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Signal selection for local module identification in linear dynamic networks: A graphical approach ⋆

Shengling Shi * **Xiaodong Cheng** ** **Bart De Schutter** *
Paul M. J. Van den Hof ***

* *Delft University of Technology, the Netherlands (e-mail: {s.shi, b.deschutter}@ tudelft.nl).*
** *Wageningen University & Research, the Netherlands (e-mail: xiaodong.cheng@wur.nl)*
*** *Eindhoven University of Technology, the Netherlands (e-mail: p.m.j.vandenhof@tue.nl)*

Abstract: In a dynamic network of interconnected transfer functions, it is not necessary to use all the node signals for estimating a local transfer function. Given the network topology, detailed conditions are available for selecting inputs and outputs in a (MIMO) predictor model that warrants consistent and minimum variance estimation of a target module through the so-called local direct method. Motivated by the existing minimum-input signal selection approach that gradually incorporates additional signals, an alternative graphical algorithm for signal selection is developed in this work by directly exploiting the complete network graph. Then, as a straightforward application of existing analytical results, graphical conditions for consistent identification are derived for the novel signal selection approach. We show by an example that in some cases, for the consistent estimation of the target module, the developed method leads to fewer selected signals than the original minimum-input method.

Keywords: System identification, identifiability, dynamic networks, interconnected systems.

1. INTRODUCTION

In this paper, we consider a particular class of linear dynamic networks, where vertices represent measurable internal signals, and directed edges denote transfer functions referred to as modules (Gonçalves and Warnick, 2008; Materassi and Innocenti, 2010; Van den Hof et al., 2013). This model class is a natural extension of the multiple-input-multiple-output (MIMO) model settings and is useful for characterizing the causal relations among the measured signals. Note that there are also alternative model classes of network models (Yu and Verhaegen, 2018).

Various problems related to data-driven modeling of dynamic networks have been addressed, e.g., network topology estimation (Zorzi, 2022), identifiability analysis (Weerts et al., 2018; Hendrickx et al., 2019; Shi et al., 2023), and the identification of a local module (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021). We will consider the last problem in this paper.

To identify a single module given the topological information of a dynamic network, it is not necessary to estimate the complete network or to use all the available signals. The major question is how to select relevant signals and a corresponding subsystem that contains the target module, such that the subsystem together with target module can be estimated successfully, i.e., typically in the sense of statistical consistency. By exploiting the network topology, the conditions for selecting the signals are typically formulated as graphical conditions (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021).

One option is to estimate the multiple-input-single-output (MISO) subsystem that contains the target module, as in Van den Hof et al. (2013); Gevers et al. (2018). An important step has been made in Dankers et al. (2016), which shows that instead of using all the inputs of the MISO subsystem, there is freedom for signal selection according to certain graphical criterion. This signal selection scheme has motivated several extensions to incorporate extra freedom for signal selection (Linder and Enqvist, 2017; Weerts et al., 2020). In contrast to the above works, a different perspective for signal selection has been developed in Jahandari and Materassi (2022), motivated by probabilistic graphical models (Koller and Friedman, 2009).

The above MISO approaches require restricted assumptions on the network, e.g., uncorrelated disturbances in the network, and thus in more general settings, it may be necessary to select an appropriate MIMO subsystem instead for estimating the target module (Ramaswamy and Van den Hof, 2021). In this setting, consistency of single module estimates typically requires a predictor model that is the result of handling four types of conditions: (a) a module invariance condition that requires the target

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module to remain invariant upon elimination (immersing) of irrelevant node signals; (b) appropriately dealing with confounding variables, i.e. correlated disturbances on predictor inputs and outputs; (c) the presence of delays in selected loops in the network, and (d) data-informativity, i.e. sufficient excitation of the predictor model inputs. Based on these conditions, several signal selection algorithms have been developed in (Ramaswamy and Van den Hof, 2021) to formulate a MIMO predictor model. These approaches are bottom-up procedures, which start from a small set of signals around the target module and then gradually expand it toward the final predictor model.

In this work, building on the basic idea of the minimum-input approach from (Ramaswamy and Van den Hof, 2021), we develop an alternative graphical approach for selecting signals to form a MIMO predictor model. For the resulting identification problem in a prediction-error framework (Ljung, 1999), graphical conditions are straightforwardly derived to ensure the consistent identification of a target module, by exploiting existing analytical tools. An example study shows that in some situations, the developed approach requires fewer signals than the original minimum-input approach.

2. DYNAMIC NETWORKS

A dynamic network model describes the causal relationships among measured *internal signals* $w(t) \in \mathbb{R}^L$, measured excitation signals $r(t) \in \mathbb{R}^K$, and unmeasured white noises $e(t) \in \mathbb{R}^P$, with K, $P \leqslant L$ (Van den Hof et al., 2013). It is formulated as

$$w(t) = G(q)w(t) + R^0 r(t) + H(q)e(t), \tag{1}$$

where q^{-1} is the delay operator, i.e. $q^{-1}w(t) = w(t-1)$; $G(q)$, $H(q)$ are matrices of unknown rational transfer operators; R^0 is known and contains a subset of columns from an identity matrix. The matrix $G(q)$ has zeros on its main diagonal. The dependencies of transfer operators on q and signals on t will be omitted for simplicity of notation. In addition, the r and e signals are called *external signals*, and the entries in G are called *modules*. Throughout this work, we assume that e is a white noise vector; r is persistently exciting and is uncorrelated with e.

In the identification setting, the network model (1) is assumed to satisfy standard assumptions in the literature, i.e., no algebraic loop [1], G and H are proper and stable, the network is well-posed (Shi et al., 2022, Assumption 1). Following Ramaswamy and Van den Hof (2021), we also assume that the covariance matrix of e is diagonal [2]. Note that the noise model H can have more rows than columns, which is more general than the square noise model in (Ramaswamy and Van den Hof, 2021).

We assume to know the structural information of (1), i.e., the fixed zeros in G, R^0, and H. This information is encoded by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of vertices that representing the (internal and external) signals, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges representing the non-zero entries in the matrices, e.g., $G_{ji} \not\equiv 0$ iff the directed edge from w_i to w_j is in $\mathcal{E}$.

[1] This means that every directed cycle in the network has a delay.

[2] This assumption can be made without loss of generality when H can be non-monic (Youla, 1961, Theorem 2).

Given measured r and w, our goal is to estimate target module G_{ji}, with input w_i and output w_j, consistently using the local direct approach (Ramaswamy and Van den Hof, 2021) of the prediction-error method (PEM) (Ljung, 1999).

3. PRELIMINARIES: SUB-PROBLEMS IN LOCAL MODULE IDENTIFICATION

In this section, we discuss several sub-problems for local module identification. The solutions to these sub-problems will be presented in the next section.

3.1 Network immersion

To identify a single module, it is not necessary to exploit the original network model (1). We can first eliminate some irrelevant signals to obtain a new network, based on which signals or subsystems can be selected for the actual identification. This signal elimination procedure is called network immersion (Dankers et al., 2016).

Given a subset $\mathcal{S}$ of selected internal signals and after eliminating the others in (1), a new network model can be obtained:

$$w_{\mathcal{S}} = \bar{G}w_{\mathcal{S}} + \bar{R}r + \bar{H}\bar{e}, \tag{2}$$

where only the subvector $w_{\mathcal{S}}$ of w remains, and the model and its sparsity pattern change after immersion. Note that $\bar{R}$ may contain zeros, ones and unknown transfer functions. Moreover, in order to obtain a noise model $\bar{H}$ suitable for identification, new noises sources in $\bar{e}$ are introduced by spectral factorization. The detailed elimination procedure can be found in (Dankers et al., 2016). Note that (1) is the special case of (2) with $w_{\mathcal{S}} = w$.

To obtain a useful immersed network for identifying G_{ji}, one important criterion is to select $\mathcal{S}$ such that the target module G_{ji} remains invariant in (2). Then it is natural to estimate this module by exploiting the new model (2) instead. A graphical condition was developed for module invariance in Dankers et al. (2016) and can be equivalently stated as follows:

Lemma 1. (Shi et al. (2022)). In $\mathcal{G}$, if $\{w_j, w_i\} \subseteq \mathcal{S}$, and if $\mathcal{S} \setminus \{w_i, w_j\}$ is a disconnecting set [3] from w_i to the other inputs of w_j, then G_{ji} is invariant in (2).

Another important problem is to obtain the graphical representation $\mathcal{G}_{\mathcal{S}}$ of the resulting immersed network model (2) from the original network graph $\mathcal{G}$.

Definition 2. In $\mathcal{G}$, given any non-empty subset $\mathcal{S}$ of internal signals, and let set $\mathcal{X}$ contain all the external signals. Then a directed path from $\mathcal{S} \cup \mathcal{X}$ to $\mathcal{S}$ is called an **unmeasured path** (with respect to $\mathcal{S}$) if it has at least one edge and its internal vertices [4] do not belong to $\mathcal{S}$.

Then the structural information of $\bar{G}$ and $\bar{R}$ in (2) can be obtained from the original graph $\mathcal{G}$ as a direct consequence of (Dankers et al., 2016, Algorithm 3).

[3] A set $\mathcal{D}$ of vertices is a disconnecting set from vertex set $\mathcal{V}_1$ to set $\mathcal{V}_2$ if when the vertices in $\mathcal{D}$ are removed, there is no directed path from $\mathcal{V}_1$ to $\mathcal{V}_2$.

[4] Internal vertices of a directed path/cycle are the ones excluding the starting and ending vertices.

Corollary 3. Let $\mathcal{G}_{\mathcal{S}}$ denote the graph of (2) and set $\mathcal{R}$ contain all excitation signals. Then in $\mathcal{G}_{\mathcal{S}}$, any vertex v_i in $\mathcal{S} \cup \mathcal{R}$ has a directed edge to a distinct vertex $w_k \in \mathcal{S}$ if in $\mathcal{G}$, v_i has an unmeasured path to w_k with respect to $\mathcal{S}$.

However, due to the spectral factorization, it is not clear how to obtain the sparsity pattern of the noise model $\bar{H}$ in (2) from the graph $\mathcal{G}$. This sub-problem will be addressed in this work.

Sub-Problem 1. Obtain the sparsity pattern of the noise model $\bar{H}$ in (2) from the original network graph $\mathcal{G}$.

3.2 Selecting subsystem for confounding noises

Given an immersed network (2) that contains the target module and given its structural information, the next issue is to select an *appropriate* subsystem that contains the module of interest. Then the consistent estimation of this subsystem can lead to the consistent estimation of the target module. Note that selecting a subsystem is equivalent to selecting its output internal signals (and the corresponding input signals).

Recall w_j is the output of the target module G_{ji}, and consider any subvector $w_{\mathcal{Y}}$ of $w_{\mathcal{S}}$ that contains w_j. Then the rows in (2) corresponding to $w_{\mathcal{Y}}$ lead to subsystem

$$w_{\mathcal{Y}} = \bar{G}_{\mathcal{Y}\mathcal{D}} w_{\mathcal{D}} + \bar{R}_{\mathcal{Y}\mathcal{K}} r_{\mathcal{K}} + \bar{R}^0_{\mathcal{Y}\mathcal{P}} r_{\mathcal{P}} + \bar{H}_{\mathcal{Y}\mathcal{U}} \bar{e}_{\mathcal{U}}, \quad (3)$$

where $w_{\mathcal{D}}$, $\bar{e}_{\mathcal{U}}$ are subvectors of $w_{\mathcal{S}}$ and $\bar{e}$ that affect $w_{\mathcal{Y}}$ via non-zero columns in (2), and the submatrices are defined according to these signals. $r_{\mathcal{K}}$ is a subvector of r and corresponds to the unknown columns in $\bar{R}$; $r_{\mathcal{P}}$ corresponds to the known non-zero columns in $\bar{R}$ (Ramaswamy, 2022).

The main differences between (3) and a standard MIMO system are (i) $w_{\mathcal{D}}$ and $w_{\mathcal{Y}}$ may have common signals and (ii) $w_{\mathcal{Y}}$ can be correlated with the signals in $w_{\mathcal{D}}$ due to noises in the network. Therefore, to consistently estimate (3) using the direct method, one sufficient condition is to ensure that there is no noise affecting both the output $w_{\mathcal{Y}}$ and the inputs in $w_{\mathcal{D}}$ that are not in $w_{\mathcal{Y}}$; otherwise, these so-called *confounding noises* lead to correlation between inputs and outputs and therefore a lack of consistency (Dankers et al., 2016).

Sub-Problem 2. Select a subset $\mathcal{Y}$ of internal signals, such that the MIMO subsystem with outputs in $\mathcal{Y}$ contains G_{ji} and does not suffer from confounding noises.

Three signal selection algorithms have been developed in Ramaswamy and Van den Hof (2021) to address the above sub-problem. This work is inspired by the basic idea in the so-called minimum-input approach there: If a signal w_k in $w_{\mathcal{D}}$ is correlated with an output signal in $w_{\mathcal{Y}}$, then w_k is taken to be an output to expand $w_{\mathcal{Y}}$. Based on this idea, we develop an alternative approach for signal selection, together with completely graphical conditions for consistent estimation. The benefit of the developed approach will be illustrated in Example 2.

3.3 Data informativity

To consistently estimate a MIMO subsystem of the form (3) using the direct method, a data informativity condition is essential. Similar to the classical informativity condition in PEM (Ljung, 1999), the existing conditions typically take the form of rank conditions on signal power spectral densities (Ramaswamy and Van den Hof, 2021): With an appropriately chosen subvector $w_{\mathcal{Y}}$ in (3), the existing sufficient condition requires

$$\Phi_{\mathcal{K}}(\omega) \succ 0 \text{ for almost all } \omega, \quad (4)$$

where $\Phi_{\mathcal{K}}(\omega)$ is the power spectral density matrix of $\mathcal{K} = \begin{bmatrix} w_{\mathcal{D}}^\top & r_{\mathcal{K}}^\top & \bar{e}_{\mathcal{U}}^\top \end{bmatrix}^\top$ in (3). It is possible to reformulate the above condition into a graphical condition by exploiting the following result from Van den Hof and Ramaswamy (2020):

Lemma 4. Given a network model (1) with a parameter value θ and a graph $\mathcal{G}$, consider any subvector $\bar{w}$ of internal signals, and let $\mathrm{Dim}(\bar{w})$ denote its dimension. If there are $\mathrm{Dim}(\bar{w})$ vertex disjoint paths [5] from all the external signals to $\bar{w}$, then $\bar{w}$ is persistently exciting for almost all θ.

The above result shows that counting the number of vertex disjoint paths in the network graph can help us verify data informativity. The result is generic, i.e., it excludes some pathological parameter values, and it has been used to derive a graphical data informativity condition to ensure (4), under particularly chosen predictor inputs and outputs (Van den Hof and Ramaswamy, 2020). In this work, following a similar reasoning, we will derive a graphical condition to guarantee informativity of the input signals in the MIMO subsystem, selected by the developed approach for handling confounding noises.

Sub-Problem 3. Derive a graphical condition on the graph $\mathcal{G}$ such that for the MIMO subsystem (3), its input signal vector $\begin{bmatrix} w_{\mathcal{D}}^\top(t) & r_{\mathcal{K}}^\top(t) & e_{\mathcal{U}}^\top(t) \end{bmatrix}^\top$ is persistently exciting generically, i.e., its spectral density matrix has full rank for almost all ω.

4. SOLUTIONS FOR THE SUB-PROBLEMS

4.1 Immersed graph for network immersion

We first address Sub-Problem 1 and derive an approach to obtain the sparsity pattern of the noise model $\bar{H}$ in the immersed network. Given a chosen subset $\mathcal{S}$ of internal signals and the graph $\mathcal{G}$ of the original network, consider the resulting immersed network (2). We first partition $\mathcal{S}$ into several subsets as

$$\mathcal{S} = \cup_{k=1}^{m+1} \mathcal{S}_k, \quad (5)$$

such that

- in $\mathcal{G}$, for any distinct $\mathcal{S}_i$ and $\mathcal{S}_j$, there is no noise that has unmeasured paths to both $\mathcal{S}_i$ and $\mathcal{S}_j$.

Then the following result holds:

Lemma 5. Given (1) with graph $\mathcal{G}$ and any subset $\mathcal{S}$ of internal signals, consider (2). If $\mathcal{S}$ is partitioned according to (5) and let $\mathcal{S}_j^e$ be the set of white noises in $\mathcal{G}$ that have an unmeasured path to $\mathcal{S}_j$, then (2) can be permuted as

[5] Two directed paths are *vertex disjoint* if they do not share any vertex, including the starting and ending vertices.

$$\begin{bmatrix} w_{\mathcal{S}_1} \\ \vdots \\ w_{\mathcal{S}_{m+1}} \end{bmatrix} = \bar{G} \begin{bmatrix} w_{\mathcal{S}_1} \\ \vdots \\ w_{\mathcal{S}_{m+1}} \end{bmatrix} + \bar{R}r + \begin{bmatrix} \bar{H}_{\mathcal{S}_1\mathcal{B}_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{H}_{\mathcal{S}_m\mathcal{B}_m} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{e}_{\mathcal{B}_1} \\ \vdots \\ \bar{e}_{\mathcal{B}_m} \end{bmatrix}, \tag{6}$$

where $\bar{H}_{\mathcal{S}_j\mathcal{B}_j}$ can be non-square, and $|\mathcal{B}_j| = \min\{|\mathcal{S}_j^e|, |\mathcal{S}_j|\}$.

The above result can be proved by following the reasoning in (Dankers et al., 2016, Lemma 1) analogously. Lemma 5 shows that by $\mathcal{G}$, the noise model of (2) admits a block diagonal structural. The last zero block row in the noise model is to incorporate the internal signals in $w_{\mathcal{S}_{m+1}}$ that are possibly noise-free. Then combining Lemma 5 and Corollary 3 leads to the complete graph $\mathcal{G}_\mathcal{S}$ of the immersed network.

4.2 Handling confounding noises

With an immersed network model (6) and its structural information, we then address Sub-Problem 2. Motivated by the minimum-input approach of (Ramaswamy and Van den Hof, 2021) as discussed in Section 3.2, we introduce an alternative simple but general approach to select a subsystem from (6) to avoid confounding noises.

In (6), the noises in $\bar{e}_{\mathcal{B}_j}$ affect the internal signals in $w_{\mathcal{S}_j}$ only. Therefore, if the target module G_{ji} belongs to the j-th block row of $\bar{G}$, i.e., its output w_j is an entry of $w_{\mathcal{S}_j}$, then we consider the j-th MIMO subsystem only:

$$w_{\mathcal{S}_j} = \bar{G}_{\mathcal{S}_j:} w_\mathcal{S} + \bar{R}_{\mathcal{S}_j:} r + \bar{H}_{\mathcal{S}_j\mathcal{B}_j} \bar{e}_{\mathcal{B}_j}, \tag{7}$$

where $\bar{G}_{\mathcal{S}_j:}$ denotes the submatrix of $\bar{G}$ determined by the rows corresponding to $w_{\mathcal{S}_j}$, and $\bar{R}_{\mathcal{S}_j:}$ is defined similarly. It is clear that in the above subsystem, there is no confounding noise: The outputs $w_{\mathcal{S}_j}$ are affected by the noises $e_{\mathcal{B}_j}$ only, while $e_{\mathcal{B}_j}$ does not affect any input signal in $w_\mathcal{S}$ except the ones in output $w_{\mathcal{S}_j}$.

In addition, note that the submatrices $\bar{G}_{\mathcal{S}_j:}$ and $\bar{R}_{\mathcal{S}_j:}$ may contain zero columns depending on the structure of the immersed network, and thus we can re-write (7) as

$$w_{\mathcal{S}_j} = \bar{G}_{\mathcal{S}_j\mathcal{D}} w_\mathcal{D} + \bar{R}_{\mathcal{S}_j\mathcal{K}} r_\mathcal{K} + \bar{R}^0_{\mathcal{S}_j\mathcal{P}} r_\mathcal{P} + \bar{H}_{\mathcal{S}_j\mathcal{B}_j} \bar{e}_{\mathcal{B}_j}, \tag{8}$$

where we have excluded the zero columns from $\bar{G}_{\mathcal{S}_j:}$, $\bar{R}_{\mathcal{S}_j:}$ and their corresponding entries from $w_\mathcal{S}$ and r. Moreover, $\bar{R}_{\mathcal{S}_j\mathcal{K}}$ consists of the columns from $\bar{R}_{\mathcal{S}_j:}$ that have an unknown transfer function, while $\bar{R}^0_{\mathcal{S}_j\mathcal{P}}$ contains the non-zero binary columns and thus is known.

Due to the absence of confounding noises in (8), under the standard conditions of PEM, i.e., system in the model set and data informativity, directly estimating (8) using PEM can lead to a consistent estimate of $\bar{G}_{\mathcal{S}_j\mathcal{D}}$ and thus of the target module as one of its entries. We formalize this estimator as follows. Assume that $H_{\mathcal{S}_j\mathcal{B}_j}$ is square, i.e., when $|\mathcal{S}_j^e| \geqslant |\mathcal{S}_j|$, and parameterize each non-zero transfer function in (8) with independent parameters, collecting into a parameter vector θ. Then the one-step ahead predictor (Ljung, 1999) for the output $w_{\mathcal{S}_j}$ is formulated as

$$\begin{aligned} \hat{w}_{\mathcal{S}_j}(t,\theta) = {} & \bar{H}^{-1}_{\mathcal{S}_j\mathcal{B}_j}(q,\theta)[\bar{G}_{\mathcal{S}_j\mathcal{D}}(q,\theta) w_\mathcal{D} + \bar{R}_{\mathcal{S}_j\mathcal{K}}(q,\theta) r_\mathcal{K} \\ & + \bar{R}^0_{\mathcal{S}_j\mathcal{P}} r_\mathcal{P}] + (I - \bar{H}^{-1}_{\mathcal{S}_j\mathcal{B}_j}(q,\theta)) w_{\mathcal{S}_j}(t), \end{aligned} \tag{9}$$

based on which, the parameter can be estimated (asymptotically as data length approaches infinity) as

$$\theta^\star \triangleq \arg\min_\theta \frac{1}{2} \bar{\mathbb{E}} \big[w_{\mathcal{S}_j}(t) - \hat{w}_{\mathcal{S}_j}(t,\theta)\big]^\top \big[w_{\mathcal{S}_j}(t) - \hat{w}_{\mathcal{S}_j}(t,\theta)\big], \tag{10}$$

where $\bar{\mathbb{E}}$ is the expectation operator for quasi-stationary signals (Ljung, 1999). This estimator leads to an asymptotic estimate $\bar{G}_{\mathcal{S}_j\mathcal{D}}(q,\theta^\star)$, and due to the absence of confounding noises, the following result can be obtained by following (Ramaswamy and Van den Hof, 2021, Theorem 2) analogously.

Lemma 6. Given an immersed network model (6) with its graph $\mathcal{G}_\mathcal{S}$, consider its j-th MIMO subsystem (8) with a real parameter value θ_0, and suppose $|\mathcal{S}_j^e| \geqslant |\mathcal{S}_j|$. If (i) $\mathcal{S}$ satisfies the condition in Lemma 1, (ii) the input signal vector $\begin{bmatrix} w_\mathcal{D}^\top & r_\mathcal{K}^\top & e_{\mathcal{B}_j}^\top \end{bmatrix}^\top$ is persistently exciting and (iii) every directed path in $\mathcal{G}_\mathcal{S}$ from $w_{\mathcal{S}_j}$ via $w_\mathcal{D}$ to $w_{\mathcal{S}_j}$ has a delay, then the parameter estimate $\theta^\star$ from (10) satisfies $G_{ji}(q,\theta^\star) = G_{ji}(q,\theta_0)$.

In the above result, condition (i) ensures that the target module is contained in the MIMO subsystem (8). Condition (ii) follows from the classical informativity condition, and condition (iii) is a generalization of the classical algebraic loop condition: Every path from an output via the input to any other output should have a delay. In addition, $|\mathcal{S}_j^e| \geqslant |\mathcal{S}_j|$ ensures that the noise model $\bar{H}_{\mathcal{S}_j\mathcal{B}_j}$ in (9) is square and monic based on Lemma 5. Under the above conditions, Lemma 6 shows that we can consistently estimate the target module G_{ji}.

However, the graphical condition (iii) in Lemma 6 is formulated on the graph $\mathcal{G}_\mathcal{S}$ of the immersed network. We can take one step further to formulate this condition directly on the graph $\mathcal{G}$ of the original network, by exploiting the connection between $\mathcal{G}_\mathcal{S}$ and $\mathcal{G}$ in Corollary 3.

Corollary 7. Lemma 6 remains to hold if condition (iii) is replaced by the condition that every directed path in $\mathcal{G}$ from $w_{\mathcal{S}_j}$ via $w_\mathcal{D}$ to $w_{\mathcal{S}_j}$ has a delay.

4.3 Graph-based data informativity

The condition (ii) in Lemma 6 for data informativity is still formulated as a rank condition on the power spectral density. In this subsection, we will address Sub-Problem 3 and formulate this condition into a more explicit graphical condition on the graph $\mathcal{G}$ of the original network.

For the data informativity of internal signals in dynamical networks, since they are generated by external signals, there are also structural aspects in the informativity conditions, i.e., the questions whether there are a sufficient number of external signals in the network, and whether these external signals can reach internal signals of interest to provide excitation (Shi et al., 2022).

The above structural aspects become clear if we combine condition (ii) in Lemma 6 and Lemma 4 to obtain a graphical data informativity condition on the immersed network: For the input vector $\mathcal{K} = \begin{bmatrix} w_\mathcal{D}^\top & r_\mathcal{K}^\top & \bar{e}_{\mathcal{B}_j}^\top \end{bmatrix}^\top$, there are $\mathrm{Dim}(\mathcal{K})$ vertex disjoint paths from the external signals to $\mathcal{K}$. Moreover, since $r_\mathcal{K}$ and $\bar{e}_{\mathcal{B}_j}$ are external signals and are already persistently exciting, we can re-write this graphical condition equivalently by only considering the internal signals in $w_{\mathcal{D}}$:

Corollary 8. Given the immersed network (6) with graph $\mathcal{G}_{\mathcal{S}}$, consider its MIMO subsystem (8), and let $\bar{\mathcal{X}}$ be the set of all external signals in $\mathcal{G}_{\mathcal{S}}$ excluding the ones in $r_{\mathcal{K}}$ and $\bar{e}_{\mathcal{B}_j}$. Then the signal vector $\begin{bmatrix} w_{\mathcal{D}}^\top & r_{\mathcal{K}}^\top & \bar{e}_{\mathcal{B}_j}^\top \end{bmatrix}^\top$ is persistently exciting for almost all θ_0 if in $\mathcal{G}_{\mathcal{S}}$, there exist $\mathrm{Dim}(w_{\mathcal{D}})$ vertex disjoint paths from $\bar{\mathcal{X}}$ to $w_{\mathcal{D}}$.

The graphical condition in Corollary 8 is formulated on the immersed graph, where the signals $r_{\mathcal{K}}$ and $\bar{e}_{\mathcal{B}_j}$ are excluded from establishing the number of vertex disjoint paths. To further formulate it directly on the graph $\mathcal{G}$ of the original network, the main issue is to determine the signals in $r_{\mathcal{K}}$ and $\bar{e}_{\mathcal{B}_j}$ by inspecting $\mathcal{G}$.

- In the immersed network, the noises in $\bar{e}_{\mathcal{B}_j}$ are inputs to $w_{\mathcal{S}_j}$. Based on Lemma 5, these noises are generated by the white noises in $\mathcal{S}_j^e$ of the original network.

Remark 9. $r_{\mathcal{K}}$ and $r_{\mathcal{P}}$ can be determined by inspecting $\mathcal{G}$ as follows. In $\mathcal{G}$, each signal in $r_{\mathcal{P}}$ should have a directed edge to some w_k in $w_{\mathcal{S}_j}$, and all the directed cycles (also called loops) from w_k to w_k should have an internal vertex in S. The above observation can be obtained analogously to the parallel path and loop condition (Dankers et al., 2016). Then $r_{\mathcal{K}}$ consists of the other excitation signals that have an unmeasured path to $w_{\mathcal{S}_j}$ in $\mathcal{G}$, based on Corollary 3.

Note that Remark 9 is a special case of (Ramaswamy, 2022, Lemma 4.1) with set $\mathcal{B} = \emptyset$ (defined there).

The above observations directly lead to the following:

Lemma 10. In $\mathcal{G}$, let set $\tilde{\mathcal{X}}$ contain all external signals excluding the noises in $\mathcal{S}_j^e$ and $r_{\mathcal{K}}$. Then $\begin{bmatrix} w_{\mathcal{D}}^\top & r_{\mathcal{K}}^\top & \bar{e}_{\mathcal{B}_j}^\top \end{bmatrix}^\top$ in (8) is persistently exciting for almost all θ_0 if in $\mathcal{G}$, there exist $\mathrm{Dim}(w_{\mathcal{D}})$ vertex disjoint paths from $\tilde{\mathcal{X}}$ to $w_{\mathcal{D}}$.

5. GRAPHICAL SIGNAL SELECTION FOR SINGLE MODULE IDENTIFICATION

Selecting appropriate signals for estimating a single target module G_{ji} involves the combination of the solutions to all the previous sub-problems. We first summarize them into the following identification algorithm:

Algorithm 1: given $\mathcal{G}$, target G_{ji}, and data

(a) Select a subset $\mathcal{S}$ of internal signals that satisfies $\{w_i, w_j\} \subseteq \mathcal{S}$ and the graphical condition in Lemma 1;
(b) Select predictor outputs: Partition $\mathcal{S}$ according to (5), and in the partition, let $\mathcal{S}_j$ denote the subset that contains w_j. Then select the signals in $\mathcal{S}_j$ to be the predictor outputs, collected into a vector $w_{\mathcal{S}_j}$. In addition, let set $\mathcal{S}_j^e$ contain the noises that have an unmeasured path to $w_{\mathcal{S}_j}$;
(c) Select predictor inputs: In $\mathcal{G}$, determine $w_{\mathcal{D}}$ that contains the internal signals in $\mathcal{S}$ having an unmeasured path (with respect to $\mathcal{S}$) to $w_{\mathcal{S}_j}$; $r_{\mathcal{K}}$ and $r_{\mathcal{P}}$ are determined graphically according to Remark 9;
(d) Construct a predictor model as in (9), where $w_{\mathcal{S}_j}$ is the predictor output, and $w_{\mathcal{D}}$, $r_{\mathcal{K}}$ and $r_{\mathcal{P}}$ are the predictor inputs;
(e) Compute a parameter estimate $\theta^\star$ as (10)

In the above algorithm, step (a) selects the remaining internal signals in $\mathcal{S}$ for the immersed network. The final selected signals for estimation are the ones in $w_{\mathcal{S}_j}$, $w_{\mathcal{D}}$, $r_{\mathcal{K}}$ and $r_{\mathcal{P}}$, which have been obtained in step (b), (c) and Remark 9 by directly inspecting the graph $\mathcal{G}$.

To ensure the consistent identification of G_{ji} using Algorithm 1, the following conditions should be ensured:

- The target module should be invariant after immersion, i.e., the selected set $\mathcal{S}$ satisfies the graphical condition in Lemma 1
- The relevant signals should be informative

The above conditions can be achieved by combining Lemma 1, Corollary 7, and Lemma 10, which directly leads to the final consistency result:

Theorem 11. Given (1) with $\mathcal{G}$, consider Algorithm 1 and suppose $|\mathcal{S}_j^e| \geqslant |\mathcal{S}_j|$ at step (b). In $\mathcal{G}$, let $\tilde{\mathcal{X}}$ be defined as in Proposition 10. Then if the following conditions hold for $\mathcal{G}$: (i) $\mathcal{S}$ satisfies the graphical condition in Lemma 1, (ii) there exist $\mathrm{Dim}(w_{\mathcal{D}})$ vertex disjoint paths from $\tilde{\mathcal{X}}$ to $w_{\mathcal{D}}$, and (iii) every directed path from $w_{\mathcal{S}_j}$ via $w_{\mathcal{D}}$ to $w_{\mathcal{S}_j}$ has a delay, then $G_{ji}(q, \theta^\star) = G_{ji}(q, \theta_0)$ for almost all θ_0.

In the above theorem, condition (i) ensures that the target module G_{ji} is indeed an entry of the estimated MIMO subsystem (8). The condition (ii) is a graphical condition for data informativity from Lemma 10, and condition (iii) is the condition on the delay from Corollary 7.

Note that all the three conditions in Theorem 11 are graphical and formulated on the graph $\mathcal{G}$ of the original graph. This is in contrast to the results in (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021) where a non-graphical data informativity condition is enforced. Compared to the graphical results in Van den Hof and Ramaswamy (2020); Van den Hof et al. (2023), our conditions are developed for the specific signal selection procedure in Algorithm 1.

Example 1. Consider the network in Fig. 1(a), and our goal is to identify G_{31}. To apply Algorithm 1, we select the green vertices in $\mathcal{S} = \{w_1, w_2, w_3\}$, which clearly satisfies the condition in Lemma 1. Then we can partition $\mathcal{S}$ as $\mathcal{S} = \{w_1\} \cup \{w_2, w_3\}$, as e_4 have unmeasured paths to both w_2 and w_3. In this partition, $\{w_2, w_3\}$ is the set $\mathcal{S}_j$ in Algorithm 1, i.e., the predictor outputs, and we have $\mathcal{S}_j^e = \{e_3, e_4\}$. Correspondingly, w_1 and w_2 are the predictor inputs for the outputs in $\{w_2, w_3\}$. Moreover, new noises $\bar{e}_2$ and $\bar{e}_3$ are introduced in Fig. 1(b) as in (6). Note that the informativity condition in Theorem 11 is also satisfied: In Fig. 1(a), there are two vertex disjoint paths from $\tilde{\mathcal{X}} = \{r_1, r_2\}$, with $\{e_3, e_4\}$ excluded, to the two predictor inputs in $\{w_1, w_2\}$.

As a comparison between Algorithm 1 and the minimum-input approach for signal selection from Ramaswamy and Van den Hof (2021), we show that in the following example, Algorithm 1 leads to a predictor with fewer signals.

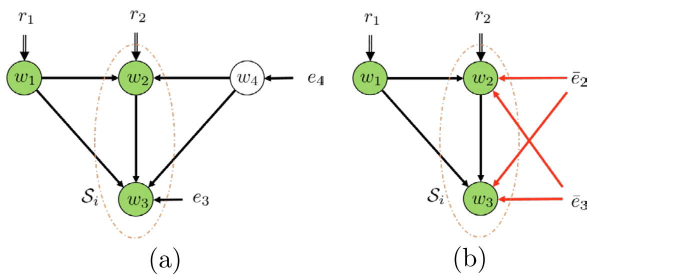

Fig. 1. Signal selection for identifying G_{31} in the original network, shown in Fig. 1(a). The immersed network with w_4 eliminated is shown in (b), where new noises $\bar{e}_2$ and $\bar{e}_3$ are introduced.

Example 2. Given the example in Fig. 2 from Fonken (2023), the goal is to select signals for identifying G_{12}. Algorithm 1 leads to a predictor model with w_1 and w_3 as predictor outputs, and w_2 as the predictor input. The path-based informativity condition is also satisfied by e_2.

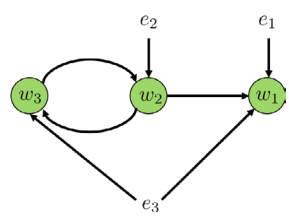

Fig. 2. A network with three internal signals (Fonken, 2023).

However, the minimum-input algorithm (Ramaswamy and Van den Hof, 2021; Van den Hof et al., 2023) requires $\{w_1, w_2, w_3\}$ as the set of predictor outputs and $\{w_2, w_3\}$ as the set of predictor inputs, see Section 7 of Van den Hof et al. (2023), while excitation signals would have to be added to w_2 and w_3 to achieve data informativity.

6. CONCLUSION

The signal selection problem for identifying a local transfer function in a linear dynamical network is considered. Motivated by the existing minimum-input approach, we have developed an alternative graphical signal selection approach that can lead to fewer selected signals in some cases. Graphical conditions for the consistent estimation have also been derived for the developed approach by applying existing analytical tools.

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