(j\omega). \tag{12}$$

The total distortion variance (noise + stochastic nonlinear distortion) $\sigma_t^2(j\omega)$ is obtained by taking the variance of the residuals over the signal realizations, while the noise distortion variance $\sigma_n^2(j\omega)$ is obtained by taking the variance of the residuals over the signal periods. The nonlinear distortion variance $\sigma_s^2(j\omega)$ is obtained as the difference between the total distortion variance and the noise distortion variance. In the frequency domain, this becomes:

$$\sigma_n^2(j\omega) = \frac{1}{M}\frac{1}{P-1}\sum_{m=1}^{M}\sum_{p=1}^{P}\left(e_i^{[m,p]}(j\omega) - \frac{1}{P}\sum_{p=1}^{P} e_i^{[m,p]}(j\omega)\right)^2 \tag{13}$$

$$\sigma_t^2(j\omega) = \frac{1}{P}\frac{1}{M-1}\sum_{p=1}^{P}\sum_{m=1}^{M}\left(e_i^{[m,p]}(j\omega) - \frac{1}{P}\sum_{m=1}^{M} e_i^{[m,p]}(j\omega)\right)^2 \tag{14}$$

$$\sigma_s^2(j\omega) = \sigma_t^2(j\omega) - \sigma_n^2(j\omega), \tag{15}$$

where the frequency domain signals $\tilde{w}_i^{[m,p]}(j\omega)$ and $\tilde{\bar{\bar{w}}}_i^{[m,p]}(j\omega)$ are obtained by taking the discrete Fourier transform of their time-domain counterparts $\tilde{w}_i^{[m,p]}(t)$ and $\tilde{\bar{\bar{w}}}_i^{[m,p]}(t)$.

5. SIMULATION EXAMPLE

A simulation example will guide the reader through the step-by-step procedure outlined in the previous section.

5.1 System

The structure of the simulated system is visualized in Figure 1. The different subsystem are described next. The linear subsystems G_{21}, G_{32} and G_{13} are first order systems of the form:

$$x_{ij}(t+1) = A_{ij}x_{ij}(t) + B_{ij}w_j(t) \tag{16}$$
$$w_i(t) = C_{ij}x_{ij}(t), \tag{17}$$

where:

$$
\begin{aligned}
A_{21} &= 0.9,\ B_{21} = 1.0,\ \ C_{21} = 0.5 \\
A_{32} &= 0.8,\ B_{32} = 0.1,\ \ C_{32} = 1.0 \\
A_{13} &= 0.3,\ B_{13} = 1.0,\ C_{13} = -0.9
\end{aligned} \tag{18}
$$

The nonlinear subsystem F_{31} is given by:

$$w_3(t) = \tanh(w_1(t-1)). \tag{19}$$

5.2 Data

The system is excited by three reference signals $r_1(t)$, $r_2(t)$ and $r_3(t)$. These signals are all three random phase multisine signals (Pintelon and Schoukens, 2012) exciting all the frequencies $]0, f_s/2[$ with a flat amplitude spectrum. The random phases are uniformly distributed between 0 and 2π. All reference signals are active simultaneosly. $M = 15$ realizations of the multisines are applied to the system, each realization contains $P = 2$ steady state periods of $N = 4096$ points per period. The multisine signals have a standard deviation of 0.1.

The noise $v_i(t)$ active on the network nodes is Gaussian white noise with a standard deviation of 0.001. For comparison, the resulting node signals $w_i(t)$ have a standard deviation of 0.2424, 0.1573 and 0.2441 respectively.

5.3 Results

The MIMO BLA is estimated from reference to node using the robust BLA estimation method. The resulting estimates are shown in Figure 3. Note that, although all the subsystems are of order 1 or lower, quite complex dynamics are observed from the reference signals to the network nodes.

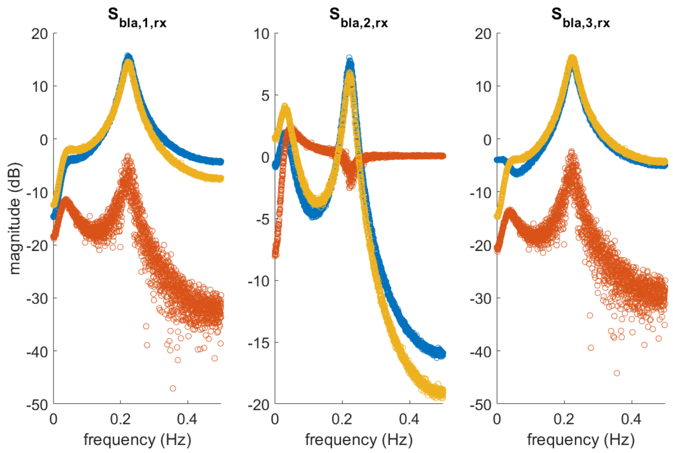

Fig. 3. The estimated BLAs from reference to node (r_1: blue, r_2: red, r_3: orange).

The nonlinearity detection and quantification analysis is performed in the MIMO BLA setting from the references to the network nodes, as is shown in Figure 4. Note that nonlinearity is detected (nonlinear distortion >> noise distortion) on all of the network nodes, although only one nonlinear subsystem is present in the networked system. Therefore, the reference to node BLA setting is not adequate to isolate the nonlinear term in the dynamic network.

The BLA is estimated in between the nodes next (Figure 5). It can be observed that the estimated BLA coincides with the true underlying linear modules. It is also clear now that the node-to-node dynamics are of low order, while a rather complex dynamic behavior was observed in Figure 3.

The nonlinearity detection and quantification analysis in the networked BLA setting shown in Figure 6 indicate that the nonlinear distortion is smaller than the noise distortion on nodes 1 and 2, indicating a linear system behavior while it is the other way around on node 3. This indicates a linear system behavior on nodes 1 and 2 and a nonlinear system behavior on node 3.

Although nonlinear behavior is detected on node 3, the presented procedure does not pinpoint which of the subsystems is nonlinear. Indeed, it can be observed from the BLA-equivalent of the nonlinear network shown in Figure 7 that the stochastic nonlinear disturbance $y_{s,31}$ impacts on node 3. However, with the given analysis, this distortion could originate both from F_{31} or G_{32}. Assigning the observed nonlinear distortion to one or more of the network modules requires further analysis beyond the scope of this paper.

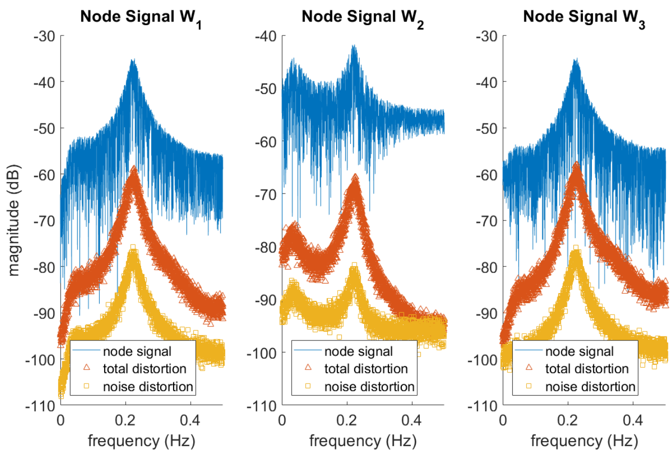

Fig. 4. The node signals and estimated noise and nonlinear distortion obtained using the MIMO BLA framework.

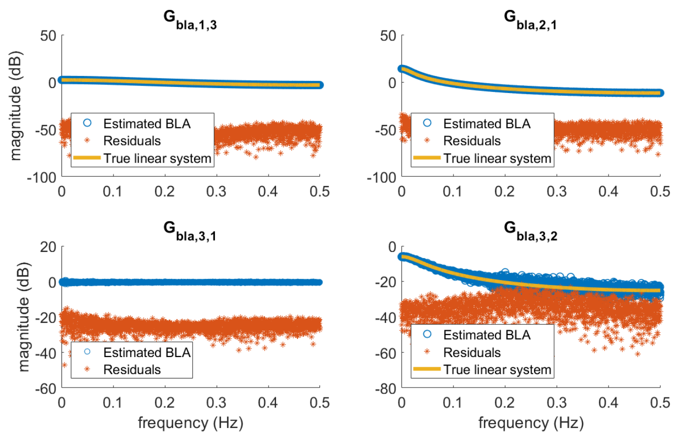

Fig. 5. The estimated BLAs of the modules present in the nonlinear dynamic network specified in Figure 1.

6. CONCLUSION

This paper presented a step-by-step procedure to detect and quantify nonlinear behavior in a dynamic network setting. The results of the proposed approach are illustrated and interpreted on a simulation example. Although nonlinearity can be detected on a node level, the presented method cannot detect which of the subsystems connecting to the nonlinear node(s) are nonlinear.

REFERENCES

Chiuso, A. and Pillonetto, G. (2012). A bayesian approach to sparse dynamic network identification. *Automatica*, 48(8), 1553 – 1565.

Cooman, A., Bronders, P., and Vandersteen, G. (2016). Distortion contribution analysis of strongly non-linear analog circuits. In *13th International Conference on Synthesis, Modeling, Analysis and Simulation Methods and Applications to Circuit Design (SMACD)*, 1–4.

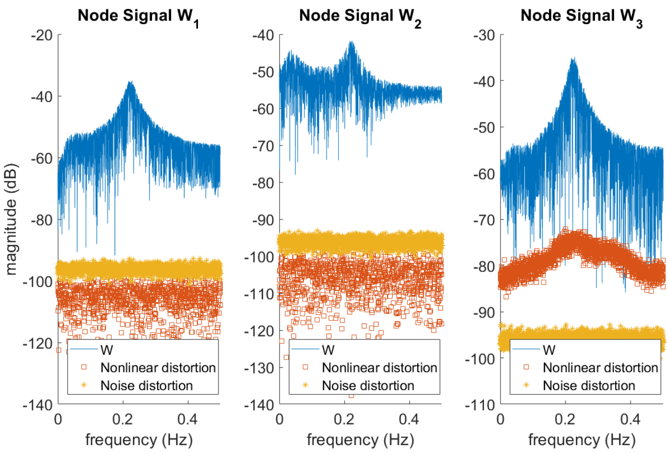

Fig. 6. The node signals and the estimated total, noise and nonlinear distortion variances. One can easily observe that only nonlinearity is detected at node 3. This corresponds with the network structure shown in Figure 1.

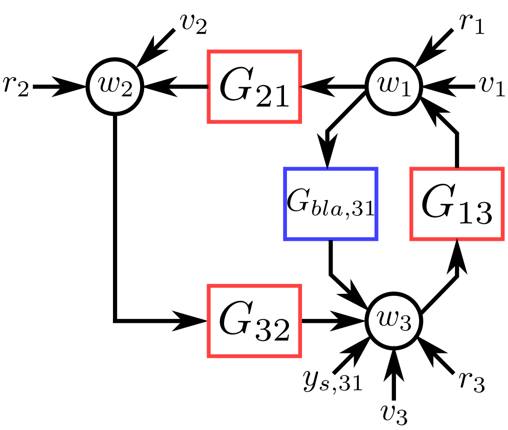

Fig. 7. The BLA representation of the example nonlinear dynamic network with 3 nodes (see Figure 1). The nonlinear subsystem F_{31} is replaced by its BLA and a stochastic nonlinear distortion source $y_{s,31}$ is added to node 3.

Dankers, A. (2014). *System Identification in Dynamic Networks*. Ph.D. thesis, Eindhoven University of Technology.

Dankers, A., Van den Hof, P., Bombois, X., and Heuberger, P. (2016). Identification of dynamic models in complex networks with prediction error methods: Predictor input selection. *IEEE Transactions on Automatic Control*, 61(4), 937–952.

Dobrowiecki, T. and Schoukens, J. (2007a). Linear approximation of weakly nonlinear MIMO systems. *IEEE Transactions on Instrumentation and Measurement*, 56(3), 887–894.

Dobrowiecki, T. and Schoukens, J. (2007b). Measuring a linear approximation to weakly nonlinear MIMO systems. *Automatica*, 43(10), 1737–1751.

Enqvist, M. and Ljung, L. (2005). Linear approximations of nonlinear FIR systems for separable input processes. *Automatica*, 41(3), 459–473.

Goncalves, J. and Warnick, S. (2008). Necessary and sufficient conditions for dynamical structure reconstruction of lti networks. *IEEE Transactions on Automatic Control*, 53(7), 1670–1674.

Linder, J. and Enqvist, M. (2017). Identification and prediction in dynamic networks with unobservable nodes. In *20th World Congress of the International Federation of Automatic Control*, 10574 – 10579.

Materassi, D. and Innocenti, G. (2010). Topological identification in networks of dynamical systems. *IEEE Transactions on Automatic Control*, 55(8), 1860–1871.

Pan, W., Yuan, Y., Ljung, L., Gonalves, J., and Stan, G. (2017). Identification of nonlinear state-space systems from heterogeneous datasets. *IEEE Transactions on Control of Network Systems*.

Pintelon, R. and Schoukens, J. (2012). *System Identification: A Frequency Domain Approach*. Wiley-IEEE Press, Hoboken, New Jersey, 2nd edition.

Pintelon, R. and Schoukens, J. (2013). FRF Measurement of Nonlinear Systems Operating in Closed Loop. *IEEE Transactions on Instrumentation and Measurement*, 62(5), 1334–1345.

Schoukens, J., Pintelon, R., and Rolain, Y. (2012). *Mastering System Identification in 100 Exercises*. John Wiley & Sons, Hoboken, New Jersey.

Van den Hof, P., Dankers, A., Heuberger, P., and Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates. *Automatica,*, Vol. 49(10), 2994–3006.

Van den Hof, P., Dankers, A., and Weerts, H. (2018). Identification in dynamic networks. *Computers & Chemical Engineering*, 109(Supplement C), 23 – 29.

Weerts, H., Van den Hof, P., and Dankers, A. (2017). Identification of dynamic networks with rank-reduced process noise. In *20th World Congress of the International Federation of Automatic Control*, 1, 10562 – 10567.

Yuan, Y., Stan, G., Warnick, S., and Goncalves, J. (2011). Robust dynamical network structure reconstruction. *Automatica*, 47(6), 1230 – 1235.