Exploiting unmeasured disturbance signals in identifiability of linear dynamic networks with partial measurement and partial excitation ⋆

Shengling Shi, Xiaodong Cheng, Paul M. J. Van den Hof

Department of Electrical Engineering, Eindhoven University of Technology, the Netherlands (e-mail: s.shi@tue.nl, x.cheng@tue.nl, p.m.j.vandenhof@tue.nl).

Abstract: Identifiability conditions for networks of transfer functions require a sufficient number of external excitation signals, which are typically measured reference signals. In this abstract, we introduce an equivalent network model structure to address the contribution of unmeasured noises to identifiability analysis in the setting with partial excitation and partial measurement. With this model structure, unmeasured disturbance signals can be exploited as excitation sources, which leads to less conservative identifiability conditions.

1. INTRODUCTION

Due to the increasing complexity of current technological systems, the study of large-scale interconnected dynamic systems receives considerable attention. Connecting to prediction-error identification methods, the most popular modeling framework is based on a network of transfer functions as introduced in (Gonçalves and Warnick, 2008; Van den Hof et al., 2013), where vertices represent the internal signals that can be measured, and directed edges denote transfer functions referred to as modules that represent the causal relations among the signals.

In this abstract we focus on *network identifiability*, which is a concept that is independent of a particular identification method chosen and reflects the ability to distinguish between network models in a network model set on the basis of measurement data. In the literature, there are two notions of network identifiability: *global identifiability* (Weerts et al., 2015, 2018; van Waarde et al., 2018) that requires models to be distinguishable from *all* other models in the model set; and *generic identifiability* (Bazanella et al., 2017; Hendrickx et al., 2019; Cheng et al., 2019; Shi et al., 2020) which requires models to be distinguishable from *almost all* models in the model set.

Identifiability conditions in the above works require a sufficient number of external excitation signals which are typically measured reference signals, as considered in (Bazanella et al., 2017; Hendrickx et al., 2019; van Waarde et al., 2018; Bazanella et al., 2019). On the other hand, in (Weerts et al., 2018) it is shown that unmeasured noise signals can also serve as excitation sources for identifiability analysis, which leads to less conservative identifiability conditions than only considering reference signals. However, this result is obtained when all internal signals are measured, which is not straightforward to be extended to a more general situation where not all vertices are excited and not all vertices are measured, i.e. with partial measurement and partial excitation. Therefore, this abstract introduces a novel approach to exploit the contribution of unmeasured noises to identifiability analysis of dynamic networks.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Dynamic networks

The dynamic network model describes the relationship among *internal signals* $\mathcal{W} = \{w_1(t), \cdots, w_L(t)\}$, a vector of measured and deterministic excitation signals $r(t)$, unmeasured stationary stochastic processes $\{v_1(t), \cdots, v_L(t)\}$, which is formulated as

$$\begin{aligned} w(t) &= G(q)w(t) + Rr(t) + v(t), \\ w_{\mathcal{C}}(t) &= Cw(t), \end{aligned} \tag{1}$$

where $G(q)$ is a matrix of rational transfer operators with delay operator q^{-1}, i.e. $q^{-1}w_i(t) = w_i(t-1)$; C is a binary matrix which extracts all the measured internal signals in $\mathcal{C} \subseteq \mathcal{W}$ from w and stacks them into vector $w_{\mathcal{C}}(t)$; R is a binary matrix that decides which internal signals are influenced by $r(t)$, i.e. each column of R has exactly one entry as 1 and the other entries as zeros, while its each row has most one entry as 1.

Let $\Phi_v(q)$ of dimension $L \times L$ denote the rational power spectral density matrix of $v(t)$ with rank $p \leqslant L$, and then a noise model for $v(t)$ can be introduced based on the spectral factorization of $\Phi_v(q)$ as

$$v(t) = H(q)e(t), \tag{2}$$

where $e(t)$ is vector of white noises with covariance matrix Λ and dimension either L or p (Weerts et al., 2018; Gevers et al., 2019); $H(q)$ is proper and stable. Combining (1) and (2) leads to a complete network model specified as

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a quadruple $M \triangleq (G(q), R, C, H(q), \Lambda)$, on which the following assumptions are made:

Assumption 1. It will be assumed that

(a) $G(q)$ is proper and hollow;
(b) $[I - G(q)]^{-1}$ is stable;
(c) The network is well-posed in the sense that all principal minors of $\lim_{z\to\infty}(I - G(z))$ are non-zero (Dankers, 2014);
(d) $H(q)$ is proper and stable;
(e) Λ is real and positive semi-definite.

Note that Assumption 1(c) ensures that every principal submatrix of $(I - G)$ has a proper inverse (Scherer, 2001), i.e. every closed-loop transfer function is proper.

In a network model, both the excitation signals r and the noise signals e are called *external signals* which are collected in the set $\mathcal{X}$. The entries in $G(q)$ are referred to as *modules*. Let set $\mathcal{Z} = \mathcal{W} \setminus \mathcal{C}$ contain all the unmeasured internal signals, and without loss of generality, w is ordered as $w = \begin{bmatrix} w_{\mathcal{C}}^\top & w_{\mathcal{Z}}^\top \end{bmatrix}^\top$, where $w_{\mathcal{C}}$ and $w_{\mathcal{Z}}$ correspond to the measured and the unmeasured internal signals. Accordingly, C is partitioned as $C = [I \ 0]$.

The external-to-internal mapping of (1) is

$$w_{\mathcal{C}} = C(I - G)^{-1}Rr + C(I - G)^{-1}He, \tag{3}$$

and a standard open-loop identification of the above model can typically lead to consistent estimates of the following objects:

$$CTR, \quad C\Phi C^T, \tag{4}$$

where $T \triangleq (I - G)^{-1}$, $\Phi \triangleq (I - G)^{-1}H\Lambda H^\star(I - G)^{-\star}$, and $H(z)^\star$ denotes $H^\top(z^{-1})$. Thus, an identifiability question arises to determine the uniqueness of modules in $G(q)$ given the objects in (4). Note that CTR leads to a subset of rows and columns of T, respectively, based on which internal signals are measured or excited.

Assumption 2. The power spectrum $C\Phi(z)C^T$ has full rank.

2.2 Model sets and identifiability

Network identifiability is defined on the basis of a network model set whose definition is given first. For a network model M and by parameterizing its entries in a rational form as $M(\theta) = (G(q, \theta), R, C, H(q, \theta), \Lambda(\theta))$, a network model set $\mathcal{M} \triangleq \{M(\theta)|\theta \in \Theta \subseteq \mathbb{R}^n\}$ is formulated, where $M(\theta)$ satisfies Assumption 1 for every $\theta \in \Theta$. Note that the dependency of transfer matrices on q and θ is often omitted for the simplicity of notation.

Concerning network identifiability, we follow the concept of global network identifiability as defined in (Weerts et al., 2018) and also consider its generic version obtained by combining it with the concept of generic identifiability introduced in (Bazanella et al., 2017; Hendrickx et al., 2019) for a different setting. In this respect, we follow an approach that is formulated in (Shi et al., 2020).

Definition 1. Given a parameterized network model set $\mathcal{M}$, consider $\theta_0 \in \Theta$ and the following implication:

$$\left.\begin{aligned} CT(z,\theta_0)R &= CT(z,\theta_1)R \\ C\Phi(z,\theta_0)C^\top &= C\Phi(z,\theta_1)C^\top \end{aligned}\right\} \Rightarrow G_{ji}(z,\theta_0) = G_{ji}(z,\theta_1), \tag{5}$$

for all $\theta_1 \in \Theta$. Then module G_{ji} is

- globally identifiable in $\mathcal{M}$ from $(w_{\mathcal{C}}, r)$ if the implication (5) holds for all $\theta_0 \in \Theta$;
- generically identifiable in $\mathcal{M}$ from $(w_{\mathcal{C}}, r)$ if the implication (5) holds for almost all $\theta_0 \in \Theta$.

In the above definition, the notion "almost all" excludes a subset of measure zero from Θ. The concept of identifiability in this definition concerns the uniqueness of a module given the first and second moment information of the measured signals. If the module is not identifiable in the model set, no identification method, that relies on the first and the second moments for estimating the module, is able to provide a unique estimate of the module.

2.3 Problem formulation

Identifiability conditions typically require a sufficient number of r signals as excitation sources. Moreover, it is shown in (Weerts et al., 2018) that when all the internal signals are measured, i.e. $C = I$, the spectrum matrix Φ in (4) admits a unique spectral factor TH under mild conditions, and thus implication (5) can be equivalently simplified by considering (TR, TH) in the LHS of (5) instead of (TR, Φ). Since the mapping from the noises to internal signals is used for identifiability analysis, the noises play the same role as $r(t)$ for identifiability analysis. However, when only a subset of internal signals is measured, the above result cannot be applied anymore as only submatrix $C\Phi C^\top$ of Φ is taken as a starting point in Definition 1. Thus, the question is how noise signals can be used as excitation sources for identifiability analysis in the current setting.

3. MAIN RESULTS

We first introduce the concept of network equivalence and then develop an equivalent network by exploiting the noise spectrum $C\Phi C^\top$. With the developed model structure, unmeasured noises can be taken into account as excitation sources in the identifiability analysis.

3.1 Equivalent network for noise excitation

Identifiability concept in (5) takes the object $(CTR, C\Phi C^\top)$ as the starting point, and this object reflects the first and the second moment of the measured signals. Therefore, we can define a concept of network equivalence based on the above object, by extending a similar concept in (Weerts et al., 2020).

Definition 2. Network models $M_1 = (G_1, R_1, C_1, H_1, \Lambda_1)$ and $M_2 = (G_2, R_2, C_2, H_2, \Lambda_2)$ are said to be (observationally) equivalent if it holds that

$$C_1T_1(z)R_1 = C_2T_2(z)R_2, \text{ and } C_1\Phi_1(z)C_1^T = C_2\Phi_2(z)C_2^T,$$

where T and Φ are defined in (4).

The above concept of equivalence characterizes two network models that can be used to model the same measured processes $(w_{\mathcal{C}}, r)$, because given measured r, the stochastic processes $w_{\mathcal{C}}$ in two equivalent models have the same mean $CTRr$ and power spectrum $C\Phi C^\top$.

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Given any network model M, it always admits an equivalent network with a simpler noise model, as shown in the following result.

Theorem 1. Any network model $M = (G, R, C, H, \Lambda)$ admits an equivalent network model as

$$\tilde{M} \triangleq (G, R, C, \left[\tilde{H}^\star\ 0\right]^\star, \tilde{\Lambda}), \tag{6}$$

where $\tilde{H} \in \mathbb{R}(q)^{c\times c}$, with $c = |\mathcal{C}|$, is minimum phase, monic, and $\tilde{\Lambda} \in \mathbb{R}^{c\times c}$ is positive semi-definite.

Proof. We first exploit the structure of the noise spectrum $C\Phi C^\top$ of M. Based on the measured signals $w_\mathcal{C}$, an immersed network model, which only represents the behavior of the measured signals, can be obtained by eliminating the unmeasured signals (called immersion or Kron reduction) (Dankers et al., 2016). We first define that

$$\bar{G} \triangleq G_{\mathcal{C}\mathcal{C}} + G_{\mathcal{C}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}G_{\mathcal{Z}\mathcal{C}}$$
$$\bar{H} \triangleq H_\mathcal{C} + G_{\mathcal{C}\mathcal{Z}}(I - G_{\mathcal{Z}\mathcal{Z}})^{-1}H_\mathcal{Z},$$

and $\bar{R}$ similarly, where, for example, $G_{\mathcal{C}\mathcal{Z}}$ represents the submatrix of G that has its rows and columns corresponding to the signals in $\mathcal{C}$ and $\mathcal{Z}$, respectively. Then the immersed network model has the following form:

$$w_\mathcal{C} = \bar{G}w_\mathcal{C} + \bar{R}r(t) + \bar{H}e(t).$$

Note that $(I-\bar{G})$ has a proper inverse because of Assumption 1(c) and consequently $(I - \bar{G}^\infty)$ being full rank. This model further leads to an external-to-internal mapping:

$$w_\mathcal{C} = (I - \bar{G})^{-1}\bar{R}r(t) + (I - \bar{G})^{-1}\bar{H}e(t). \tag{7}$$

Based on (3) and (7), it can be found that

$$C\Phi C^T = (I - \bar{G})^{-1}\bar{H}\Lambda\bar{H}^*(I - \bar{G})^{-*}, \tag{8}$$

where it holds that

$$C(I - G)^{-1}C^T = (I - \bar{G})^{-1}. \tag{9}$$

In addition, $\bar{H}\Lambda\bar{H}^\star$ can be re-factorized into $\tilde{H}\tilde{\Lambda}\tilde{H}^\star$ (Gevers et al., 2019), which together with (8) and (9) leads to

$$C\Phi C^T = C(I - G)^{-1}\begin{bmatrix}\tilde{H}\\0\end{bmatrix}\tilde{\Lambda}\left[\tilde{H}^\star\ 0\right](I - G)^{-*}C^T.$$

The above equation implies that the external-to-output mapping of (6), i.e.

$$w_\mathcal{C} = C(I - G)^{-1}Rr + C(I - G)^{-1}\begin{bmatrix}\tilde{H}\\0\end{bmatrix}\tilde{e},$$

leads to the same object $(CTR, C\Phi C^\top)$ as (3), which concludes the proof. ■

Based on the above result, the measured process $(w_\mathcal{C}, r)$ that is modeled by M can be equivalently modeled by $\tilde{M}$ in (6), which has the same matrices G, R, C and the unmeasured internal signals noise-free. In addition, $\tilde{M}$ has a transformed noise signal $\tilde{e}$ with the covariance matrix $\tilde{\Lambda}$. This noise model is simpler than the one in M, and more importantly, $\tilde{M}$ keeps the G matrix invariant as in M. This invariance of G matrix is particularly important for the identifiability analysis and the identification of network modules.

The equivalence between M and $\tilde{M}$ is obtained due to the freedom in transforming the unmeasured internal signals and modeling the noises, since the objects in (4) only reflects the properties of the measured processes. Therefore, $\tilde{M}$ may describe different unmeasured processes from $w_\mathcal{Z}$ in M due to the possible change in its stochastic properties. However, for the simplicity of notation, we still use $w_\mathcal{Z}$ and $\mathcal{X}$ to denote the unmeasured internal signals and the external signals in $\tilde{M}$, respectively.

3.2 Identifiability for the equivalent network

Since a network M and its corresponding $\tilde{M}$ are equivalent and contain the same G matrix, both of them can be used to model the same data set, i.e. the measured $(w_\mathcal{C}, r)$, for the identification of the modules in a dynamic network (1). In the previous section, it is discussed that $\tilde{M}$ in (6) can potentially be a better option due to its simpler noise model. In this section, we further show that the noise model of $\tilde{M}$ is also beneficial for the identifiability analysis.

From now on, let $\mathcal{M}$ denote a model set obtained from the parameterization of $\tilde{M}$ with a parameter θ, as described in Section 2.2. It can be found that under mild conditions, the power spectrum $C\Phi C^T$ of $\tilde{M}$ admits a unique spectral factor $T\left[\tilde{H}^\star\ 0\right]^\star$, which implies that the identifiability concept can be simplified as follows.

Assumption 3. In network model set $\mathcal{M}$, $G(q,\theta)$ is parameterized to be strictly proper.

Proposition 1. For a network model set $\mathcal{M}$ that satisfies Assumptions 2, 3 and defining

$$T_{\mathcal{W}\mathcal{X}} \triangleq (I - G)^{-1}X, \quad X \triangleq [R \quad \begin{bmatrix}\tilde{H}\\0\end{bmatrix}], \tag{10}$$

implication (5) for $\mathcal{M}$ can be equivalently formulated as

$$CT_{\mathcal{W}\mathcal{X}}(q,\theta_0) = CT_{\mathcal{W}\mathcal{X}}(q,\theta_1) \Rightarrow G_{ji}(\theta_0) = G_{ji}(\theta_1), \tag{11}$$

for all $\theta_1 \in \Theta$.

Proof. The proof is analogous to the proof for Proposition 1 in (Weerts et al., 2018). ■

The above result indicates that the mappings from both r and $\tilde{e}$ to the measured internal signals can be used for analyzing identifiability in $\mathcal{M}$, and thus the unmeasured noise signal $\tilde{e}$ plays the same role as the measured $r(t)$ for the identifiability analysis. In this case, we say that $\tilde{e}$ signals act as excitation sources for the identifiability analysis.

By contrast, when a model set of a general network model is considered, the simplification in Proposition 1 cannot be achieved. Therefore, identifiability of this model set involves the spectrum $C\Phi C^\top$ and the mapping from r to $w_\mathcal{C}$, i.e. only r signals can be used as excitation signals, as the approach considered in (Hendrickx et al., 2019; Bazanella et al., 2019).

3.3 Consequence for the graphical identifiability test

Due to the advantages of $\tilde{M}$ over a general model M, we regard the model set $\mathcal{M}$ of $\tilde{M}$ as the standard network model set for identifying the modules in the setting with partial measurement and partial excitation.

We develop a graphical identifiability test to show the consequence of using $\mathcal{M}$. Note that some entries in the network matrices $G(q,\theta), R, \tilde{H}(q,\theta)$ in $\mathcal{M}$ may be fixed to zeros, which reflect the modeling assumptions of the user; and all the non-zero entries in G and H are parameterized by θ.

Assumption 4. The transfer functions in $\mathcal{M}$ are parameterized independently.

The sparsity patterns of the network matrices in $\mathcal{M}$ lead to a graphical representation $\mathcal{G}$ of $\mathcal{M}$, with a vertex set $\mathcal{V} \triangleq \mathcal{X} \cup \mathcal{W}$, which denotes all the external and internal signals, and the set of directed edges $\mathcal{E}$ representing the entries that are not fixed to zero, e.g., a directed edge from w_i to w_j exists iff $G_{ji}(q, \theta)$ is not fixed to zero.

In addition, we introduce several graphical concepts. Given two vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$, a vertex set $\mathcal{D}$ is said to be a $\mathcal{V}_1 - \mathcal{V}_2$ disconnecting set if $\mathcal{D}$ intersects all the directed paths from $\mathcal{V}_1$ to $\mathcal{V}_2$. Two directed paths are vertex disjoint if they do not share any vertex, including the starting and ending vertices. We use $b_{\mathcal{V}_1 \to \mathcal{V}_2}$ to denote the maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$. The link between vertex disjoint paths and generic identifiability is originally investigated in (Hendrickx et al., 2019), for the setting where all internal signals are excited by r signals.

Having the above graphical concepts, a graphical test for generic identifiability of a single module with partial excitation and partial measurement can be obtained.

Theorem 2. Consider a model set $\mathcal{M}$ that satisfies Assumptions 2, 3, 4, where set $\mathcal{N}_j^-$ contains all inputs of w_j in $\mathcal{W}$ and set $\mathcal{X}_j$ contains all r signals and the noises without directed edges to w_j. Then G_{ji} is generically identifiable in $\mathcal{M}$ from $(w_{\mathcal{C}}, r)$ if there exists a $\mathcal{X}_j - \mathcal{N}_j^- \setminus \{w_i\}$ disconnecting set $\mathcal{D} \subseteq \mathcal{W}$ such that

(1) $b_{\mathcal{X}_j \to \{w_i\} \cup \mathcal{D}} = |\mathcal{D}| + 1$;
(2) $\{w_i, w_j\} \cup \mathcal{D} \subseteq \mathcal{C}$.

Proof. This result is a direct extension of Theorem 4 in (Shi et al., 2020) which assumes all internal signals are measured, and thus the proof is omitted. ■

The above result shows that when the output w_j is measured, it is sufficient to measure and excite the signals in $\{w_i\} \cup \mathcal{D}$ instead of all the inputs of w_j. More importantly, by making use of $\tilde{M}$, the excitation for $\{w_i\} \cup \mathcal{D}$ comes from both r and unmeasured $\tilde{e}$. By contrast, we can consider only r signals and thus replace $\mathcal{X}_j$ by the set of all r signals, when a general model set is considered.

4. CONCLUSION

In this abstract we develop an approach to exploit noise excitation for identifiability analysis of linear dynamic networks, in the setting with partial measurement and partial excitation. By introducing the concept of equivalent networks, a novel network model structure has been developed, and with this model structure, the unmeasured noises have been exploited as excitation sources in the identifiability analysis.

REFERENCES

Bazanella, A.S., Gevers, M., and Hendrickx, J.M. (2019). Network identification with partial excitation and measurement. In *Proc. 58th IEEE Conf. Decis Control (CDC)*, 5500–5506. IEEE.

Bazanella, A.S., Gevers, M., Hendrickx, J.M., and Parraga, A. (2017). Identifiability of dynamical networks: which nodes need be measured? In *Proc. 56th IEEE Conf. Decis Control (CDC)*, 5870–5875. IEEE.

Cheng, X., Shi, S., and Van den Hof, P.M.J. (2019). Allocation of excitation signals for generic identifiability of dynamic networks. In *Proc. 58th IEEE Conf. Decis Control (CDC)*, 5507–5512. doi:10.1109/CDC40024.2019.9029218.

Dankers, A.G. (2014). System identification in dynamic networks. *PhD dissertation.*

Dankers, A.G., Van den Hof, P.M.J., Bombois, X., and Heuberger, P.S.C. (2016). Identification of dynamic models in complex networks with prediction error methods: Predictor input selection. *IEEE Trans. Autom. Control*, 61(4), 937–952.

Gevers, M.R., Bazanella, A.S., and Pimentel, G.A. (2019). Identifiability of dynamical networks with singular noise spectra. *IEEE Transactions on Automatic Control*, 64(6), 2473–2479.

Gonçalves, J. and Warnick, S. (2008). Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks. *IEEE Trans. Autom. Control*, 53(7), 1670–1674.

Hendrickx, J.M., Gevers, M., and Bazanella, A.S. (2019). Identifiability of dynamical networks with partial node measurements. *IEEE Trans. Autom. Control*, 64(6), 2240–2253.

Scherer, C. (2001). Theory of robust control. *Delft University of Technology*, 1–160.

Shi, S., Cheng, X., and Van den Hof, P.M. (2020). Generic identifiability of subnetworks in a linear dynamic network: the full measurement case. *arXiv preprint arXiv:2008.01495.*

Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods basic methods for consistent module estimates. *Automatica*, 49(10), 2994–3006.

van Waarde, H.J., Tesi, P., and Camlibel, M.K. (2018). Necessary and sufficient topological conditions for identifiability of dynamical networks. *arXiv preprint arXiv:1807.09141.*

Weerts, H.H.M., Dankers, A.G., and Van den Hof, P.M.J. (2015). Identifiability in dynamic network identification. *IFAC-PapersOnLine*, 48-28, 1409–1414. Proc. 17th IFAC Symposium on System Identification, Beijing, China.

Weerts, H.H.M., Linder, J., Enqvist, M., and Van den Hof, P.M.J. (2020). Abstractions of linear dynamic networks for input selection in local module identification. *Automatica*, 117. doi:10.1016/j.automatica.2020.108975.

Weerts, H.H.M., Van den Hof, P.M.J., and Dankers, A.G. (2018). Identifiability of linear dynamic networks. *Automatica*, 89, 247–258.