Signal selection for local module identification in linear dynamic networks: A graphical approach ⋆

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Abstract: In a dynamic network of interconnected transfer functions, it is not necessary to make use of all the node signals in the network for estimating a local transfer function. Given the network topology, detailed conditions are available for selecting inputs and outputs in a predictor model that warrants consistent and minimum variance estimation of a target module through the so-called local direct method. These conditions are derived from four aspects: module invariance, confounding variable handling, loop delay conditions and data-informativity. In this paper it is shown that for the most general (MIMO) identification approach to the problem, all these conditions can be merged into one set of graphical conditions, and result in a particular algorithm in which the conditions can directly be verified by inspecting the network graph.

Keywords: System identification, identifiability, dynamic networks, interconnected systems.

1. INTRODUCTION

In this paper, we consider a particular class of linear dynamic networks, where vertices represent measurable internal signals, and directed edges denote transfer functions referred to as modules (Gonçalves and Warnick, 2008; Materassi and Innocenti, 2010; Van den Hof et al., 2013). This model class is a natural extension of the single-input-single-output (SISO) and the multiple-input-multiple-output (MIMO) model settings and is useful for characterizing the causal relations among the measured signals. Note that there are also alternative model classes of network models, e.g. in the form of state-space models (Haber and Verhaegen, 2014; Yu and Verhaegen, 2018).

Various problems related to data-driven modeling of dynamic networks have been addressed in the literature, e.g., the estimation of the network topology (Zorzi, 2022), the identifiability analysis (Weerts et al., 2018; Hendrickx et al., 2019), and the identification of a local module (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021). We will consider the last problem in this paper.

To identify a single module given the topological information of a dynamic network, it is not necessary to estimate the complete network or to use all the available signals. Therefore, the major question is how to select relevant signals and a corresponding subsystem that contains the target module, such that the subsystem together with target module can be estimated successfully, i.e., typically in the sense of statistical consistency. By exploiting the network topology, the conditions for selecting the signals are typically formulated as graphical conditions (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021).

One option is to estimate the multiple-input-single-output (MISO) subsystem that contains the target module, as considered in Van den Hof et al. (2013); Gevers et al. (2018). An important step has been made in Dankers et al. (2016), which shows that instead of using all the inputs of the MISO subsystem, there is freedom for signal selection according to certain graphical criterion. This signal selection scheme has motivated several extensions to incorporate extra freedom for signal selection (Linder and Enqvist, 2017; Weerts et al., 2020). In contrast to the above works, a different perspective for signal selection has been developed in Materassi and Salapaka (2020), motivated by probabilistic graphical models (Koller and Friedman, 2009). Necessary and sufficient graphical conditions for consistent estimation in the MISO setting are presented in Jahandari and Materassi (2022).

The above MISO approaches require restricted assumptions on the data-generating network, e.g., uncorrelated disturbances in the network, and thus in more general settings, it may be necessary to select an appropriate MIMO subsystem instead for estimating the target module (Ramaswamy and Van den Hof, 2021).

In this setting, consistency of single module estimates typically requires a predictor model that is the result

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of handling four types of conditions: (a) a module invariance condition that requires the target module to remain invariant upon elimination (immersing) of irrelevant node signals; (b) appropriately dealing with confounding variables, i.e. correlated disturbances on predictor inputs and outputs; (c) the presence of delays in selected loops in the network, and (d) data-informativity, i.e. sufficient excitation of the predictor model inputs. While for all four conditions, separate results have been formulated, in this paper we will develop a result that brings them all into one graphical formulation, while the condition on full rank disturbances on the network will be further relaxed.

This paper is structured as follows. We will first introduce the dynamic network model in Section 2, and then the different conditions or sub-problems of local module identification are introduced in Section 3. The developed graphical solutions to these sub-problems are presented in Section 4. Combining the different solutions leads to the final algorithm in Section 5, which allows a simple graphical verification of the consistency conditions, and mimics the so-called "minimum-input case" algorithm in Ramaswamy and Van den Hof (2021).

2. DYNAMIC NETWORKS

A dynamic network model describes the causal relationships among measured *internal signals* $w(t) \in \mathbb{R}^L$, measured excitation signals $r(t) \in \mathbb{R}^K$, and unmeasured white noises $e(t) \in \mathbb{R}^P$, with K, $P \leqslant L$ (Van den Hof et al., 2013). It is formulated as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t), \tag{1}$$

where q^{-1} is the delay operator, i.e. $q^{-1}w(t) = w(t-1)$, and $G(q)$, $R(q)$, and $H(q)$ are matrices of rational transfer operators. The matrix $G(q)$ has zeros on its main diagonal. The dependencies of transfer operators on q and signals on t will be omitted for simplicity of notation. In addition, the r and e signals are called *external signals*, and the entries in G are called *modules*. Throughout this work, we assume that e is a white noise vector; r is persistently exciting and is uncorrelated with e.

In the identification setting, the network model (1) is assumed to satisfy standard assumptions in the literature, i.e., no algebraic loop [1], G, R and H are proper and stable, the network is well-posed. To avoid the technical burden, the details of these assumptions can be found in (Shi et al., 2022, Assumption 1). Following Ramaswamy and Van den Hof (2021), we also assume that the covariance matrix of e is diagonal [2]. Note that the noise model H can have more rows than columns, i.e., the reduced-rank setting, which is more general than the square noise model in (Ramaswamy and Van den Hof, 2021).

In addition, we assume to have prior knowledge of the structural information of the data-generation network (1), i.e., the structural zeros in the matrices G, R, and H. This information is encoded by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of vertices that represent all the (internal and external) signals, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges representing the non-zero entries in the matrices, e.g., $G_{ji} \not\equiv 0$ iff the directed edge from w_i to w_j is in $\mathcal{E}$.

Given measured r and w, our goal is to estimate one single module G_{ji}, with input w_i and output w_j, consistently using the standard local direct approach (Van den Hof et al., 2013) of the prediction-error method (PEM) (Ljung, 1999). While this can be done trivially by estimating the complete network model (1) and using all the measured signals, we aim to estimate G_{ji} using fewer measured signals, and the main issue is to select relevant node signals as predictor inputs and predictor outputs in PEM for a consistent estimation.

3. PRELIMINARIES: SUB-PROBLEMS IN LOCAL MODULE IDENTIFICATION

In this section, we discuss several important sub-problems for local module identification. The solutions to these sub-problems will be presented in the next section and then combined to address the local module identification problem.

3.1 Network immersion

When the target is to identify a single module, it may not be necessary to exploit the original network model (1). We can first eliminate some irrelevant signals in (1) to obtain a new network, based on which signals or subsystems can be selected for the actual identification. This signal elimination procedure is called network immersion (Dankers et al., 2016).

Given a subset $\mathcal{S}$ of selected internal signals and after eliminating the others in (1), a new network model can be obtained:

$$w_{\mathcal{S}} = \bar{G}w_{\mathcal{S}} + \bar{R}r + \bar{H}\bar{e}, \tag{2}$$

where only the subvector $w_{\mathcal{S}}$ of w remains, and the model and its sparsity pattern change after immersion. Moreover, in order to obtain a noise model $\bar{H}$ suitable for identification, new noises sources in $\bar{e}$ are introduced by spectral factorization. The detailed elimination procedure is presented in Appendix A for completeness.

Remark 1. Note that (1) is the special case of (2) when we choose $w_{\mathcal{S}} = w$ and $\bar{e} = e$, i.e., directly using the original network without eliminating any signal. Therefore, all results for immersed networks can also be applied to the original network (1).

To obtain a useful immersed network for identifying G_{ji}, one important criterion is to select $\mathcal{S}$ such that the target module G_{ji} remains invariant after the signal elimination, which is the so-called *module invariance* property (Dankers et al., 2016). Then it is natural to estimate this module by exploiting the new model (2) instead. As investigated in (Dankers et al., 2016), a graphical condition called *parallel path and loop condition* can be used to ensure the module invariance property, and this condition can be equivalently stated as follows:

Lemma 2. (Shi et al. (2022)). Given a target module G_{ji} and a subset $\mathcal{S}$ of internal signals, consider the resulting immersed network (2). If $\{w_j, w_i\} \subseteq \mathcal{S}$, and if $\mathcal{S} \setminus \{w_i, w_j\}$

[1] This means that every directed cycle in the network has a delay.
[2] This assumption can be made without loss of generality when H can be non-monic (Youla, 1961, Theorem 2).

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is a disconnecting set [3] from w_i to the other inputs of w_j, then G_{ji} is invariant in (2).

Another important problem related to network immersion is to obtain the graphical representation $\mathcal{G}_\mathcal{S}$ of the resulting immersed network model (2) from the original network graph $\mathcal{G}$. This is highly relevant because when identifying the target module using the immersed network, the availability of $\mathcal{G}_\mathcal{S}$ allows us to exploit its structural information. To this end, we first introduce a graphical concept.

Definition 3. Given a graph $\mathcal{G}$ of a dynamic network, consider any non-empty subset $\mathcal{S}$ of internal signals, and let set $\mathcal{X}$ contain all the external signals. Then a directed path from $\mathcal{S}\cup\mathcal{X}$ to $\mathcal{S}$ is called an **unmeasured path** (with respect to $\mathcal{S}$) if it has at least one edge and its internal vertices [4] do not belong to $\mathcal{S}$.

Then the structural information of $\bar{G}$ and $\bar{R}$ in (2) can be obtained from the original graph $\mathcal{G}$ as a direct consequence of (Dankers et al., 2016, Algorithm 3).

Corollary 4. Given an original network (1) with graph $\mathcal{G}$ and any subset $\mathcal{S}$ of internal signals, consider the resulting immersed network (2), and let $\mathcal{G}_\mathcal{S}$ denote its graph. Let set $\mathcal{R}$ denote the set of all reference signals. Then in $\mathcal{G}_\mathcal{S}$, any vertex v_i in $\mathcal{S}\cup\mathcal{R}$ has a directed edge to a distinct vertex $w_k \in \mathcal{S}$ if in $\mathcal{G}$, v_i has an unmeasured path to w_k with respect to $\mathcal{S}$.

However, due to the spectral factorization, it is not clear how to obtain the sparsity pattern of the noise model $\bar{H}$ in (2) from the graph $\mathcal{G}$. This sub-problem will be addressed in this work.

Sub-Problem 1. Given any subset $\mathcal{S}$ of internal signals, consider the immersed network (2). Obtain the sparsity pattern of the noise model $\bar{H}$ from the original network graph $\mathcal{G}$.

3.2 Selecting subsystem for confounding noises

Given an immersed network (2) that still contains the target module and given its structural information, the next issue for single module identification is to select an *appropriate* subsystem that contains the module of interest. Then the consistent estimation of this subsystem can lead to the consistent estimation of the target module. Note that selecting a subsystem is equivalent to selecting its output internal signals (and the corresponding input signals).

Recall w_j is the output of the target module G_{ji}, and consider any subvector $w_\mathcal{Y}$ of $w_\mathcal{S}$ that contains w_j. Then the rows in (2) corresponding to $w_\mathcal{Y}$ lead to the following MIMO subsystem:

$$w_\mathcal{Y} = \bar{G}_{\mathcal{Y}\mathcal{D}} w_\mathcal{D} + \bar{R}_{\mathcal{Y}\mathcal{U}} r_\mathcal{U} + \bar{H}_{\mathcal{Y}\mathcal{U}} \bar{e}_\mathcal{U}, \tag{3}$$

where $w_\mathcal{D}$, $r_\mathcal{U}$, $\bar{e}_\mathcal{U}$ are subvectors of $w_\mathcal{S}$, r, and $\bar{e}$ that affect $w_\mathcal{Y}$ via non-zero columns in (2), and the submatrices are defined according to these signals.

The main differences between (3) and a standard MIMO system are (i) $w_\mathcal{D}$ and $w_\mathcal{Y}$ may have common signals and (ii) $w_\mathcal{Y}$ can be correlated with the signals (different from $w_\mathcal{Y}$) in $w_\mathcal{D}$ due to noises in the network. Therefore, to consistently estimate (3) using the direct method, one sufficient condition is to ensure that there is no noise affecting both the output $w_\mathcal{Y}$ and the inputs in $w_\mathcal{D}$ that are not in $w_\mathcal{Y}$; otherwise, these so-called *confounding noises* lead to correlation between inputs and outputs and therefore a lack of consistency (Van den Hof et al., 2017).

Sub-Problem 2. Select a subset $\mathcal{Y}$ of internal signals, such that the MIMO subsystem with outputs in $\mathcal{Y}$ contains the target module G_{ji} and does not suffer from confounding noises.

This problem has been considered in Ramaswamy and Van den Hof (2021), where several different approaches have been proposed. We consider the basic idea in the so-called minimum-input approach of Ramaswamy and Van den Hof (2021): If a signal w_k in $w_\mathcal{D}$ is correlated with an output signal in $w_\mathcal{Y}$, then w_k is also taken to be an output to expand the vector $w_\mathcal{Y}$. This is a bottom-to-top approach, i.e., starting from a small-size MIMO subsystem and gradually expanding this subsystem. Based on this basic idea, we will take a global approach in this work where we directly explore the network graph and decide the subsystem. This approach can be more easily incorporated with the solutions to other sub-problems, leading to completely graphical conditions for consistent estimation.

3.3 Data informativity

To consistently estimate a MIMO subsystem of the form (3) using the direct method, a data informativity condition is essential. Similar to the classical informativity condition in PEM (Ljung, 1999), the existing data-informativity conditions for local module identification typically takes the form of rank conditions on signal power spectral densities (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021): With an appropriate chosen subvector $w_\mathcal{Y}$ in (3), the existing sufficient condition requires

$$\Phi_\mathcal{K}(\omega) \succ 0 \text{ for almost all } \omega, \tag{4}$$

where $\Phi_\mathcal{K}(\omega)$ is the power spectral density matrix of the input signal vector in (3):

$$\mathcal{K} = \left[w_\mathcal{D}^\top \; r_\mathcal{U}^\top \; \bar{e}_\mathcal{U}^\top\right]^\top .$$

It is possible to reformulate the above condition into a graphical condition by exploiting the following general result:

Lemma 5. (Van den Hof and Ramaswamy (2020)). Given a network model (1) with a parameter value θ and a graph $\mathcal{G}$, consider any subvector $\bar{w}$ of internal signals, and let $\text{Dim}(\bar{w})$ denote its dimension. If there are $\text{Dim}(\bar{w})$ vertex disjoint paths [5] from all the external signals to $\bar{w}$, then $\bar{w}$ is persistently exciting for almost all θ.

The above result shows counting the number of vertex disjoint paths in the network graph can help us verify data informativity. The result is generic, i.e., it excludes some pathological parameter values, and it has been used

[3] A set $\mathcal{D}$ of vertices is a disconnecting set from vertex set $\mathcal{V}_1$ to set $\mathcal{V}_2$ if when the vertices in $\mathcal{D}$ are removed, there is no directed path from $\mathcal{V}_1$ to $\mathcal{V}_2$.

[4] Internal vertices of a directed path are the ones excluding the starting and ending vertices.

[5] Two directed paths are *vertex disjoint* if they do not share any vertex, including the starting and ending vertices.

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to derive a graphical data informativity condition to ensure (4), under particularly chosen predictor inputs and outputs (Van den Hof and Ramaswamy, 2020).

In this work, following a similar reasoning, we will derive a graphical condition to guarantee informativity of the input signals in the MIMO subsystem, selected by the developed approach for handling confounding noises.

Sub-Problem 3. Derive a graphical condition on the graph $\mathcal{G}$ such that for the MIMO subsystem (3), its input signal vector $\left[w_{\mathcal{D}}^{\top}(t)\ r_{\mathcal{U}}^{\top}(t)\ e_{\mathcal{U}}^{\top}(t)\right]^{\top}$ is persistently exciting, i.e., its spectral density matrix has full rank for almost all ω.

4. SOLUTIONS FOR THE SUB-PROBLEMS

4.1 Immersed graph for network immersion

We first address Sub-Problem 1 and derive an approach to obtain the sparsity pattern of the noise model $\bar{H}$ in the immersed network. Given a chosen subset $\mathcal{S}$ of internal signals and the graph $\mathcal{G}$ of the original network, consider the resulting immersed network (2). We first partition $\mathcal{S}$ into several subsets as

$$\mathcal{S} = \cup_{k=1}^{m+1} \mathcal{S}_i, \tag{5}$$

such that

- in $\mathcal{G}$, for any distinct $\mathcal{S}_i$ and $\mathcal{S}_j$, there is no noise that has unmeasured paths to both $\mathcal{S}_i$ and $\mathcal{S}_j$.

Then the following result holds:

Lemma 6. Given an original network (1) with graph $\mathcal{G}$ and any subset $\mathcal{S}$ of internal signals, consider the resulting immersed network (2). If the set $\mathcal{S}$ is partitioned according to (5) and let $\mathcal{S}_i^e$ be the set of white noises in $\mathcal{G}$ that have an unmeasured path to $\mathcal{S}_i$, then (2) can be permuted such that its noise model admits a block diagonal structure as

$$\begin{bmatrix} w_{\mathcal{S}_1} \\ \vdots \\ w_{\mathcal{S}_{m+1}} \end{bmatrix} = \bar{G} \begin{bmatrix} w_{\mathcal{S}_1} \\ \vdots \\ w_{\mathcal{S}_{m+1}} \end{bmatrix} + \bar{R}r + \begin{bmatrix} \bar{H}_{\mathcal{S}_1\mathcal{B}_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{H}_{\mathcal{S}_m\mathcal{B}_m} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{e}_{\mathcal{B}_1} \\ \vdots \\ \bar{e}_{\mathcal{B}_m} \end{bmatrix}, \tag{6}$$

where $\bar{H}_{\mathcal{S}_i\mathcal{B}_i}$ can be non-square, and $|\mathcal{B}_i| = \min\{|\mathcal{S}_i^e|, |\mathcal{S}_i|\}$.

The proof of the above result is presented in Appendix B. Lemma 6 shows that by inspecting the graph $\mathcal{G}$ of the original network, the noise model of the immersed network (2) admits a block diagonal structural as in (6). The last zero block row in the noise model is to incorporate the internal signals in $w_{\mathcal{S}_{m+1}}$ that are possibly noise-free. Then combining Lemma 6 and Corollary 4 leads to the complete graph $\mathcal{G}_{\mathcal{S}}$ of the immersed network.

4.2 Handling confounding noises

With an immersed network model (6) and its structural information, we then address Sub-Problem 2. Motivated by the minimum-input approach of (Ramaswamy and Van den Hof, 2021) as discussed in Section 3.2, we introduce a simple but general approach to select a subsystem from (6) to avoid confounding noises.

In (6), the noises in $\bar{e}_{\mathcal{B}_i}$ affect the internal signals in $w_{\mathcal{S}_i}$ only. Therefore, if the target module G_{ji} belongs to the i-th block row of $\bar{G}$, i.e., its output w_j is an entry of $w_{\mathcal{S}_i}$, then we consider the i-th MIMO subsystem only:

$$w_{\mathcal{S}_i} = \bar{G}_{\mathcal{S}_i:} w_{\mathcal{S}} + \bar{R}_{\mathcal{S}_i:} r + \bar{H}_{\mathcal{S}_i\mathcal{B}_i}\bar{e}_{\mathcal{B}_i}, \tag{7}$$

where $\bar{G}_{\mathcal{S}_i:}$ denotes the submatrix of $\bar{G}$ determined by the rows corresponding to $w_{\mathcal{S}_i}$, and $\bar{R}_{\mathcal{S}_i:}$ is defined similarly. It is clear that in the above subsystem, there is no confounding noise: The outputs $w_{\mathcal{S}_i}$ are affected by the noises $e_{\mathcal{B}_i}$ only, while $e_{\mathcal{B}_i}$ do not affect any input signal in $w_{\mathcal{S}}$ except the ones in output $w_{\mathcal{S}_i}$.

In addition, note that the submatrices $\bar{G}_{\mathcal{S}_i:}$ and $\bar{R}_{\mathcal{S}_i:}$ may have an additional structure: They may contain zero columns depending on the structure of the immersed network. In this case, we can re-write (7) equivalently as

$$w_{\mathcal{S}_i} = \bar{G}_{\mathcal{S}_i\mathcal{D}} w_{\mathcal{D}} + \bar{R}_{\mathcal{S}_i\mathcal{R}_i} r_{\mathcal{R}_i} + \bar{H}_{\mathcal{S}_i\mathcal{B}_i}\bar{e}_{\mathcal{B}_i}, \tag{8}$$

where we have excluded the zero columns from $\bar{G}_{\mathcal{S}_i:}$, $\bar{R}_{\mathcal{S}_i:}$ and their corresponding entries from $w_{\mathcal{S}}$ and r.

Due to the absence of confounding noises in (8), under the standard conditions of PEM, i.e., system in the model set and data informativity, directly estimating (8) using PEM can lead to a consistent estimate of $\bar{G}_{\mathcal{S}_i\mathcal{D}}$ and thus the target module as its entry. We formalize this estimator as follows.

Assume that the noise model $H_{\mathcal{S}_i\mathcal{B}_i}$ is square, i.e., when $|\mathcal{S}_i^e| \geqslant |\mathcal{S}_i|$, and parameterize each non-zero transfer function in (8) with independent parameters, collecting into a parameter vector θ. Then the one-step ahead predictor (Ljung, 1999) for the output $w_{\mathcal{S}_i}$ is formulated as

$$\begin{aligned} \hat{w}_{\mathcal{S}_i}(t,\theta) = {} & \bar{H}^{-1}_{\mathcal{S}_i\mathcal{B}_i}(q,\theta)[\bar{G}_{\mathcal{S}_i\mathcal{D}}(q,\theta)w_{\mathcal{D}} + \bar{R}_{\mathcal{S}_i\mathcal{R}_i}(q,\theta)r_{\mathcal{R}_i}] \\ & + (I - \bar{H}^{-1}_{\mathcal{S}_i\mathcal{B}_i}(q,\theta))w_{\mathcal{S}_i}(t), \end{aligned} \tag{9}$$

based on which, the parameter can be estimated (asymptotically as data length approaches infinity) as

$$\begin{aligned} \theta^{\star} & = \arg\min_{\theta} \bar{V}(\theta) \\ & \triangleq \arg\min_{\theta} \frac{1}{2}\bar{\mathbb{E}}\left[w_{\mathcal{S}_i}(t) - \hat{w}_{\mathcal{S}_i}(t,\theta)\right]^{\top}\left[w_{\mathcal{S}_i}(t) - \hat{w}_{\mathcal{S}_i}(t,\theta)\right], \end{aligned} \tag{10}$$

where $\bar{\mathbb{E}}$ is the expectation operator for quasi-stationary signals (Ljung, 1999). This estimator leads to an asymptotic estimate $\bar{G}_{\mathcal{S}_i\mathcal{D}}(q,\theta^{\star})$, and due to the absence of confounding noises, the following result can be obtained by following (Ramaswamy and Van den Hof, 2021, Theorem 2) analogously.

Lemma 7. Given an immersed network network model (6) with its graph $\mathcal{G}_{\mathcal{S}}$, consider its i-th MIMO subsystem (8) with a real parameter value θ_0, and suppose $|\mathcal{S}_i^e| \geqslant |\mathcal{S}_i|$. If (i) $\mathcal{S}$ satisfies the condition in Lemma 2, (ii) the input signal vector $\left[w_{\mathcal{D}}^{\top}\ r_{\mathcal{R}_i}^{\top}\ e_{\mathcal{B}_i}^{\top}\right]^{\top}$ is persistently exciting and (iii) every directed path in $\mathcal{G}_{\mathcal{S}}$ from $w_{\mathcal{S}_i}$ via $w_{\mathcal{D}}$ to $w_{\mathcal{S}_i}$ has a delay, then the parameter estimate $\theta^{\star}$ from (10) satisfies $G_{ji}(q,\theta^{\star}) = G_{ji}(q,\theta_0)$

In the above result, condition (i) ensures that the target module is contained in the MIMO subsystem (8). Condition (ii) follows from the classical informativity condition, and condition (iii) is a generalization of the classical algebraic loop condition: Every path from an output via the input to any other output should have a delay. In addition, $|\mathcal{S}_i^e| \geqslant |\mathcal{S}_i|$ ensures that the noise model $\bar{H}_{\mathcal{S}_i\mathcal{B}_i}$ in (9) is square and monic based on Lemma 6. Under the

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above conditions, Lemma 7 shows that we can consistently estimate the target module G_{ji}.

However, the graphical condition (iii) in Lemma 7 is formulated on the graph $\mathcal{G}_{\mathcal{S}}$ of the immersed network. We can take one step further to formulate this condition directly on the graph $\mathcal{G}$ of the original network, by exploiting the connection between $\mathcal{G}_{\mathcal{S}}$ and $\mathcal{G}$ in Corollary 4.

Corollary 8. Given a network (1) and its graph $\mathcal{G}$, consider a subset $\mathcal{S}$ of internal signals and the resulting immersed network (6). Then in the setting of Lemma 7, if (i) $\mathcal{S}$ satisfies the condition in Lemma 2 (ii) the input signal vector $\left[w_{\mathcal{D}}^\top \; r_{\mathcal{R}_i}^\top \; e_{\mathcal{B}_i}^\top\right]^\top$ is persistently exciting and (iii) every directed path in $\mathcal{G}$ from $w_{\mathcal{S}_i}$ via $w_{\mathcal{D}}$ to $w_{\mathcal{S}_i}$ has a delay, then the parameter estimate $\theta^\star$ from (10) satisfies $G_{ji}(q,\theta^\star) = G_{ji}(q,\theta_0)$

4.3 Graph-based data informativity

The condition (ii) in Corollary 8 for data informativity is still formulated as a rank condition on the power spectral density. In this subsection, we will address Sub-Problem 3 and formulate this condition into a more explicit graphical condition on the graph $\mathcal{G}$ of the original network.

For the data informativity of internal signals in dynamical networks, since they are generated by external signals, there are also structural aspects in the informativity conditions, i.e., the questions whether there are a sufficient number of external signals in the network, and whether these external signals can reach internal signals of interest to provide excitation (Shi et al., 2022).

The above structural aspects become clear if we combine condition (ii) in Corollary 8 and Lemma 5 to obtain a graphical data informativity condition on the immersed network: For the input vector $\mathcal{K} = \left[w_{\mathcal{D}}^\top \; r_{\mathcal{R}_i}^\top \; \bar{e}_{\mathcal{B}_i}^\top\right]^\top$, there are $\mathrm{Dim}(\mathcal{K})$ vertex disjoint paths from the external signals to $\mathcal{K}$. Moreover, since $r_{\mathcal{R}_i}$ and $\bar{e}_{\mathcal{B}_i}$ are external signals and are already persistently exciting, we can re-write this graphical condition equivalently by only considering the internal signals in $w_{\mathcal{D}}$:

Corollary 9. Given the immersed network (6) with graph $\mathcal{G}_{\mathcal{S}}$, consider its MIMO subsystem (8), and let $\bar{\mathcal{X}}$ be the set of all external signals in $\mathcal{G}_{\mathcal{S}}$ excluding the ones in $r_{\mathcal{R}_i}$ and $\bar{e}_{\mathcal{B}_i}$. Then the signal vector $\left[w_{\mathcal{D}}^\top \; r_{\mathcal{R}_i}^\top \; \bar{e}_{\mathcal{B}_i}^\top\right]^\top$ is persistently exciting for almost all θ_0 if in $\mathcal{G}_{\mathcal{S}}$, there exist $\mathrm{Dim}(w_{\mathcal{D}})$ vertex disjoint paths from $\bar{\mathcal{X}}$ to $w_{\mathcal{D}}$.

The graphical condition in Corollary 9 is formulated on the immersed graph, where the signals $r_{\mathcal{R}_i}$ and $\bar{e}_{\mathcal{B}_i}$ are excluded from establishing the number of vertex disjoint paths. To further formulate it directly on the graph $\mathcal{G}$ of the original network, the main issue is to determine the signals in $r_{\mathcal{R}_i}$ and $\bar{e}_{\mathcal{B}_i}$ by inspecting $\mathcal{G}$:

- $r_{\mathcal{R}_i}$ is input to $w_{\mathcal{D}}$ in the immersed network, and thus based on Corollary 4, every signal in $r_{\mathcal{R}_i}$ has an unmeasured path to $w_{\mathcal{D}}$ in $\mathcal{G}$;
- In the immersed network, the noises in $\bar{e}_{\mathcal{B}_i}$ are inputs to $w_{\mathcal{D}}$. Based on Lemma 6 and its proof, these noises are generated by the white noises in $\mathcal{S}_i^e$, defined in Lemma 6, of the original network.

The above observations directly lead to the following result:

Lemma 10. Given a network (1) with its graph $\mathcal{G}$ and a subset $\mathcal{S}$ of internal signals, consider the resulting immersed network (6) and its MIMO subsystem (8). In $\mathcal{G}$, let $\tilde{\mathcal{X}}$ contain the set of all external signals excluding the noises in $\mathcal{S}_i^e$ and the reference signals having an unmeasured path to $w_{\mathcal{D}}$. Then $\left[w_{\mathcal{D}}^\top \; r_{\mathcal{R}_i}^\top \; \bar{e}_{\mathcal{B}_i}^\top\right]^\top$ is persistently exciting for almost all θ_0 if in $\mathcal{G}$, there exist $\mathrm{Dim}(w_{\mathcal{D}})$ vertex disjoint paths from $\tilde{\mathcal{X}}$ to $w_{\mathcal{D}}$.

5. GRAPHICAL SIGNAL SELECTION FOR SINGLE MODULE IDENTIFICATION

Selecting appropriate internal and external signals for estimating a single target module G_{ji} involves the combination of the solutions to all the previous sub-problems. We first summarize them into the following identification algorithm:

Algorithm 1: given network graph $\mathcal{G}$, target G_{ji}, and data

(a) Select a subset $\mathcal{S}$ of internal signals that satisfies $\{w_i, w_j\} \subseteq \mathcal{S}$ and the graphical condition in Lemma 2;
(b) Select predictor outputs: Partition $\mathcal{S}$ according to (5), and in the partition, let $\mathcal{S}_i$ denote the subset that contains w_j. Then select the signals in $\mathcal{S}_i$ to be the predictor outputs, collected into a vector $w_{\mathcal{S}_i}$. In addition, let set $\mathcal{S}_i^e$ contain the noises that have an unmeasured path to $w_{\mathcal{S}_i}$;
(c) Select predictor inputs: In $\mathcal{G}$, determine $w_{\mathcal{D}}$ and $r_{\mathcal{R}_i}$, which contain the internal signals in $\mathcal{S}$ and reference signals that have an unmeasured path (with respect to $\mathcal{S}$) to $w_{\mathcal{S}_i}$;
(d) Construct a predictor model as in (9), where $w_{\mathcal{S}_i}$ is the predictor output, and $w_{\mathcal{D}}$ and $r_{\mathcal{R}_i}$ are the predictor inputs;
(e) Compute a parameter estimate $\theta^\star$ as (10)

In the above algorithm, step (a) selects the remaining internal signals in $\mathcal{S}$ for the immersed network. The final selected signals for estimation are the ones in $w_{\mathcal{S}_i}$, $w_{\mathcal{D}}$, and $r_{\mathcal{R}_i}$, which have been obtained in step (b) and (c) by directly inspecting the graph $\mathcal{G}$.

To ensure the consistent identification of the target module G_{ji} using Algorithm 1, the following conditions should be ensured for the estimator:

- The target module should be invariant after immersion, i.e., the selected set $\mathcal{S}$ satisfies the graphical condition in Lemma 2
- The relevant signals should be informative

The above conditions can be achieved by combining Lemma 2, Corollary 8, and Lemma 10, which directly leads to the final consistency result:

Theorem 11. Given a network (1) with $\mathcal{G}$, consider Algorithm 1 and suppose $|\mathcal{S}_i^e| \geqslant |\mathcal{S}_i|$ at step (b). In $\mathcal{G}$, let $\tilde{\mathcal{X}}$ contain the set of all external signals excluding the noises in $\mathcal{S}_i^e$ and the reference signals having an unmeasured path to $w_{\mathcal{D}}$. Then if the following conditions hold for $\mathcal{G}$: (i) $\mathcal{S}$ satisfies the graphical condition in Lemma 2, (ii) there

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exist $\text{Dim}(w_\mathcal{D})$ vertex disjoint paths from $\tilde{\mathcal{X}}$ to $w_\mathcal{D}$, and (iii) every directed path from $w_{\mathcal{S}_i}$ via $w_\mathcal{D}$ to $w_{\mathcal{S}_i}$ has a delay, then it holds that $G_{ji}(q,\theta^\star) = G_{ji}(q,\theta_0)$ for almost all θ_0.

In the above theorem, condition (i) ensures that the target module G_{ji} is indeed an entry of the estimated MIMO subsystem (8). The condition (ii) is a graphical condition for data informativity from Lemma 10, and condition (iii) is the condition on the delay from Corollary 8.

Note that all the three conditions in Theorem 11 are graphical and formulated on the graph $\mathcal{G}$ of the original graph. This is in contrast to the results in (Dankers et al., 2016; Ramaswamy and Van den Hof, 2021) where a non-graphical data informativity condition is enforced. Compared to the graphical results in Van den Hof and Ramaswamy (2020), our conditions are developed for the specific signal selection procedure in Algorithm 1.

Example 12. Consider the original network in Fig. 1(a), and our goal is to identify the target module G_{31} with input w_1 and output w_3. To apply Algorithm 1, we select the green vertices in $\mathcal{S} = \{w_1, w_2, w_3\}$, which clearly satisfies the condition in Lemma 2. Then we can partition $\mathcal{S}$ as $\mathcal{S} = \{w_1\} \cup \{w_2\} \cup \{w_3\}$, as the noises e_3 and e_4 only have unmeasured paths to w_3. Since $\{w_3\}$ contains the output of the target module, $\{w_3\}$ is the set $\mathcal{S}_i$ in Algorithm 1, i.e., the predictor output, and we have $\mathcal{S}_i^e = \{e_3, e_4\}$. Correspondingly, w_1 and w_2 are the predictor inputs, i.e., the entries in $w_\mathcal{D}$, as they have an unmeasured path to w_3 in Fig. 1(a). Indeed, after w_4 is eliminated and in the resulting immersed network, shown in Fig. 1(b), w_2 becomes an input to w_3 according to Corollary 4.

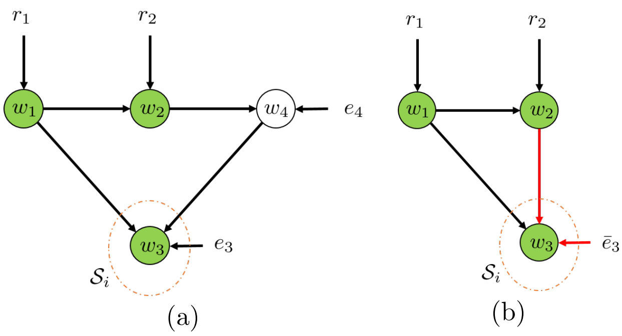

Fig. 1. Signal selection for identifying G_{31} in the original network, shown in Fig. 1(a). The immersed network with e_4 eliminated is shown in (b), where w_2 becomes an input to w_3, and a new noise source $\bar{e}_3$ is introduced.

Note that the informativity condition in Theorem 11 is also satisfied: In Fig. 1(a), there are two vertex disjoint paths from $\tilde{\mathcal{X}} = \{r_1, r_2\}$, with $\{e_3, e_4\}$ excluded, to the two predictor inputs in $\{w_1, w_2\}$.

6. CONCLUSION

The problem of selecting signals for identifying a local transfer function in a linear dynamical network is considered in this work. This problem involves several subproblems, including network immersion, data informativity, and the handling of confounding noises. Motivated by the basic idea in the minimum-input approach of Ramaswamy and Van den Hof (2021), we have presented a simple and systematic graphical approach to select predictor inputs and outputs to address the above sub-problems. Moreover, graphical conditions have been developed such that using the selected signals in PEM can estimate the target module consistently.

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Appendix A. NETWORK IMMERSION PROCEDURE

The network immersion procedure, originally introduced in (Dankers et al., 2016), is presented here for completeness. Let set $\mathcal{W}$ contain all the internal signals and $\mathcal{Z} = \mathcal{W}\backslash\mathcal{S}$ contain the unselected internal signals. The network model (1) can be written as $(I-G)w = [Rr\ He]$ and further permuted as

$$\begin{bmatrix} I-G_{\mathcal{SS}} & -G_{\mathcal{SZ}} \\ -G_{\mathcal{ZS}} & I-G_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{S}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} R_{\mathcal{S}\star}r + H_{\mathcal{S}:}e \\ R_{\mathcal{Z}\star}r + H_{\mathcal{Z}:}e \end{bmatrix}.$$

Applying block Gaussian elimination to the above system of linear equations and considering only the upper block row leads to

$$w_{\mathcal{S}} = G_{pre}w_{\mathcal{S}} + R_{pre}r + H_{pre}e, \tag{A.1}$$

where

$$G_{pre} = G_{\mathcal{SS}} + G_{\mathcal{SZ}}(I-G_{\mathcal{ZZ}})^{-1}G_{\mathcal{ZS}}, \tag{A.2}$$

$$H_{pre} = H_{\mathcal{S}:} + G_{\mathcal{SZ}}(I-G_{\mathcal{ZZ}})^{-1}H_{\mathcal{Z}:}, \tag{A.3}$$

and R_{pre} is defined similarly. Note that the above signal elimination procedure is essentially the so-called Kron reduction (Dorfler and Bullo, 2012). However, (A.1) is not an appropriate network model as G_{pse} may contain non-zero diagonal entries, and H_{pse} may have more columns than rows and is not suitable for identification setting. Therefore, the following two steps are required to reformulate (A.1) into an appropriate network model:

(1) Remove the non-zero diagonal entries of G_{pse}:
Let a diagonal matrix F contain the diagonal entries of G_{pse}, and then (A.1) leads to

$$w_{\mathcal{S}} = \bar{G}w_{\mathcal{S}} + \bar{R}r + (I-F)^{-1}H_{pse}x, \tag{A.4}$$

where $\bar{G} = (I-F)^{-1}(G_{pse}-F)$ now has zero diagonal entries, and $\bar{R} = (I-F)^{-1}R_{sep}$.

(2) Re-model the noises:
Reformulate the above model as

$$w_{\mathcal{S}} = \bar{G}w_{\mathcal{S}} + \bar{R}r + \bar{H}\bar{e},$$

where $(\bar{H}, \bar{e})$ is the spectral factor obtained from the spectral factorization of the spectrum of $(I-F)^{-1}H_{pse}e$ based on (Weerts et al., 2018, Lemma 1).

It can be found that (2) is a valid network model that satisfies the standard assumptions.

Appendix B. PROOF OF LEMMA 6

In (A.3) and following (Dankers et al., 2016, Lemma 1), for any $w_p \in \mathcal{S}$ and any noise e_k, $H_{pk} \neq 0$ if $(e_k, w_p) \in \mathcal{G}$, and $G_{p\mathcal{Z}}(I-G_{\mathcal{ZZ}})^{-1}H_{\mathcal{Z}k} \neq 0$ if there is a path from e_k to w_p with only internal vertices in $\mathcal{Z}$. Combing the above fact and the definition of unmeasured paths, $[H_{pre}]_{\{w_p\}\{e_k\}}$, i.e., the entry in H_{pre} corresponds to w_p and e_k, is non-zero if there exists an unmeasured path from e_k to w_p. Therefore, according to the definition of the confounding sets, H_{pre} can be permuted into

$$\begin{bmatrix} H_{\mathcal{S}_1\mathcal{S}_1^e} & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & H_{\mathcal{S}_m\mathcal{S}_m^e} & \vdots \\ 0 & \cdots & \ddots & 0 \end{bmatrix}, \tag{B.1}$$

Then based on the spectral factorization in (Weerts et al., 2018, Lemma 1), the factorization of each block spectrum, i.e. $H_{\mathcal{S}_i\mathcal{S}_i^e}\Lambda_i H^\star_{\mathcal{S}_i\mathcal{S}_i^e}$ with Λ_i the covariance of the noises in $\mathcal{S}_i^e$, leads to the new noises in $\bar{\mathcal{B}}_i$, with cardinality $|\min(|\mathcal{S}_i|, |\mathcal{S}_i^e|)|$, and the new noise model in (6) (after permutation).

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