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Data-driven distributed control: Virtual reference feedback tuning in dynamic networks

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Abstract— **The problem of constructing controllers from measurement data is commonly performed for a centralized controller architecture with central data collection. In this paper, the problem of synthesizing a *distributed* controller from data is considered, with the objective to optimize a model-reference control criterion. To solve this problem we utilize virtual reference feedback tuning (VRFT): a data-driven method for the design of controllers which does not use a system model. We establish an explicit ideal distributed controller that solves the model-reference control problem for a structured reference model. On the basis of input-output data collected from the interconnected system, a virtual experiment setup is constructed which leads to a network identification problem. We formulate a prediction-error identification criterion that has the same global optimum as the model-reference criterion, when the controller class contains the ideal distributed controller. The method is demonstrated for an academic example network consisting of nine subsystems, for which also the influence of the controller interconnection structure on the achieved closed-loop performance is illustrated.**

I. Introduction

Control of interconnected systems is a challenging problem. In the first place due to the spatial distribution or dimensionality, which prohibits the use of centralized controller design and implementation. Moreover, for many practical control applications, such as smart grids, smart buildings or industrial processes, dynamical models are not readily available, while measurement data is available with increased ease [1]. A relevant question is how to directly exploit the available data for distributed controller synthesis, without necessarily using a system model.

Indeed, most of the methods to design distributed controllers are based on a model of the interconnected system, e.g. distributed model predictive control [2], distributed H_2 [3] and H_∞ [4] control, and scalable control of positive systems [5]. From a model-based perspective, a logical procedure is the data-driven modelling of the interconnected system and subsequent synthesis of the distributed controller based on the obtained model. The interplay between identification and control has been investigated in the field of 'identification for control' [6], [7] to improve the achieved closed-loop performance of single- and multi-variable systems. The development of data-driven modelling of interconnected systems opens the way for dedicated control-oriented identification experiments for distributed controller design. See e.g. [8] for the experiment design for locally-controlled multi-agent systems.

For data-driven control, the step of modelling the interconnected system may be circumvented, however, and the design of the distributed controller could be performed on the basis of data. Several methods have been developed for data-driven controller design, see e.g. [9] for an overview. A common feature of these methods is that they are based on the model-reference paradigm, wherein a reference model describes the desired behavior of the closed-loop system [10]. Subsequent tuning of a parametrized controller based on input-output measurements leads to a closed-loop system that is optimal with respect to the model-reference criterion. These methods can be iterative or 'one-shot'. In the iterative feedback tuning [11] and correlation-based tuning [12] methods, the controller parameters are iteratively updated, while a new experiment is performed during each iteration. Virtual reference feedback tuning (VRFT) [13] is a method in which the model-reference criterion is optimized without iterations using a single batch of measurement data. Recent developments of 'one-shot' data driven methods include multi-variable VRFT [14], [15], unbiased multi-variable VRFT [16], optimal multi-variable controller identification [17] and asymptotically exact multi-variable controller tuning [18]. The step to the distributed case has not been made yet to the best of the authors' knowledge, while the potential of data-driven methods for interconnected systems is large [1].

In this paper, following the model-reference paradigm, we specify the desired behavior for the closed-loop system in terms of a *structured* reference model. The accompanying control problem is to find a distributed controller which minimizes the model reference criterion. A controller that admits the same interconnection structure as the process and achieves the desired closed-loop behavior exactly, is called an ideal distributed controller, analogous to the ideal controller for the standard model-reference problem [10]. We will first provide an explict ideal distributed controller. Then, by considering VRFT in a distributed setting, we perform data-driven control via dynamic network identification methods [19]. The identification is enabled by a virtual reference framework, by generating a virtual network setup via the structured reference model. We will provide conditions for which the distributed control problem can be reduced to a network identification problem and, hence, an ideal distributed controller can be obtained from data.

The remainder of the paper is organized as follows. In Section II, we introduce the interconnected system, controller

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class and reference model. The ideal distributed controller is discussed in Section III. Section IV presents the developed data-driven approach to distributed control. A simulation example is provided in Section V. Conclusions are summarized in Section VI.

II. Preliminaries

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality L and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbour set of vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. To each vertex $i \in \mathcal{V}$, we associate a linear discrete-time system with dynamics

$$y_i(t) = G_i(q)u_i(t) + \sum_{j \in \mathcal{N}_i} W_{ij}(q)s_{ij}(t),$$
$$o_{ij}(t) = F_{ij}(q)y_i(t), \qquad j \in \mathcal{N}_i,$$

with G_i, W_{ij}, F_{ij} rational transfer functions, q the forward shift defined as $qx(t) = x(t+1)$, $u_i : \mathbb{Z} \to \mathbb{R}$ is the control input, $y_i : \mathbb{Z} \to \mathbb{R}$ the output, and $o_{ij}, s_{ij} : \mathbb{Z} \to \mathbb{R}$ are variables through which the systems at vertices $(i,j) \in \mathcal{E}$ are interconnected. The problem that we consider is that of reference tracking, i.e., for each system it is desired that the output y_i tracks a reference signal r_i. The tracking error for system i is defined as $e_i := r_i - y_i$. By stacking all incoming and outgoing interconnection variables of system i in vectors s_i and o_i, that is $s_i := \mathrm{col}_{j \in \mathcal{N}_i} s_{ij}$ and $o_i := \mathrm{col}_{j \in \mathcal{N}_i} o_{ij}$, we arrive at the following description for the system at vertex i, denoted $\mathcal{P}_i$:

$$\mathcal{P}_i : \begin{cases} y_i &= G_i(q)u_i + W_i(q)s_i, \\ o_i &= F_i(q)y_i, \\ e_i &= r_i - y_i, \end{cases} \tag{1}$$

where $W_i := \mathrm{row}_{j \in \mathcal{N}_i} W_{ij}$ and $F_i := \mathrm{col}_{j \in \mathcal{N}_i} F_{ij}$, and the time t is omitted for brevity. The interconnection of system $\mathcal{P}_i$ and $\mathcal{P}_j$, $(i,j) \in \mathcal{E}$, is defined by

$$s_{ij} = o_{ji} \quad \text{and} \quad s_{ji} = o_{ij}. \tag{2}$$

In the model reference approach to controller design, performance specifications are specified through a desired closed-loop system, called the reference model, describing a transfer from the reference to the output. For the design of the distributed controller, we follow the model reference approach by specifying an *interconnected* system that describes how the closed-loop system should behave. This interconnected system can be interpreted as a structured analogy to the well-known reference model [9]. It is described by

$$\mathcal{K}_i : \begin{cases} y_i^d &= T_i(q)r_i + Q_i(q)k_i, \\ p_i &= P_i(q)y_i^d, \end{cases} \tag{3}$$

where $Q_i := \mathrm{row}_{j \in \mathcal{N}_i} Q_{ij}$ and $P_i := \mathrm{col}_{j \in \mathcal{N}_i} P_{ij}$ and the interconnection variables are similarly partitioned as for $\mathcal{P}_i$, i.e., $k_i := \mathrm{col}_{j \in \mathcal{N}_i} k_{ij}$ and $p_i := \mathrm{col}_{j \in \mathcal{N}_i} p_{ij}$. For each pair $(i,j) \in \mathcal{E}$ the interconnection of $\mathcal{K}_i$ and $\mathcal{K}_j$ is defined by

$$k_{ij} = p_{ji} \quad \text{and} \quad k_{ji} = p_{ij}. \tag{4}$$

Hence, $\mathcal{K}_i$ and $\mathcal{K}_j$ can only be interconnected if $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected. A particular case of such a reference model occurs when a decoupled closed-loop system is desired, i.e., $Q_{ij} = 0$ and $P_{ij} = 0$, $i,j = 1, 2, \ldots, L$.

For the control of the interconnected system described by (1) and (2), we consider that each system $\mathcal{P}_i$ is associated with a (parametrized) controller $\mathcal{C}_i$, which is a linear discrete-time system that has the tracking error e_i as an input, control input u_i as an output and is interconnected with other controllers $\mathcal{C}_j$ through interconnection variables η_{ij}, ζ_{ij}:

$$\mathcal{C}_i(\rho_i) : \begin{cases} u_i = C_{ii}(q,\rho_i)e_i + \sum_{j \in \mathcal{N}_i} C_{ij}(q,\rho_i)\eta_{ij}, \\ \zeta_{ij} = K_{ij}(q,\rho_i)e_i + \sum_{h \in \mathcal{N}_i} K_{ijh}(q,\rho_i)\eta_{ih},\ j \in \mathcal{N}_i. \end{cases}$$

The interconnection of $\mathcal{C}_i$ and $\mathcal{C}_j$, $(i,j) \in \mathcal{E}$ is defined by

$$\eta_{ij} = \zeta_{ji} \quad \text{and} \quad \eta_{ji} = \zeta_{ij}. \tag{5}$$

By defining $\eta_i := \mathrm{col}_{j \in \mathcal{N}_i} \eta_{ij}$ and $\zeta_i := \mathrm{col}_{j \in \mathcal{N}_i} \zeta_{ij}$, we compactly represent controller i by

$$\mathcal{C}_i(\rho_i) : \begin{bmatrix} u_i \\ \zeta_i \end{bmatrix} = C_i(q,\rho_i) \begin{bmatrix} e_i \\ \eta_i \end{bmatrix}. \tag{6}$$

It is assumed that each controller matrix is parametrized linearly, i.e., $[C_i(q,\rho_i)]_{ij} = \rho_i^\top \overline{C}_{ij}(q)$ for some vector of transfer functions $\overline{C}_{ij}$. The family of parametrized controllers for node i is $\mathscr{C}_i := \{C_i(q,\rho_i) \,|\, \rho_i \in \mathbb{R}^{l_i}\}$.

A. Problem formulation

The model reference approach to distributed-controller design is exemplified in Figure 1 for $L = 2$: on the right, we have the plant as the interconnection of the systems depicted in blue, which are controlled by the interconnected controllers, depicted in orange. On the left, we have the structured reference model. The objective is to minimize the difference between the outputs y_i and y_i^d by tuning the vectors ρ_i that parametrize the controllers $\mathcal{C}_i(\rho_i)$, $i = 1, \ldots, L$. The controller synthesis problem is then defined as follows:

$$\min_{\rho_1,\ldots,\rho_L} J_{\mathrm{MR}}(\rho_1, \ldots, \rho_L), \tag{7}$$
$$J_{\mathrm{MR}}(\rho_1, \ldots, \rho_L) := \sum_{i=1}^{L} \bar{E}[y_i^d(t) - y_i(t)]^2,$$

with $\bar{E} := \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E$ and E the expectation operator.

The controller that we consider is defined on the same graph as the plant. Such a control architecture was utilized in the works where the idea of distributed control was first introduced [4]. We will show in Section III that there exists a controller with such an architecture that solves the controller synthesis problem exactly, i.e., that there exists $\rho_1^d, \ldots, \rho_L^d$ such that $J_{\mathrm{MR}}(\rho_1^d, \ldots, \rho_L^d) = 0$. In Section IV, we will show how to obtain this controller from data.

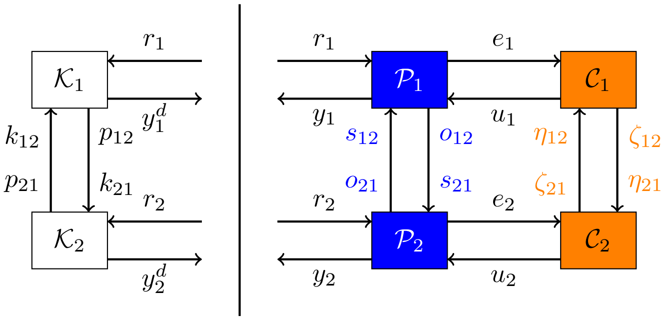


Fig. 1: Structured reference model (left) and closed-loop network with distributed controller (right) for $L = 2$.

B. Well-posedness

By stacking all incoming and outgoing interconnection variables of the interconnected system described by (1) and (2), i.e., $s := \text{col}(s_1, \dots, s_L)$ and $o := \text{col}(o_1, \dots, o_L)$, we can write

$$y = Gu + Ws, \quad o = Fy, \quad s = \Delta o,$$

with $G = \text{diag}(G_1, \dots, G_L)$, $W = \text{diag}(W_1, \dots, W_L)$, $F = \text{diag}(F_1, \dots, F_L)$ and the matrix Δ defined by aggregating (2) for all corresponding index pairs. Elimination of the interconnection variables yields the input-output behavior of the interconnected system: $y = (I - W\Delta F)^{-1}Gu$. Analogously, for the structured reference model described by (3) and (4) one has $y^d = (I - Q\Delta P)^{-1}Tr$. It is therefore assumed that $\det(I - W\Delta F) \neq 0$ and $\det(I - Q\Delta P) \neq 0$. Furthermore, it is assumed that the structured reference model is such that $y^d \neq r$ for all non-zero r, i.e., such that $\det((I - Q\Delta P)^{-1}T - I) \neq 0$.

III. Ideal distributed controller synthesis

A controller that admits the same structure as the interconnected system and for which the closed-loop network matches the structured reference model exactly, i.e., $y_i = y_i^d$ for all $i = 1, \dots, L$, is called an *ideal distributed controller*. In this section we will construct such a distributed controller.

First, however, let us have a look at the unstructured ideal controller. For a process $y = G_0(q)u$, $e = r - y$, and reference model $y^d = T_d(q)r$, the ideal controller is $u = C_d(q)e$ [17], with

$$C_d(q) = G_0^{-1}(q)T_d(q)(I - T_d(q))^{-1}. \tag{8}$$

Controller (8) does not directly yield an ideal distributed controller, since it is not clear how to separate C_d into sub-controllers $\mathcal{C}_i^d$ as in (6) with an interconnection structure that matches the process' interconnection structure, i.e., such that two controllers $\mathcal{C}_i^d$ and $\mathcal{C}_j^d$ are interconnected if and only if $(i,j) \in \mathcal{E}$. An exceptional case occurs when the reference model T_d is diagonal (corresponding to $Q_{ij} = 0$ and $P_{ij} = 0$ for all $(i,j) \in \mathcal{E}$) and one observes that $G_0^{-1} = G^{-1}(I - W\Delta F)$ for the interconnected system. Then $[C_d]_{ij} = 0$ for $(i,j) \notin \mathcal{E}$ and a 'distributed' implementation of C_d can be realized by the communication of tracking errors e_i among nodes. However, this is not possible if there are non-zero couplings Q_{ij}, P_{ij} in (3) such that T_d is non-diagonal; non-diagonal reference models can be crucial to obtain a good performance with restrictive controller classes, e.g. in the case of high-order couplings or when the process has transmission zeros outside the unit circle [20].

We will now describe a procedure to obtain an ideal distributed controller. The main idea of the construction is to interconnect a subsystem $\mathcal{K}_i$ of the structured reference model with a subsystem $\mathcal{P}_i$ of the interconnected system. The distributed controller is constructed by interconnecting the local controllers through their interconnection variables. More precisely, by the interconnection of $\mathcal{K}_i$ and $\mathcal{P}_i$, we mean that both equations (1) and (3) hold with $y_i^d = y_i$, i.e.,

$$\begin{bmatrix} y_i \\ o_i \\ e_i \\ y_i^d \\ p_i \end{bmatrix} = \begin{bmatrix} G_i u_i + W_i s_i \\ F_i y_i \\ r_i - y_i \\ T_i r_i + Q_i k_i \\ P_i y_i^d \end{bmatrix} \quad \text{and} \quad y_i^d = y_i. \tag{9}$$

Elimination of the variables y_i^d, y_i and r_i in (9) yields a local controller $\mathcal{C}_i^d$, described by (denoting x_i by x_i^c for the interconnection variables to distinguish controller variables from plant variables)

$$\mathcal{C}_i^d: \tag{10}$$

$$\begin{bmatrix} u_i \\ o_i^c \\ p_i^c \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{T_i}{G_i(1-T_i)} & -\dfrac{1}{G_i}W_i & \dfrac{1}{G_i(1-T_i)}Q_i \\ \dfrac{T_i}{1-T_i}F_i & 0 & \dfrac{1}{1-T_i}F_iQ_i \\ \dfrac{T_i}{1-T_i}P_i & 0 & \dfrac{1}{1-T_i}P_iQ_i \end{bmatrix}}_{=:C_i^d(q)} \begin{bmatrix} e_i \\ s_i^c \\ k_i^c \end{bmatrix}.$$

The ideal distributed controller is described by the local controllers (10) and the interconnections

$$\begin{bmatrix} s_{ij}^c \\ k_{ij}^c \end{bmatrix} = \begin{bmatrix} o_{ji}^c \\ p_{ji}^c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s_{ji}^c \\ k_{ji}^c \end{bmatrix} = \begin{bmatrix} o_{ij}^c \\ p_{ij}^c \end{bmatrix}. \tag{11}$$

Remark III.1 The upper-left element of C_i^d and the transfer function C_d in (8) are equal for $G_i = G$ and $T_i = T_d$. Hence, in the absence of interconnection variables ($\mathcal{E} = \emptyset$), each local controller reduces to the ideal controller (8).

Theorem III.1 *The closed-loop network described by* (1) - (2) *and the distributed controller* (10) - (11) *satisfies*

$$y_i = y_i^d, \qquad i = 1, \dots, L.$$

Proof: Let the control variables (u_i, e_i) and controller interconnection variables $(s_i^c, o_i^c, k_i^c, p_i^c)$ satisfy (10) for all i and (11) for all $(i,j) \in \mathcal{E}$, i.e., $s^c = \Delta o^c$ and $k^c = \Delta p^c$. We will first show that there exist latent variables $r_i^c : \mathbb{Z} \to \mathbb{R}$ and $y_i^c : \mathbb{Z} \to \mathbb{R}$ for each i, so that

$$\begin{bmatrix} y_i^c \\ o_i^c \\ e_i \\ y_i^c \\ p_i^c \end{bmatrix} = \begin{bmatrix} G_i u_i + W_i s_i^c \\ F_i y_i^c \\ r_i^c - y_i^c \\ T_i r_i^c + Q_i k_i^c \\ P_i y_i^c \end{bmatrix}. \tag{12}$$

Define $y_i^c := G_i u_i + W_i s_i$ and $r_i^c := e_i + y_i^c$. We then have to show that $y_i^c = T_i r_i^c + Q_i k_i^c$, $o_i^c = F_i y_i^c$ and $p_i^c = P_i y_i^c$. By (10) we have that

$$u_i = \frac{T_i}{G_i(1-T_i)} e_i + \frac{1}{G_i(1-T_i)} Q_i k_i^c - \frac{1}{G_i} W_i s_i^c$$
$$\Leftrightarrow$$
$$(1-T_i) G_i u_i = T_i e_i + Q_i k_i^c - (1-T_i) W_i s_i^c,$$

which, by the definition of y_i^c, is equivalent with

$$(1-T_i) y_i^c = T_i e_i + Q_i k_i^c \tag{13}$$

and hence, by the definition of r_i^c, $y_i^c = T_i r_i^c + Q_i k_i^c$. By (13), it follows that

$$y_i^c = \frac{T_i}{1-T_i} e_i + \frac{1}{1-T_i} Q_i k_i^c$$

and thus, by (10), we have

$$o_i^c = \frac{T_i}{1-T_i} F_i e_i + \frac{1}{1-T_i} F_i Q_i k_i^c = F_i y_i^c$$

and

$$p_i^c = \frac{T_i}{1-T_i} P_i e_i + \frac{1}{1-T_i} P_i Q_i k_i^c = P_i y_i^c.$$

Next, define $y^c := \mathrm{col}(y_1^c, \dots, y_L^c)$ and $u := \mathrm{col}(u_1, u_2, \dots, u_L)$. It follows by (12) that $y^c = Gu + Ws^c$ and $o^c = Fy$, such that, by $s^c = \Delta o^c$

$$y^c = (I - W\Delta F)^{-1} G u. \tag{14}$$

Similarly, define $r^c := \mathrm{col}(r_1^c, \dots, r_L^c)$ to obtain

$$y^c = (I - Q\Delta P)^{-1} T r^c, \tag{15}$$

by (12), with $Q = \mathrm{diag}(Q_1, \dots, Q_L)$, $P = \mathrm{diag}(P_1, \dots, P_L)$ and $T = \mathrm{diag}(T_1, \dots, T_L)$. Then by (14), (15) and $e = r^c - y^c$, the controller satisfies

$$u = G^{-1}(I - W\Delta F)(I - Q\Delta P)^{-1} T \times (I - (I - Q\Delta P)^{-1} T)^{-1} e. \tag{16}$$

Finally, the process $y = (I - W\Delta F)^{-1} G u$ with $e = r - y$ and the controller (16) yield

$$y = (I - Q\Delta P)^{-1} T r = y_d,$$

which concludes the proof. ■

Although the ideal distributed controller will yield the desired behavior for the closed-loop network, some observations are in place related to its practical implementation. Given a stable structured reference model, the ideal distributed controller will yield a stable closed-loop network in the sense that the transfer $r \to y_d$ is stable. As it is the case for the single-process ideal controller [10], however, internal stability is not guaranteed. From (10), the following guidelines for attaining stable ideal controllers can be obtained:

- If G_i has non-minimum phase zeros, then these must also be zeros of T_i, W_{ij} and Q_{ij}.
- The unstable poles of W_{ij} must also be poles of G_i.
- The unstable poles of F_{ij} must also be zeros of T_i.

Another concern is that the local controllers should be causal, i.e., that the elements of the transfer matrices $C_i^d(q)$ have a non-negative relative degree. By analyzing (10), we observe that the relative degree of T_i, Q_{ij} and W_{ij} must be larger than or equal to the relative degree of G_i.

In the sequel, it will be assumed that the family of parametrized controllers for each i is rich enough to describe the ideal controller $\mathcal{C}_i^d$, i.e., that the ideal distributed controller belongs to the parametrized class of distributed controllers. This is formalized in the following assumption, where, for ease of exposition, we introduce the permutation matrices $P_i := \mathrm{diag}(1, \overline{P}_i)$, $i = 1, \dots, L$, such that $\mathrm{col}(s_i^c, k_i^c) = \overline{P}_i \, \mathrm{col}_{j \in \mathcal{N}_i} \mathrm{col}(s_{ij}^c, k_{ij}^c)$.

Assumption III.1 $P_i^\top C_i^d P_i \in \mathscr{C}_i$ for each $i = 1, \dots, L$.

We associate $\rho_1^d, \dots, \rho_L^d$ with the ideal distributed controller, such that $P_i^\top C_i^d P_i = C_i(\rho_i^d)$ for $i = 1, \dots, L$. The following simple example briefly illustrates the ideal distributed controller constructed in this section.

Example III.1 Consider two coupled processes

$$y_1(t) = G_1(q) u_1(t) + G_{12}(q) y_2(t),$$
$$y_2(t) = G_2(q) u_2(t) + G_{21}(q) y_1(t),$$

with transfer functions

$$G_1(q) = \frac{c_1}{q - a_1}, \quad G_{12}(q) = \frac{d_1}{q - a_1},$$
$$G_2(q) = \frac{c_2}{q - a_2}, \quad G_{21}(q) = \frac{d_2}{q - a_2}.$$

The objective is to let the closed-loop interconnected system behave as two decoupled processes with first-order dynamics, according to

$$y_i^d(t) = T_i(q) r_i(t), \quad T_i(q) = \frac{1 - \gamma_i}{q - \gamma_i}, \quad i = 1, 2. \tag{17}$$

Now, via (10), we find that the ideal distributed controller is described by

$$\begin{bmatrix} u_1 \\ o_1^c \end{bmatrix} = \begin{bmatrix} C_{11}^d & C_{12}^d \\ K_{12}^d & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ s_1^c \end{bmatrix} \text{ and } \begin{bmatrix} u_2 \\ o_2^c \end{bmatrix} = \begin{bmatrix} C_{22}^d & C_{21}^d \\ K_{21}^d & 0 \end{bmatrix} \begin{bmatrix} e_2 \\ s_2^c \end{bmatrix}$$

with $e_i = r_i - y_i$, the interconnections $s_1^c = o_2^c$, $s_2^c = o_1^c$, and

$$C_{11}^d(q) = \frac{1-\gamma_1}{c_1} \frac{q - a_1}{q - 1}, \quad C_{12}^d(q) = -\frac{d_1}{c_1},$$
$$K_{12}^d(q) = \frac{1 - \gamma_1}{q - 1},$$
$$C_{22}^d(q) = \frac{1-\gamma_2}{c_2} \frac{q - a_2}{q - 1}, \quad C_{21}^d(q) = -\frac{d_2}{c_2},$$
$$K_{21}^d(q) = \frac{1 - \gamma_2}{q - 1}.$$

Observe that the control input u_1 has two contributions: one from a local controller C_{11}^d, which is a PI controller, and one from a communicated signal $s_1^c = o_2^c$ multiplied by a static gain C_{12}^d. The communicated signal is a filtered version of e_2, with filter K_{21}^d, which is an integrator. This reasoning applies analogously to control input u_2.

IV. DATA-DRIVEN DISTRIBUTED CONTROLLER SYNTHESIS

In the remainder of this paper, we shall consider the case where $F_{ij} = 1$ for all $i = 1, \ldots, L$, $j \in \mathcal{N}_i$. This implies that all systems are directly coupled through the outputs y_i:

$$y_i = G_i(q)u_i + \sum_{j \in \mathcal{N}_i} W_{ij}(q)y_j, \quad i = 1, \ldots, L, \tag{18}$$

compactly written as $y = W_I y + Gu$, with $[W_I]_{ij} = W_{ij}$. The interconnected system can always be represented as in (18) without changing the transfer $u \to y$, by replacing W_{ij} in (18) by $\bar{W}_{ij} = W_{ij}F_{ji}$, since $s_{ij} = F_{ji}(q)y_j$ by (1), (2).

The problem that is considered in this section, is the one of solving the model-reference control problem (7) using data, when models for the processes $\mathcal{P}_i$ are unavailable. The problem is stated as follows: Given input-output data $\{u_i, y_i\}$, $i = 1, \ldots, L$, determine the parameter vectors $\rho_1, \rho_2, \ldots, \rho_L$ that yield the best performance according to the criterion $J_{\text{MR}}(\rho_1, \ldots, \rho_L)$.

We address this problem by adapting the VRFT method to the interconnected-system setting for the identification of the ideal distributed controller. The basic principle of VRFT is to generate a virtual reference signal and a virtual tracking error $\bar{e}$ from data, such that the ideal controller can be identified as the dynamical relationship between $\bar{e}$ and u [10]. VRFT is not iterative and allows for the direct minimization of the model-reference control objective by casting the control problem as a prediction-error identification problem [10]. These attractive properties are retained for the design of distributed controllers for interconnected systems. We will first consider the computation of virtual reference signals according to the structured reference model and consequently relate the distributed control problem to a network identification problem.

A. Virtual reference generation

Consider data $\{u_i, y_i\}$, $i = 1, \ldots, L$ collected from the network (18). This data can be obtained in closed loop with a stabilizing controller or in open loop if the network is stable, i.e., if $(I - W_I)^{-1}G$ is stable. For the reference model described by (3), (4), we recall that $y_d = (I - Q\Delta P)^{-1}Tr$. Now, given $y_1, y_2, \ldots, y_L$, consider the computation of the *virtual reference* signals $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ according to the structured reference model as

$$\begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix} = (I - Q\Delta P)^{-1}T \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_L \end{bmatrix}. \tag{19}$$

Then $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ are such that, when the network (18) is in closed loop with the ideal distributed controller, fictitiously, the measured outputs $y_1, y_2, \ldots, y_L$ are the corresponding outputs. Solving (19) requires the data $y_1, y_2, \ldots, y_L$ to be collected by a central governor. Because central data collection is not favourable, we propose to generate the virtual reference signals locally. This can always be done for the considered reference model, by determining the virtual reference signals $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_L$ and the *virtual interconnection* signals $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_L$ according to (3) and (4) so that

$$y_i = T_i\bar{r}_i + \sum_{j \in \mathcal{N}_i} Q_{ij}\bar{p}_{ji},$$
$$\bar{p}_{ij} = P_{ij}y_i, \quad j \in \mathcal{N}_i.$$

Given a virtual reference signal $\bar{r}_i$, the corresponding virtual tracking error and, hence, the input to the ideal controller, is $\bar{e}_i = \bar{r}_i - y_i$. The virtual reference generation can thus be distributed, as summarized in Algorithm 1.

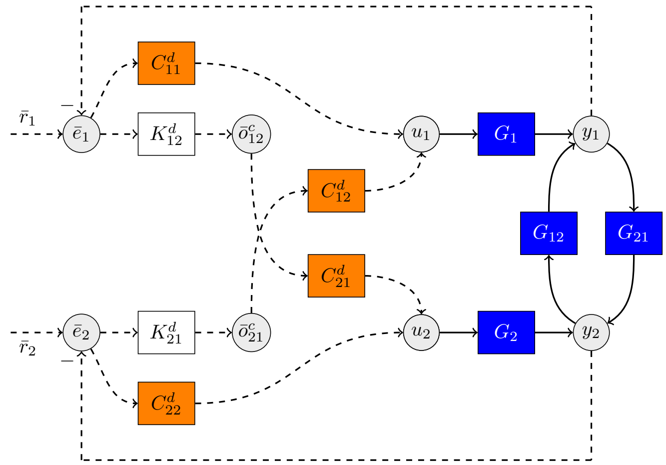


Fig. 2: Virtual experiment setup for identification of the ideal distributed controller in Example III.1.

Algorithm 1 Distributed virtual reference computation

Input: Reference model transfer functions T_i, Q_i, P_i and output data y_i for $i = 1, \ldots, L$
Output: Virtual signals $\bar{r}_i$, $\bar{e}_i$, $\bar{p}_i$ for $i = 1, \ldots, L$

for $i = 1$ to L **do**
  Compute $\bar{p}_i$ such that $\bar{p}_i(t) = P_i(q)y_i(t)$.
end for
for $i = 1$ to L **do**
  Receive $\bar{p}_{ji}$ from nodes $j \in \mathcal{N}_i$. Compute $\bar{r}_i$ such that

$$T_i(q)\bar{r}_i(t) = y_i(t) - \sum_{j \in \mathcal{N}_i} Q_{ij}(q)\bar{p}_{ji}(t).$$

  $\bar{e}_i \leftarrow \bar{r}_i - y_i$
end for
return $\bar{r}_i$, $\bar{e}_i$, $\bar{p}_i$, $i = 1, \ldots, L$

B. Distributed controller identification

Let us return to Example III.1. Figure 2 shows the constructed virtual network that is obtained by following Algorithm 1. The task of determining the controllers $\mathcal{C}_1^d$ and $\mathcal{C}_2^d$ now essentially becomes a dynamic network identification problem [19], where $\{C_{11}^d, C_{12}^d, K_{12}^d\}$ and $\{C_{22}^d, C_{21}^d, K_{21}^d\}$ are the modules to be identified (strictly speaking $\{C_{11}^d, C_{12}^d\}$ and $\{C_{22}^d, C_{21}^d\}$, since K_{12}^d and K_{21}^d are known). The signals u_1 and u_2 are directly available from the measurements, while $\bar{e}_1$ and $\bar{e}_2$ are virtual and obtained by Algorithm 1. The virtual controller interconnection signals $\bar{o}^c_{ij}$ are obtained by filtering $\bar{e}_i$ as $\bar{o}^c_{12} = K^d_{12}\bar{e}_1$ and $\bar{o}^c_{21} = K^d_{21}\bar{e}_2$.

To illustrate the identification, consider the parametrized models $\{C_{11}(\rho_1), C_{12}(\rho_1)\}$, $\{C_{22}(\rho_2), C_{21}(\rho_2)\}$, the predictors

$$\hat{u}_1(\rho_1) = C_{12}(\rho_1)\bar{o}^c_{21} + C_{11}(\rho_1)\bar{e}_1, \quad (20)$$

$$\hat{u}_2(\rho_2) = C_{21}(\rho_2)\bar{o}^c_{12} + C_{22}(\rho_2)\bar{e}_2 \quad (21)$$

and the identification criterion

$$J_{\text{VR}}(\rho_1, \rho_2) = \bar{E}[\varepsilon_1(\rho_1)]^2 + \bar{E}[\varepsilon_2(\rho_2)]^2$$

with $\varepsilon_i := u_i - u_i(\rho_i)$. We will now analyse the minima of J_{VR}. Since $\bar{o}^c_{12} = K^d_{12}\bar{e}_1$ and $\bar{o}^c_{21} = K^d_{21}\bar{e}_2$, it follows that

$$\varepsilon_1(\rho_1) = (C^d_1 - C_1(\rho_1))\bar{e}_1 + (C^d_{12} - C_{12}(\rho_1))K^d_{21}\bar{e}_2,$$
$$\varepsilon_2(\rho_2) = (C^d_2 - C_2(\rho_1))\bar{e}_2 + (C^d_{21} - C_{21}(\rho_2))K^d_{12}\bar{e}_1.$$

Then, since $\bar{e} = (T^{-1} - I)(I - W_I)^{-1}Gu$, where $T = \text{diag}(T_1, T_2)$, the prediction errors are

$$\begin{bmatrix}\varepsilon_1(\rho_1)\\ \varepsilon_2(\rho_2)\end{bmatrix} = \begin{bmatrix} C^d_{11} - C_{11}(\rho_1) & (C^d_{12} - C_{12}(\rho_1))K^d_{21} \\ (C^d_{21} - C_{21}(\rho_2))K^d_{12} & C^d_{22} - C_{22}(\rho_2)\end{bmatrix} \times (T^{-1} - I)(I - W_I)^{-1}Gu.$$

It now appears that a global minimum of J_{VR} is $(\rho_1, \rho_2) = (\rho^d_1, \rho^d_2)$ and that this minimum is unique if the control input signal $u = \text{col}(u_1, u_2)$ from the experiment is persistently exciting of a sufficient order. Hence, the global minimum of $J_{\text{VR}}(\rho_1, \rho_2)$ is then the same as the global minimum of $J_{\text{MR}}(\rho_1, \rho_2)$, where J_{VR} is quadratic in ρ when the models are parametrized linearly in ρ. The distributed-controller synthesis problem is therefore reformulated as a network identification problem.

In a more general situation, interconnection signals are present in the structured reference model and the signals $\bar{p}_i$ will serve as additional predictor inputs for the identification, which also leads to a more comprehensive setup than the one depicted in Figure 2. The latter reasoning for Example III.1 leads to the following result for a general interconnected system:

Theorem IV.1 *Consider the predictor* $\hat{u}_i(\rho_i) = C_{ii}(\rho_i)\bar{e}_i + \sum_{j\in\mathcal{N}_i} C^W_{ij}(\rho_i)\bar{o}^c_{ji} + C^Q_{ij}(\rho_i)\bar{p}_{ji}$ *with* $\bar{o}^c_{ji} = (1-T_j)^{-1}T_j\bar{e}_j + \sum_{h\in\mathcal{N}_j}(1-T_j)^{-1}Q_{jh}\bar{p}_{hj}$. *The identification criterion*

$$J^{\text{VR}}_i(\rho_i) = \bar{E}[u_i - \hat{u}_i(\rho_i)]^2$$

has a global minimum point at ρ^d_i *and this minimum is unique if the spectrum of* $w_i = \text{col}(\bar{e}_i, \text{col}_{j\in\mathcal{N}_i}\bar{o}^c_{ji}, \text{col}_{j\in\mathcal{N}_i}\bar{p}_{ji})$, *denoted* $\Phi_{w_i}(\omega)$, *is positive definite for all* $\omega \in [-\pi, \pi]$.

Proof: First, we note that $\bar{p}_{ji} = p^c_{ji}$ and $\bar{o}^c_{ji} = o^c_{ji}$, where p^c_{ji} and o^c_{ji} satisfy (10) and (11) for $e_i = \bar{e}_i$, $i = 1, \ldots, L$. Consequently, by applying Corollary 1 in [19], it follows that ρ^d_i is the unique global minimum point of J_{VR}. ■

When the reference model is decoupled, it is possible to translate the spectrum condition directly to the spectrum of the input signals:

Corollary IV.1 *Let* $P_i = 0$, $Q_i = 0$ *and consider the predictors* $\hat{u}_i(\rho_i) = C_{ii}(\rho_i)\bar{e}_i + \sum_{j\in\mathcal{N}_i} C_{ij}(\rho_i)\bar{o}^c_{ji}$, $i = 1, 2, \ldots, L$. *The identification criterion*

$$J_{\text{VR}}(\rho_1, \ldots, \rho_L) = \sum_{i=1}^{L} \bar{E}[u_i - \hat{u}_i(\rho_i)]^2$$

has a global minimum point at $(\rho^d_1, \ldots, \rho^d_L)$ *and this minimum is unique if* $\Phi_u(\omega)$ *is positive definite for all* $\omega \in [-\pi, \pi]$.

Remark IV.1 The condition on Φ_u in Corollary IV.1 can be realized by appropriate experiment design. The condition on Φ_{w_i} in Theorem IV.1, however, cannot always be realized by an appropriate design of u, depending on the system interconnection. For instance, consider Example III.1, but now with non-zero Q_i, P_i. Then the number of entries of $w_1 = \text{col}(\bar{e}_1, \bar{o}_{21}, \bar{p}_{21})$ is larger than the number of inputs in $u = \text{col}(u_1, u_2)$, hence $\Phi_{w_1}(\omega)$ cannot be positive definite. For an example for which the condition can be satisfied, consider a system on a path graph $\mathcal{G}$ with $L = 5$. Then the number of entries of w_i, $i = 1, \ldots, 5$, is not larger than the number of inputs in $u = \text{col}(u_1, \ldots, u_5)$. The addition of noise in the set-up is possibly helpful to satisfy the excitation conditions, but this goes beyond the current framework and is a topic for future research.

Each identification criterion $J^{\text{VR}}_i(\rho_i)$, $i = 1, \ldots, L$, can be minimized separately. The required predictor inputs for node i are the virtual signals obtained in Algorithm 1, which are available locally ($\bar{e}_i$) or communicated by nodes $j \in \mathcal{N}_i$ ($\bar{o}^c_{ji}$ and $\bar{p}_{ji}$). Of course, J^{VR}_i is not to be considered in practice since it involves expectations; for a finite number (N) of data the solution ρ^d_i is obtained by minimizing $\bar{J}^{\text{VR}}_i(\rho_i) = \sum_{t=1}^{N}(u_i(t) - \hat{u}_i(t, \rho_i))^2$. Observe the following difference with respect to multi-variable VRFT [16]. Instead of identifying the transfer from $\bar{e}$ to u, we identify the local controller dynamics C^d_i by exploiting the structure of the interconnected system. An analogy can be made with the difference between multi-variable process identification and dynamic network identification [19].

V. Numerical example

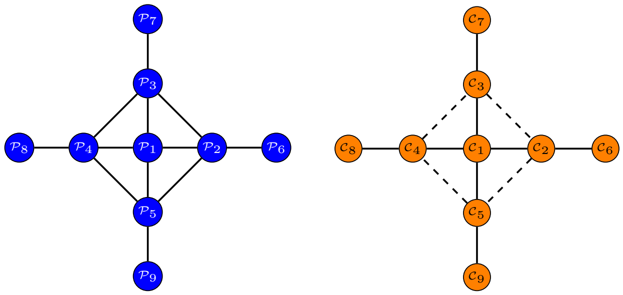
Consider the interconnected system (18) with $L = 9$ and the interconnection structure depicted in Figure 3a. The transfer functions describing the dynamics are of order one and given by

$$G_i = \frac{1}{q - a_i}, \quad W_{ij} = \frac{0.1}{q - a_i}, \quad i = 1, \ldots, 9,$$

with $a_i \in (0, 1)$. It is desired to decouple the interconnected system and to have the same step response for every output channel. Hence the reference model is chosen as $y^d_i = T^d_i(q)r_i$, where

$$T_i(q) = \frac{0.4}{q - 0.6}, \quad i = 1, \ldots, 9.$$

We collect the data $\{u_i(t), y_i(t), t = 1, 2, \ldots, 100\}$ from (18) in open-loop, with mutually uncorrelated white-noise input signals u_i having a standard deviation of $\sigma_{u_i} = 1$. Hence,



(a) Interconnected system (b) Distributed controller

Fig. 3: Graph for the interconnected system (a) and distributed controller (b) that satisfies Assumption III.1 (solid and dashed edges), with a reduced number of communication links (solid edges) and a decentralized controller (no edges).

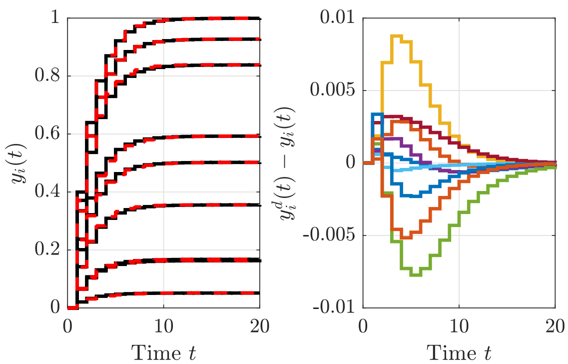


Fig. 4: Left: Closed-loop response of the network with the data-driven distributed controller (red) and the desired response of the structured reference model (black). Right: Response errors with respect to the desired outputs.

we are in the situation of Corollary IV.1. Each controller C_i, $i = 1, \ldots, 9$, is parametrized such that Assumption III.1 holds. Since there is no noise present in the output, it is clear that the optimization of $\bar{J}_i^{\mathrm{VR}}$ with predictors (20) yields the parameters ρ_i^d and therefore a model-reference cost J_{MR} equal to zero.

Next, we will analyze the situation where noise affects the system, by considering disturbed outputs $\tilde{y}_i(t) = y_i(t) + v_i(t)$ for the synthesis, with v_i white-noise processes with standard deviations $\sigma_{v_i} = 0.1$ that are mutually uncorrelated and uncorrelated with u_i. The method of generating virtual references and predictors is kept the same. The resulting distributed controller is interconnected with the plant and a step reference is applied to each subsystem simultaneously, with an amplitude between zero and one. Figure 4 shows the output response of the closed-loop network in red together with the response of the reference model (in black) on the left. We observe only a minor difference between the responses, due to the noise added to the data for identification, as shown in Figure 4 on the right.

The distribution of the error between the achieved closed-loop network with the identified distributed controller and the structured reference model resulting from 100 Monte Carlo runs is presented in Figure 5. Because $P_i^\top C_i^d P_i \in \mathscr{C}_i$ for all i, the error between the achieved closed-loop system and the reference model is only due to the noise. In practice, the ideal distributed controller does not always belong to the considered class of controllers, e.g. due to the existing controller architecture or equipment. To illustrate such a situation for our example, that is, a situation where Assumption III.1 does not hold, consider the case where four communication links between controllers $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_5$ are not present. These links are illustrated in Figure 3b by the dashed edges. The implication is that in the parametrization $C_{ij}(\rho_i) = 0$ for the corresponding edges $(i, j) \in \mathcal{E}$ and, e.g., $\hat{u}_2(\rho_2) = C_{22}\bar{e}_2 + C_{21}(\rho_2)\bar{o}_{12}^c + C_{26}(\rho_2)\bar{o}_{62}^c$. Note that links are thus not removed *a posteriori*, but the interconnection structure for the distributed controller is induced by the controller class. As shown by Figure 5, there is a significant performance degradation, because the controller class is not 'rich' enough, although the graph for the controller remains connected. We finally consider the data-driven synthesis of a decentralized controller, corresponding to structure without any interconnections between controllers, i.e., $C_{ij}(\rho_i) = 0$ for all $(i, j) \in \mathcal{E}$. The resulting discrepancy between reference model and closed-loop network is plotted in Figure 5 and shows a further decrease in performance.

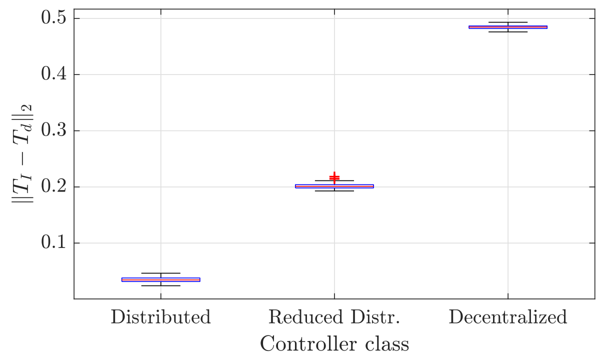


Fig. 5: Distribution of the achieved performance for various controller classes, where T_I and T_d denote the transfers $r \to y$ and $r \to y_d$, respectively. From left to right: a distributed controller satisfying Assumption III.1, a controller with four communication links removed and a decentralized controller.

VI. Concluding Remarks

In this paper it has been shown how to perform data-driven synthesis of a distributed controller for an interconnected system defined on arbitrary graph. In the extension of VRFT for a single process to the interconnected system case, the two main steps involved are (i) the extension of the ideal controller to an ideal distributed controller and (ii) the data-driven modelling of this controller via the identification of modules in a (virtual) dynamic network. The distributed controller can be both synthesized (identified) and implemented in a distributed way, allowing the application to large-scale interconnected systems for which multi-variable VRFT may not be feasible.

The developed data-driven distributed control method allows for several interesting extensions. For example, one can impose reduced orders for the controller modules or remove interconnection links in the controller, which corresponds to changing the controller class. In future work, we will deal with the analysis and filter design for such non-ideal, but practically relevant, cases.

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