# Data-driven distributed controller synthesis in the presence of noise: an optimal controller identification approach

Tom R.V. Steentjes, Mircea Lazar, Paul M.J. Van den Hof

***Abstract*— In this paper, the data-driven *distributed* controller synthesis is considered for interconnected systems that are subject to unmeasured disturbances. The considered problem is the optimization of a model-reference control criterion, where the reference model is described by an interconnected system. We provide a method to determine the optimal distributed controller by performing network identification in an augmented network. Sufficient conditions are provided for which the data-driven method solves the distributed model-reference control problem, whereas state-of-the-art methods for data-driven distributed control can only provide performance guarantees in the absence of disturbances. The effectiveness of the method is demonstrated via a simple network example consisting of two interconnected systems.**

## I. Introduction

Data-driven methods for controller design are applicable to systems for which a mathematical description is not available. A common property of these methods is that they are based on the model-reference paradigm [1], wherein the control performance is determined with respect to a user-specified model describing the desired bevavior for the closed-loop system. State-of-the-art data-driven methods that solve the model-reference control problem using a single batch of data are virtual reference feedback tuning (VRFT) [2], optimal controller identification (OCI) [3], [4] and asymptotically exact controller tuning [5].

The aforementioned methods are typically applicable to (multi-variable) isolated or small-scale systems. For interconnected systems, the controllers are not implementable due to their lumped nature or cannot be synthesized due to the dimensionality of the system. A sensible approach for interconnected systems would be to synthesize a *distributed* controller directly from data. Data-driven distributed control removes the need for identifying large-scale models, as required by model-based distributed controller synthesis methods [6], [7] and provides the scalability that is not provided by data-driven methods for lumped systems, such as VRFT, OCI, but also data-enabled predictive control [8] or control enabled by data-based linear matrix inequalities [9].

Data-driven distributed control is an only just emerging field, with few contributions for predictive control [10] and model-reference control [11]. In [11], a data-driven distributed controller design method was introduced, based on virtual reference feedback tuning in dynamic networks. By constructing a virtual reference network, the distributed model-reference control problem can be equivalently stated as a network identification problem [11], [12].

The interconnected system considered in [11] is not influenced by any uncontrollable and unmeasured inputs. Even for scalar systems, however, VRFT inherently introduces a bias in the controller estimates when disturbances affect the system [13], leading to a degraded closed-loop performance. This problem can be solved by using an instrumental variable (IV) approach in case the controller model is linear with respect to the parameters. In the general case, however, it is not straightforward how to obtain unbiased estimates. Furthermore, depending on the choice of IV, the introduction of IVs can require additional experiments on the system [13] and increase the parameter variance with a negative effect on the control performance.

In this paper, we solve the distributed model reference control problem in the case that the interconnected system is subject to unmeasured exogenous inputs. By using the direct method for identification in dynamic networks [12], we provide sufficient conditions for the consistent estimation of a distributed controller introduced in [11], that solves the considered control problem. The local nature of the identification problems imply a decentralized computation of the distributed controller is possible. The method generalizes OCI to interconnected systems and provides a solution to the control problem introduced in [11] for a class of interconnected systems with disturbances.

The remainder of this paper is organized as follows: in Section II we provide some preliminaries, state the distributed control problem and recall an ideal distributed controller. Section III provides the problem formulation. In Section IV, we present the main result: a method to solve the control problem for systems subject to disturbances. Section V provides a numerical example to show the effectiveness of the developed method. Conclusions are summarized in Section VI.

## II. Preliminaries

### A. Dynamical network and distributed controller

Consider a simple and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality $L$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbour set of vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. The graph $\mathcal{G}$ describes the structure of a network of linear discrete-time systems, where the dynamics

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associated with vertex $i \in \mathcal{V}$ are described by

$$y_i(t) = G_i(q)u_i(t) + \sum_{j\in\mathcal{N}_i} G_{ij}(q)y_j(t) + H_i(q)e_i(t), \quad (1)$$

with $u_i : \mathbb{Z} \to \mathbb{R}$ the control input, $y_i : \mathbb{Z} \to \mathbb{R}$ the output, $e_i$ an unmeasured zero-mean white-noise process and $q$ the forward shift defined as $qx(t) = x(t+1)$. The process $e_i$ is assumed to be uncorrelated to all $e_j$, $j \in \mathcal{V} \setminus \{i\}$ and to all $u_j$, $j \in \mathcal{V}$. The rational transfer functions $G_i$, $G_{ij}$ and $H_i$, $(i,j) \in \mathcal{E}$, describe the local dynamics, coupling dynamics and noise dynamics, respectively. The noise filter $H_i$ is assumed to be monic, stable and minimum phase. The network can be compactly written as

$$y = G_I y + Gu + He, \quad (2)$$

where $G = \operatorname{diag} G_1, \ldots, G_L$, $H = \operatorname{diag} H_1, \ldots, H_L$ and

$$G_I = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \cdots & G_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1} & G_{L2} & \cdots & 0 \end{bmatrix}.$$

Furthermore, $y = (I - G_I)^{-1}(Gu + He)$, where it is assumed that $\mathcal{G} := (I - G_I)^{-1}$ exists.

Considering a reference tracking problem for the network, each system is equipped with a reference signal $r_i$ and the corresponding tracking error $z_i := r_i - y_i$:

$$\mathcal{P}_i : \begin{cases} y_i &= G_i u_i + \sum_{j\in\mathcal{N}_i} G_{ij} y_j + H_i e_i, \\ z_i &= r_i - y_i. \end{cases} \quad (3)$$

The dynamical network is operating in closed-loop with a distributed controller that consists of local controllers

$$\mathcal{C}_i(\rho_i) : \begin{cases} u_i = C_{ii}(q,\rho_i) z_i + \sum_{j\in\mathcal{N}_i} C_{ij}(q,\rho_i)\eta_{ij}, \\ \zeta_{ij} = K_{ij}(q,\rho_i)e_i + \sum_{h\in\mathcal{N}_i} K_{ijh}(q,\rho_i)\eta_{ih}, \; j \in \mathcal{N}_i. \end{cases}$$

Each controller is parametrized by a parameter vector $\rho_i$. The controller class for node $i$ is defined as $\mathscr{C}_i := \{C_i(q,\rho_i) \,|\, \rho_i \in \mathbb{R}^{l_i}\}$. Controllers $\mathcal{C}_i$ and $\mathcal{C}_j$ are interconnected if and only if $(i,j) \in \mathcal{E}$. The interconnection is defined by constraints on the controller interconnection signals $\eta_{ij}$ and $\zeta_{ij}$:

$$\eta_{ij} = \zeta_{ji} \quad \text{and} \quad \eta_{ji} = \zeta_{ij}, \quad (i,j) \in \mathcal{E}.$$

With the definitions $\eta_i := \operatorname{col}_{j\in\mathcal{N}_i} \eta_{ij}$ and $\zeta_i := \operatorname{col}_{j\in\mathcal{N}_i} \zeta_{ij}$, we compactly represent $\mathcal{C}_i$ by

$$\mathcal{C}_i(\rho_i) : \begin{bmatrix} u_i \\ \zeta_i \end{bmatrix} = C_i(q,\rho_i) \begin{bmatrix} z_i \\ \eta_i \end{bmatrix}. \quad (4)$$

### B. Distributed model reference control

Model reference control in dynamic networks considers the synthesis of a structured controller such that the closed-loop network dynamics are optimal with respect to a structured reference model. This reference model is described by subsystems $\mathcal{K}_i$, $i \in \mathcal{V}$:

$$\mathcal{K}_i : \begin{cases} y_i^d &= T_i(q) r_i + \sum_{j\in\mathcal{N}_i} Q_{ij}(q) k_{ij}, \\ p_{ij} &= P_{ij}(q) y_i^d, \quad j \in \mathcal{N}_i. \end{cases} \quad (5)$$

The interconnection of $\mathcal{K}_i$ and $\mathcal{K}_j$ is defined by

$$k_{ij} = p_{ji} \quad \text{and} \quad k_{ji} = p_{ij}. \quad (6)$$

Subsystems $\mathcal{K}_i$ and $\mathcal{K}_j$ can only be interconnected if $(i,j) \in \mathcal{E}$. A particular case of a structured reference model is a decoupled reference model, i.e., $Q_{ij} = 0$ and $P_{ij} = 0$, $(i,j) \in \mathcal{V}$. For well-posedness, we assume that the reference model satisfies $\det(I - Q\Delta P) \neq 0$ and is such that $y^d \neq r$ for all non-zero $r$, i.e., $\det((I - Q\Delta P)^{-1}T - I) \neq 0$.

Given $e_i = 0$ for all $i \in \mathcal{V}$, the distributed model reference control problem is

$$\min_{\rho_1,\ldots,\rho_L} J_{\text{MR}}(\rho_1,\ldots,\rho_L) = \min_{\rho_1,\ldots,\rho_L} \sum_{i=1}^{L} \bar{E}[y_i^d(t) - y_i(t)]^2, \quad (7)$$

where $\bar{E} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E$ and $E$ is the expectation. A distributed controller that solves (7) was developed in [11]. Define the column vector $\mathbf{1} := \operatorname{col}(1,\ldots,1)$ and, for $i \in \mathcal{V}$, define $G_{iI} := \operatorname{row}_{j\in\mathcal{N}_i} G_{ij}$, $Q_i := \operatorname{row}_{j\in\mathcal{N}_i} Q_{ij}$ and $P_i := \operatorname{col}_{j\in\mathcal{N}_i} P_{ij}$.

**Proposition II.1** *Consider $e_i = 0$ for all $i \in \mathcal{V}$ and consider a distributed controller described by the subsystems*

$$\mathcal{C}_i^d : \begin{bmatrix} u_i \\ o_i^c \\ p_i^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_i}{G_i(1-T_i)} & -\frac{1}{G_i}G_{iI} & \frac{1}{G_i(1-T_i)}Q_i \\ \frac{T_i}{1-T_i}\mathbf{1} & 0 & \frac{1}{1-T_i}\mathbf{1}Q_i \\ \frac{T_i}{1-T_i}P_i & 0 & \frac{1}{1-T_i}P_iQ_i \end{bmatrix}}_{=:C_i^d(q)} \begin{bmatrix} z_i \\ s_i^c \\ k_i^c \end{bmatrix}, \quad (8)$$

*for $i \in \mathcal{V}$ and the controller interconnections described by*

$$\begin{bmatrix} s_{ij}^c \\ k_{ij}^c \end{bmatrix} = \begin{bmatrix} o_{ji}^c \\ p_{ji}^c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s_{ji}^c \\ k_{ji}^c \end{bmatrix} = \begin{bmatrix} o_{ij}^c \\ p_{ij}^c \end{bmatrix}, \quad (9)$$

*for $(i,j) \in \mathcal{E}$. The network* (2) *in closed-loop with the distributed controller* (8)-(9) *satisfies*

$$y_i = y_i^d, \quad i \in \mathcal{V}.$$

We refer the reader to [11] for the proof and the corresponding result [11, Theorem III.1]. The controller described by (8)-(9) provides a solution to (7), by showing the existence of a minimizing argument $(\rho_1^d,\ldots,\rho_L^d)$. Details on how to obtain $(\rho_1^d,\ldots,\rho_L^d)$ from $C_i^d$, $i \in \mathcal{V}$, can be found in Section III in [11].

## III. Problem formulation

The problem that we consider in this paper is to determine the ideal controller described by (8)-(9) in the case that the network described by (2) is unknown, i.e., in the case that the transfer functions $G_i$, $G_{ij}$ and $H_i$, $(i,j) \in \mathcal{E}$, are unknown. The local controller modules $\mathcal{C}_i^d$ contain known modules, depending solely on the reference model dynamics $\mathcal{K}_i$ and unknown modules describing the top row in (8):

$$u_i = C_{ii}^d(q) z_i + \sum_{j\in\mathcal{N}_i} C_{ij}^d(q) s_{ij}^c + \sum_{j\in\mathcal{N}_i} C_{ij}^{Td}(q) k_{ij}^c. \quad (10)$$

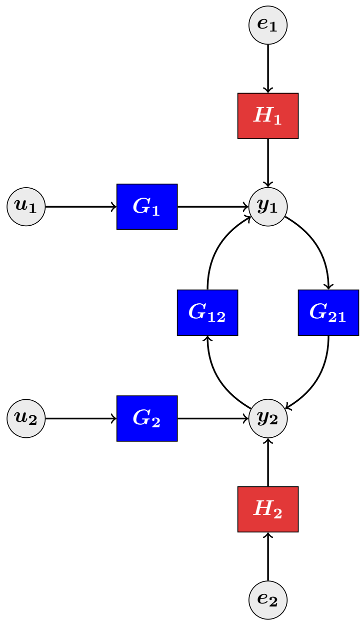


Fig. 1: Two output coupled process with noise represented by a dynamic network.

Given data collected from the network (2), the transfer functions $C_{ii}^d$, $C_{ij}^d$ and $C_{ij}^{Td}$ can be determined through VRFT in dynamic networks [11], in the case that no noise is present, i.e., $e_i = 0$ for all $i \in \mathcal{V}$. The application of VRFT for systems with noise leads to biased estimates, both for the single-process case [13] and the network case [11], leading to a degraded closed-loop performance. The bias of distributed controller estimates with VRFT is illustrated in an example.

**Example III.1** Consider a network consisting of two subsystems, as depicted in Figure 1, described by

$$y_1 = G_1 u_1 + G_{12} y_2 + H_1 e_1, \tag{11}$$
$$y_2 = G_2 u_2 + G_{21} y_1 + H_2 e_2. \tag{12}$$

We choose the reference model to be decoupled:

$$y_1^d = T_1 r_1, \quad y_2^d = T_2 r_2. \tag{13}$$

The ideal distributed controller is then described by

$$\begin{bmatrix} u_1 \\ o_1^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_1}{G_1(1-T_1)} & -\frac{G_{12}}{G_1} \\ \frac{T_1}{1-T_1} & 0 \end{bmatrix}}_{=:\begin{bmatrix} C_{11}^d & C_{12}^d \\ K_{12}^d & 0 \end{bmatrix}} \begin{bmatrix} z_1 \\ s_1^c \end{bmatrix}, \tag{14}$$

$$\begin{bmatrix} u_2 \\ o_2^c \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{T_2}{G_2(1-T_2)} & -\frac{G_{21}}{G_2} \\ \frac{T_2}{1-T_2} & 0 \end{bmatrix}}_{=:\begin{bmatrix} C_{22}^d & C_{21}^d \\ K_{21}^d & 0 \end{bmatrix}} \begin{bmatrix} z_2 \\ s_2^c \end{bmatrix}, \tag{15}$$

with the interconnections $s_1^c = o_2^c$ and $s_2^c = o_1^c$. In the distributed VRFT (DVRFT) approach to distributed controller synthesis [11], virtual reference signals and virtual tracking errors $\bar{r}_i$ and $\bar{z}_i$ are computed from data, satisfying

$$y_i = T_i \bar{r}_i \quad \text{and} \quad \bar{z}_i := \bar{r}_i - y_i, \quad i = 1, 2. \tag{16}$$

Using the definitions for the virtual signals and ideal controller, the processes (11) and (12) are rewritten as

$$u_1 = C_{11}^d \bar{z}_1 + C_{12}^d \bar{o}_2^c - G_1^{-1} H_1 e_1, \tag{17}$$
$$u_2 = C_{22}^d \bar{z}_2 + C_{21}^d \bar{o}_1^c - G_2^{-1} H_2 e_2, \tag{18}$$

where

$$\bar{o}_i^c := \frac{T_i}{1-T_i} \bar{z}_i = y_i, \quad i = 1, 2.$$

Controller module 1, for example, is then obtained by minimizing the identification criterion

$$J_1^{\mathrm{VR}}(\rho_1) = \bar{E}[u_1 - \hat{u}_1(\rho_1)]^2 = \bar{E}[u_1 - \rho_1^\top \varphi_1]^2,$$

where the controller model is chosen to be linear in the parameters:

$$\hat{u}_1(\rho_1) = C_{11}(\rho_1)\bar{z}_1 + C_{12}(\rho_1)\bar{o}_{21}^c = \rho_1^\top \varphi_1,$$

with $\rho_1 := \mathrm{col}(\rho_{11}, \rho_{12})$ and the regression vector $\varphi_1 := \mathrm{col}(\bar{C}_{11}\bar{z}_1, \bar{C}_{12}\bar{o}_2^c)$. Any global minimizer $\rho_1^*$ of $J_1^{\mathrm{VR}}$ satisfies the normal equation [13]

$$\bar{E}[\varphi_1 \varphi_1^\top]\rho_1^* = \bar{E}[\varphi_1 u_1].$$

If $\bar{E}[\varphi_1 \varphi_1^\top]^{-1}$ exists, then

$$\rho_1^* = \bar{E}[\varphi_1 \varphi_1^\top]^{-1} \bar{E}[\varphi_1 u_1]. \tag{19}$$

Let $\rho_i^d$ be the parameter that corresponds to $C_{ii}^d$ and $C_{ij}^d$. The difference between $\rho_i^d$ and the parameter estimate for finite data $\hat{\rho}_i := [\sum_{t=1}^N \varphi_i \varphi_i^\top]^{-1} \sum_{t=1}^N \varphi_i(t) u_i(t)$ is

$$\hat{\rho}_i - \rho_i^d = (\rho_i^* - \rho_i^d) + (\hat{\rho}_i - \rho_i^*),$$

where $\rho_i^* - \rho_i^d$ is the *bias error* and $\hat{\rho}_i - \rho_i^*$ the *variance error* [13].

**Lemma III.1** *The bias error of the estimator $\hat{\rho}_i$, $i = 1, 2$ is*

$$\bar{E}[\varphi_i \varphi_i^\top]^{-1} \bar{E}[\varphi_i G_i^{-1} H_i e_i].$$

*Proof: Substitution of (17) in (19) yields*

$$\begin{aligned} \rho_1^* &= \bar{E}[\varphi_1 \varphi_1^\top]^{-1} \bar{E}[\varphi_1 \left( C_{11}^d \bar{z}_1 + C_{12}^d \bar{o}_{21}^c - G_1^{-1} H_1 e_1 \right)] \\ &= \rho_1^d - \bar{E}[\varphi_1 \varphi_1^\top]^{-1} \bar{E}[\varphi_1 G_1^{-1} H_1 e_1], \end{aligned} \tag{20}$$

*where the linearity of the expectation operator was used to obtain the last equality. The bias expression for $i = 2$ can be obtained analogously.* ■

**Remark III.1** If there is process noise present at process $i \in \{1, 2\}$, i.e. $e_i \neq 0$, this will only influence the quality of the controller parameter estimate $\hat{\rho}_i$ in terms of bias by Lemma III.1. The interpretation is that $e_i$ causes an errors-in-variables type of problem for DVRFT, while $e_j$, $j \neq i$, only serve as excitation sources in the network.

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Due to the bias, controller parameter estimates will in general not minimize the cost function (7) and hence lead to a reduced closed-loop performance if the process is subject to noise. The problem that is considered in this paper is: how to obtain consistent distributed controller estimates in the presence of noise?

Optimal controller identification (OCI) [3] and VRFT in conjunction with instrumental variables (IVs) [13], [14] are two methods for obtaining consistent estimates of controllers for lumped systems. The use of IVs can lead to estimators with a high variance, however, which in turn affects the controller performance (illustrated for an example network in Section V). As such, in what follows we consider the identification of auxiliary controller modules in order to obtain consistent estimates of the ideal controllers modules $C_i^d$, $i \in \mathcal{V}$, by generalizing the OCI method to dynamical networks.

## IV. Main result: Indirect distributed controller identification

In order to pose an identification problem, we start by rewriting the network dynamics in terms of the ideal distributed controller dynamics. The approach of rewriting the dynamics of a single-input single-output system in terms of an ideal controller for prediction-error identification was introduced in [3].

### A. Decoupled reference model

Let us first consider the case where the reference model is decoupled, i.e., for each $i \in \mathcal{V}$,

$$\mathcal{K}_i : \quad y_i^d = T_i(q) r_i.$$

By (8), we observe that the transfer functions in (10) are

$$C_{ii}^d = \frac{T_i}{G_i(1-T_i)}, \quad C_{ij}^d = -\frac{G_{ij}}{G_i}, \quad (i,j) \in \mathcal{E}.$$

Hence, we can write the network dynamics (1) in terms of the ideal distributed controller and the reference model as

$$G_i = \frac{1}{C_{ii}^d} \frac{T_i}{1-T_i}$$

and

$$G_{ij} = -C_{ij}^d G_i = -\frac{C_{ij}^d}{C_{ii}^d} \frac{T_i}{1-T_i}.$$

Models for the network's transfer functions can then be written in terms of the controller parameters as

$$G_i(\rho_i) := \frac{1}{C_{ii}(\rho_i)} \frac{T_i}{1-T_i} \text{ and } G_{ij}(\rho_i) := -\frac{C_{ij}(\rho_i)}{C_{ii}(\rho_i)} \frac{T_i}{1-T_i}.$$

We can thus rewrite the network dynamics (1) as

$$y_i = \frac{1}{C_{ii}^d} \frac{T_i}{1-T_i} u_i - \sum_{j \in \mathcal{N}_i} \frac{C_{ij}^d}{C_{ii}^d} \frac{T_i}{1-T_i} y_j + H_i e_i,$$

or

$$y_i = \bar{C}_{ii}^d \bar{u}_i + \sum_{j \in \mathcal{N}_i} \bar{C}_{ij}^d \bar{y}_{ij} + H_i e_i, \tag{21}$$

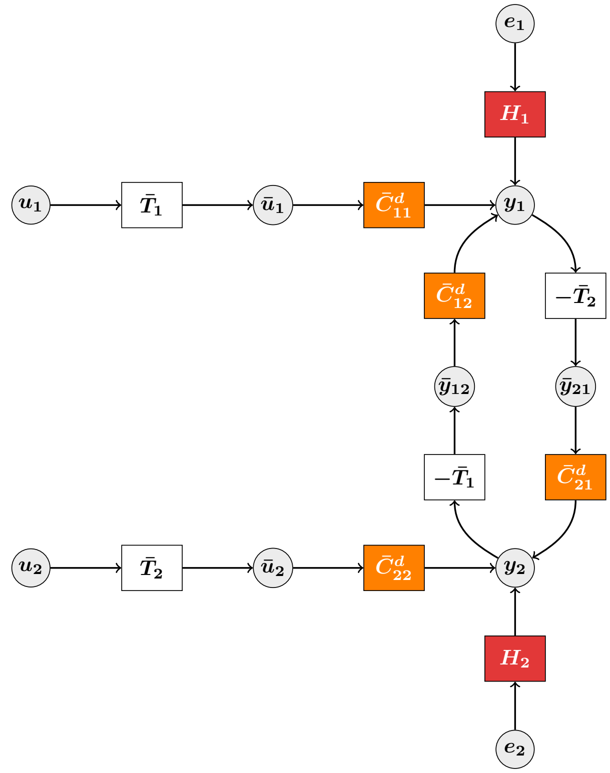


Fig. 2: The dynamic network represented by modules of the ideal distributed controller and reference model.

with

$$\bar{C}_{ii}^d := \frac{1}{C_{ii}^d}, \quad \bar{C}_{ij}^d := \frac{C_{ij}^d}{C_{ii}^d},$$

and the signals

$$\bar{u}_i := \frac{T_i}{1-T_i} u_i = \bar{T}_i u_i, \tag{22}$$

$$\bar{y}_{ij} := -\frac{T_i}{1-T_i} y_j = -\bar{T}_i y_j. \tag{23}$$

In the case that $T_i$ is proper, $\bar{T}_i$, $i \in \mathcal{V}$, will be proper and the dynamical relations (21)-(23) can be interpreted as an (augmented) dynamic network:

$$\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \bar{C}_I^d & \bar{C}^d \\ \bar{T}_{\mathcal{N}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:\bar{C}_{\mathcal{G}}} \begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \bar{T} \end{bmatrix} u + \begin{bmatrix} H \\ 0 \\ 0 \end{bmatrix} e, \tag{24}$$

with the matrices $\bar{C}_I^d := \text{diag}_{i \in \mathcal{V}} \text{row}_{j \in \mathcal{N}_i} \bar{C}_{ij}^d$, $\bar{C}_d := \text{diag}(\bar{C}_{11}^d, \ldots, \bar{C}_{LL}^d)$, $\bar{T} := \text{diag}(\bar{T}_1, \ldots, \bar{T}_L)$, $\bar{T}_{\mathcal{N}} := \text{col}_i \text{col}_{j \in \mathcal{N}_i} -\bar{T}_i \mathbf{e}_j$, with $\mathbf{e}_i$ the $i$-th unit vector, and $\bar{T} := \text{diag}(\bar{T}_1, \ldots, \bar{T}_L)$. This augmented network is visualized in Figure 2 for $L = 2$.

To write the model, define $\bar{C}_{ii}(\rho_i) := \frac{1}{C_{ii}(\rho_i)}$ and $\bar{C}_{ij}(\rho_i) := \frac{C_{ij}(\rho_i)}{C_{ii}(\rho_i)}$, $(i,j) \in \mathcal{E}$, such that

$$y_i(\theta_i) = \bar{C}_{ii}(\rho_i) \bar{u}_i + \sum_{j \in \mathcal{N}_i} \bar{C}_{ij}(\theta_i) \bar{y}_{ij} + H_i(\theta_i) e_i, \tag{25}$$

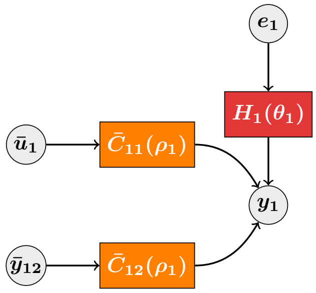


Fig. 3: Model with the auxiliary controller modules for $i = 1$, $L = 2$.

where $\theta_i = \mathrm{col}(\rho_i, \eta_i)$, with $\eta_i$ parameters for the noise model $H_i(\theta_i)$. Model (25) is visualized in Figure 3 for $i = 1$, $L = 2$.

By using the direct method for dynamic network identification [12], the estimates are obtained as

$$\hat{\theta}_i = \arg\min_{\theta_i} V_i(\theta_i), \quad V_i(\theta_i) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon_i^2(t, \theta_i), \tag{26}$$

with the prediction error defined by

$$\varepsilon_i(t, \theta_i) := y_i(t) - \hat{y}_i(t, \theta_i)$$

and the predictor

$$\hat{y}_i(t, \theta_i) := H_i^{-1}(\theta_i)\left(\bar{C}_{ii}(\rho_i)\bar{u}_i + \sum_{j\in\mathcal{N}_i} \bar{C}_{ij}(\rho_i)\bar{y}_{ij}\right) + \left(1 - H_i^{-1}(\theta_i)\right) y_i.$$

By definition of the auxiliary controller models, the controller estimates are then

$$C_{ii}(\hat{\rho}_i) = \frac{1}{\bar{C}_{ii}(\hat{\rho}_i)}, \quad C_{ij}(\hat{\rho}_i) = \bar{C}_{ij}(\hat{\rho}_i)C_{ii}(\hat{\rho}_i), \quad (i,j) \in \mathcal{E}.$$

Under weak assumptions, the estimator $\hat{\theta}_i$ converges asymptotically in $N$ [15]:

$$\hat{\theta}_i \to \theta_i^* \quad \text{w.p. 1 as} \quad N \to \infty,$$

where $\theta_i^* = \arg\min_{\theta_i} \bar{V}_i(\theta_i)$ with $\bar{V}_i(\theta_i) := \bar{E}[\varepsilon_i^2(t, \theta_i)]$.

**Theorem IV.1** *Suppose the following conditions hold:*

- *the spectral density of* $\bar{\xi}_i := \mathrm{col}(\bar{u}_i, \bar{y}_{i1}, \dots, \bar{y}_{iL})$, $\Phi_{\bar{\xi}_i}(\omega)$, *is positive definite for all* $\omega \in [-\pi, \pi]$,
- *there exists a* $\theta_i^d$ *such that* $C_i(\theta_i^d) = C_i^d$ *and* $H_i(\theta_i^d) = H_i$,
- $\mathcal{G}_{ji}$ *contains a delay for every* $j \in \mathcal{N}_i$.

*Then it holds that* $C_{ii}(\theta_i^*) = C_{ii}^d$, $H_i(\theta_i^*) = H_i$ *and* $C_{ij}(\theta_i^*) = C_{ij}^d$, $j \in \mathcal{N}_i$.

*Proof:* *We will first show that the minimum of the objective function* $\bar{V}_i$ *is* $\sigma_{e_i}^2 = Ee_i^2$. *By the definition of the predictor and prediction error, we have*

$$\bar{V}_i(\theta_i) = \bar{E}\left(H_i(\theta_i)^{-1}\left(v_i + \sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{y}_{ij} + \Delta\bar{C}_{ii}(\theta_i)\bar{u})_i\right)\right)^2.$$

*Then, by (24) it follows that*

$$\bar{V}_i(\theta_i) = \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(v_i - \sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i y_j + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i\Bigg)\Bigg)^2$$

*and by (2):*

$$\begin{aligned}\bar{V}_i(\theta_i) &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(v_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\quad - \sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i \sum_{k\in\mathcal{V}} \mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg)\Bigg)^2 \\ &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(\Delta H_i(\theta_i)e_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\quad - \sum_{j\in\mathcal{N}_i}\Delta\bar{C}_{ij}(\theta_i)\bar{T}_i \sum_{k\in\mathcal{V}} \mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg) + e_i\Bigg)^2.\end{aligned}$$

*Since both* $H_i$ *and* $H_i(\theta_i)$ *are monic,* $\Delta H_i(\theta_i)$ *is strictly proper. Hence,* $\Delta H_i(\theta_i)e_i$ *is uncorrelated with* $e_i$. *Also* $-\sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i \sum_{k\in\mathcal{V}} \mathcal{G}_{jk}G_k u_k$ *is uncorrelated with* $e_i$, *since it is a filtered linear combination of* $u_k$, $k \in \mathcal{V}$, *which are uncorrelated with* $e_i$ *by assumption. Moreover,* $-\sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i \sum_{k\in\mathcal{V}} \mathcal{G}_{jk}H_k e_k$ *is uncorrelated with* $e_i$, *since* $\Delta\bar{C}_{ij}(\theta_i)$ *is proper and* $\mathcal{G}_{ji}$ *is strictly proper for all* $j \in \mathcal{N}_i$. *Hence,*

$$\begin{aligned}\bar{V}_i(\theta_i) &= \bar{E}\Bigg(H_i(\theta_i)^{-1}\Bigg(\Delta H_i(\theta_i)e_i + \Delta\bar{C}_{ii}(\theta_i)\bar{T}_i u_i \\ &\quad - \sum_{j\in\mathcal{N}_i} \Delta\bar{C}_{ij}(\theta_i)\bar{T}_i \sum_{k\in\mathcal{V}} \mathcal{G}_{jk}(G_k u_k + H_k e_k)\Bigg)\Bigg)^2 \\ &\quad + \bar{E}e_i^2 \\ &\geq \sigma_{e_i}^2.\end{aligned}$$

*It remains to show that* $\bar{V}_i(\theta_i) = \sigma_{e_i}^2 \Rightarrow \theta_i = \theta_i^d$. *This follows from the first two conditions and is shown in [12, Appendix B]. This part is excluded for brevity.* ■

Positive definiteness of $\Phi_{\bar{\xi}_i}$ is implied by sufficient excitation of the filtered input $\bar{u}_i = \bar{T}_i u_i$ and the signals $\bar{y}_{ij}$, $j \in \mathcal{N}_i$. The condition on $\Phi_{\bar{\xi}_i}$ can be translated to conditions on external signals $u_j$, $e_j$, $j \in \mathcal{V}$, and conditions on the augmented network topology, as described in [16]. While the excitation conditions are sufficient, they can be rather conservative. We point out the results on data informativity for centralized control problems in [17], which open the way for the formulation of necessary excitation conditions for the distributed model reference control problem.

### *B. Dealing with a coupled reference model*

In the case of a coupled reference model, the ideal distributed controller (8)-(9) contains additional transfer function modules $C_{ij}^{Td}$, $(i,j) \in \mathcal{E}$ that are unknown. These modules are not identified in the procedure described in Section IV-A. We will now briefly describe an extension of the procedure in Section IV-A to obtain consistent estimates of the additional modules $C_{ij}^{Td}$, $(i,j) \in \mathcal{E}$.

Consider again the example network (11)-(12) with a coupled reference model

$$y_1^d = T_1 r_1 + Q_{12} k_{12}, \quad p_{12} = P_{12} y_1^d,$$
$$y_2^d = T_2 r_2 + Q_{21} k_{21}, \quad p_{21} = P_{21} y_2^d.$$

where $k_{12} = p_{21}$ and $k_{21} = p_{12}$. The top row in (8) is given by (10), with the modules defined in (14)-(15) and

$$C_{12}^{Td} = \frac{Q_{12}}{G_1(1-T_1)}, \quad C_{21}^{Td} = \frac{Q_{21}}{G_2(1-T_2)}.$$

Now, the network (11)-(12) can be transformed into the augmented network (24) as depicted in Figure 2, but by defining $\bar{C}_{12}^{Td} := \frac{1}{C_{12}^{Td}}$, $\bar{C}_{21}^{Td} := \frac{1}{C_{21}^{Td}}$ and the variables

$$\bar{u}_{12} := \frac{Q_{12}}{1-T_1} u_1 = \bar{Q}_{12} u_1, \quad \bar{u}_{21} := \frac{Q_{21}}{1-T_2} u_2 = \bar{Q}_{21} u_2,$$

we can write the network (11)-(12) also as

$$y_1 = \bar{C}_{12}^{Td} \bar{u}_{12} + \bar{C}_{12}^{d} \bar{y}_{12} + H_1 e_1,$$
$$y_2 = \bar{C}_{21}^{Td} \bar{u}_{21} + \bar{C}_{21}^{d} \bar{y}_{21} + H_2 e_2,$$

as visualized in Figure 4.

The unknown modules $C_{12}^{Td}$ and $C_{21}^{Td}$ can now be determined by posing network identification problems for the modules $\bar{C}_{12}^{Td}$ and $\bar{C}_{21}^{Td}$, shown in orange in Figure 4. The modules $\bar{C}_{12}^{d}$ and $\bar{C}_{21}^{d}$ are now depicted in white and assumed to be estimated consistently *a priori* as described in Section IV-A.

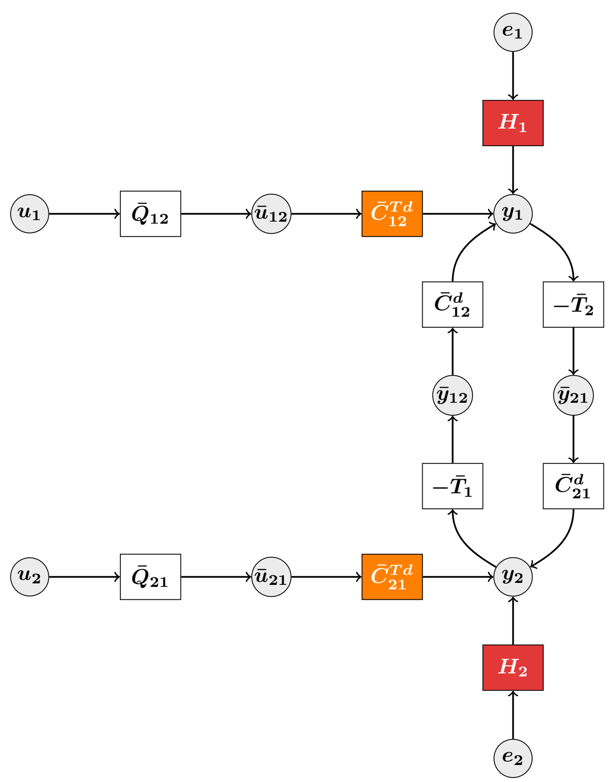


Fig. 4: The dynamic network represented by modules of the ideal distributed controller and reference model.

For a general network (2) we have

$$y_i = \bar{C}_{ij}^{Td} \bar{u}_{ij} + \sum_{k \in \mathcal{N}_i} \bar{C}_{ik}^{d} \bar{y}_{ik} + H_i e_i, \quad (i,j) \in \mathcal{E}. \tag{27}$$

Consider now the identification of $C_{ij}^{Td}$ as follows. For a given node $i \in \mathcal{V}$, let $C_{ij}^d$, $j \in \mathcal{N}_i$, and $H_i$ be given. We consider the following model for (27):

$$y_{ij}(\rho_{ij}^c) = \bar{C}_{ij}^{c}(\rho_{ij}^c) \bar{u}_{ij} + \sum_{k \in \mathcal{N}_i} \bar{C}_{ik}^{d} \bar{y}_{ik} + H_i e_i, \quad (i,j) \in \mathcal{E}.$$

Define the prediction error $\varepsilon_{ij}^c(t, \rho_{ij}^c) := y_i(t) - \hat{y}_{ij}(t, \rho_{ij}^c)$ with

$$\hat{y}_{ij}(t, \rho_{ij}^c) := H_i^{-1} \left( \bar{C}_{ij}^{c}(\rho_{ij}^c) \bar{u}_{ij} + \sum_{k \in \mathcal{N}_i} \bar{C}_{ik}^{d} \bar{y}_{ik} \right) + (1 - H_i^{-1}) y_i.$$

Then by minimizing the identification criterion $V_{ij}(\rho_{ij}^c) := \frac{1}{N} \sum_{t=1}^{N} \varepsilon_{ij}^c(t, \rho_{ij}^c)^2$ for each $j \in \mathcal{N}_i$, the parameter estimates $\hat{\rho}_{ij}^c$ are obtained. The asymptotic estimates are $\rho_{ij}^* = \arg\min_{\rho_{ij}^c} \bar{E}[\varepsilon_{ij}^c(t, \rho_{ij}^c)^2]$, $j \in \mathcal{N}_i$.

**Corollary IV.1** *Suppose the following conditions hold:*

- $\Phi_{\bar{u}_{ij}}(\omega)$ *is positive definite for all* $\omega \in [-\pi, \pi]$,
- *there exists* $\rho_{ij}^d$ *such that* $C_{ij}^c(\rho_{ij}^d) = C_{ij}^{Td}$.

*Then it holds that* $C_{ij}^c(\rho_{ij}^*) = C_{ij}^{Td}$.

Notice that one can equivalently minimize $\sum_{j \in \mathcal{N}_i} V_{ij}(\rho_{ij}^c)$. Moreover, no additional experiment is required for data acquisition.

## V. Numerical Example

Consider the two-node network described by (11)-(12), with transfer functions

$$G_1(q) = \frac{c_1}{q - a_1}, \quad G_{12}(q) = \frac{d_1}{q - a_1}, \quad H_1 = 1,$$
$$G_2(q) = \frac{c_2}{q - a_2}, \quad G_{21}(q) = \frac{d_2}{q - a_2}, \quad H_2 = 1,$$

where $a_1 = 0.5$, $a_2 = 0.2$, $c_1 = c_2 = 1$ and $d_1 = d_2 = 0.1$. The objective is to let the closed-loop interconnected system behave as two decoupled processes with first-order dynamics, according to

$$y_i^d(t) = T_i(q) r_i(t), \quad T_i(q) = \frac{1-\gamma_i}{q-\gamma_i}, \quad i = 1, 2, \tag{28}$$

with $\gamma_1 = \gamma_2 = 0.8$.

As described in Section III, the ideal distributed controller is described by (14)-(15), with the interconnections $s_1^c = o_2^c$, $s_2^c = o_1^c$, and

$$C_{11}^d(q) = \frac{1-\gamma_1}{c_1} \frac{q - a_1}{q - 1}, \quad C_{12}^d(q) = -\frac{d_1}{c_1},$$
$$K_{12}^d(q) = \frac{1-\gamma_1}{q-1},$$
$$C_{22}^d(q) = \frac{1-\gamma_2}{c_2} \frac{q - a_2}{q - 1}, \quad C_{21}^d(q) = -\frac{d_2}{c_2},$$
$$K_{21}^d(q) = \frac{1-\gamma_2}{q-1}.$$

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For the experiment, consider that $u_1$ and $u_2$ are Gaussian white-noise signals with unit variance and $e_1$ and $e_2$ are (unmeasured) Guassian white-noise signals with variance $\sigma_e^2 = 0.25$. As analyzed Section III, the noise will cause a bias in the controller parameter estimates when the distributed virtual reference feedback tuning (DVRFT) method is applied directly. For the distributed optimal controller identification (DOCI) method, described in Section IV-B, we expect consistent estimates and hence an improved closed-loop performance.

We first represent the network as shown in Figure 2, where

$$\bar{C}_{11}^d = \frac{c_1}{1-\gamma_1}\frac{q-1}{q-a_1}, \quad \bar{C}_{12}^d = -\frac{d_1}{c_1}\frac{q-1}{q-a_1}.$$

The modules are therefore parametrized as

$$\bar{C}_{11}(\theta_1) = \theta_{1a}\frac{1-q^{-1}}{1-\theta_{1b}q^{-1}}, \quad \bar{C}_{12}(\theta_1) = \theta_{1c}\frac{1-q^{-1}}{1-\theta_{1b}q^{-1}},$$
$$H_1(\theta_1) = 1,$$

so that there exists $\theta_1^d$ such that $\bar{C}_{11}^d = \bar{C}_{11}(\theta_1^d)$, $\bar{C}_{12}^d = \bar{C}_{12}(\theta_1^d)$ and $H_1 = H_1(\theta_1)$. By forming the predictor

$$\hat{y}_1(t|t-1;\theta_1) := \bar{C}_{11}(\rho_1)\bar{u}_1 + \bar{C}_{12}(\rho_1)\bar{y}_2$$

and minimizing $V_1(\theta_1)$ in (26) for $N = 100$ samples, we find the estimate $\hat{\theta}_1$. The estimate $\hat{\theta}_2$ for controller 2 is obtained by following an analogous procedure. Note that $V_1$ and $V_2$ are not quadratic functions in the parameters. The corresponding DVRFT cost functions are quadratic and the optimization problems therefore have explicit solutions.

The distributed controller resulting from DOCI leads to a closed-loop network with a step response shown in Figure 5 and a frequency response shown in Figure 6 in green of the transfer $r \to y$, for 20 experiments. For comparison, we synthesize a DVRFT controller using the same data via the method described in [11, Section IV]. The corresponding responses are shown in Figure 5 and 6 in yellow. Note that the controller classes are chosen such that the ideal controller belongs to the controller class for each method, which leads to a quadratic cost and non-quadratic cost function for DVRFT and DOCI, respectively. We observe that the distributed controller synthesized via DOCI leads to a closed-loop network with a response that is closer to the reference model compared to the controller synthesized via VRFT. As discussed in Section III, DVRFT leads to biased controller estimates when the noise terms $e_1$ and $e_2$ are non-zero. This bias is illustrated in Figure 7b, where the parameter estimates for controller 1 are plotted for 100 experiments. The parameter estimates for controller 1 with DOCI are plotted in Figure 7a. Finally, Figure 8 shows the distribution of the achieved performance for DOCI and DVRFT in the presence of noise. As described in the introduction, VRFT can yield consistent estimates when instrumental variables (IVs) are used. The construction of IVs for the example network was performed using an additional experiment [13], *mutatis mutandis*. The mean value of the performance of DVRFT with IVs is significantly lower compared to DVRFT, while the variance is significantly higher. We observe that the mean value as well as the variance of the performance are considerably lower for DOCI. Hence, although both DOCI and DVRFT with IVs yield consistent estimates, the increased variance due to IVs in DVRFT yields an overall poor performance compared to DOCI.

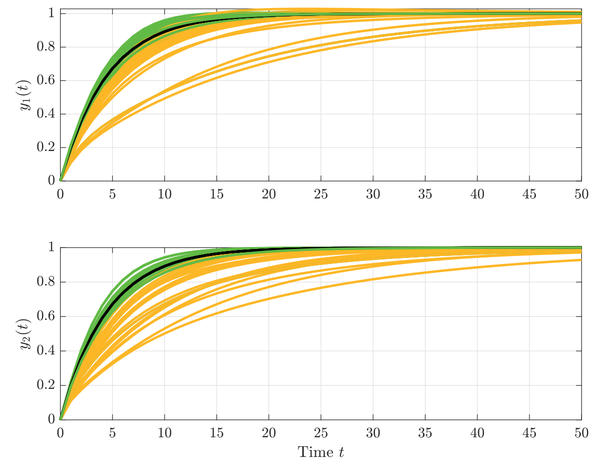



Fig. 5: Step response of the closed-loop network for 20 experiments with DOCI (green), DVRFT (yellow) and the desired closed-loop network (black).

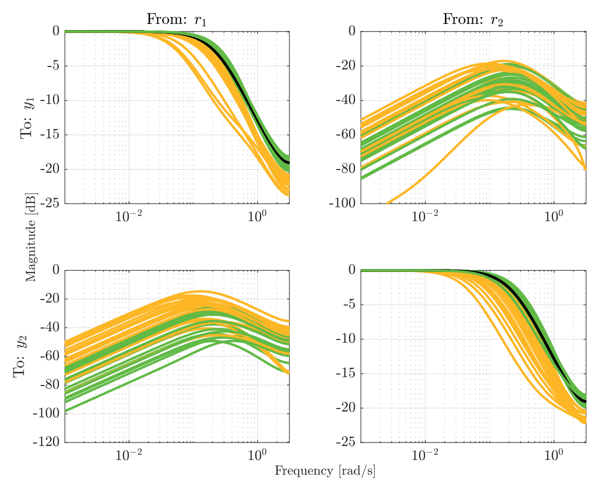
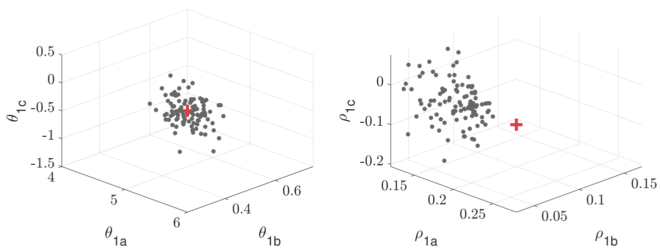



Fig. 6: Frequency response of the closed-loop network for 20 experiments with DOCI (green), DVRFT (yellow) and the desired closed-loop network (black). Notice that the desired network is decoupled, thus the corresponding transfers $r_i \to y_j^d$, $i \neq j$, are identical to zero.

## VI. Conclusions

We have developed a data-driven method for the construction of a distributed controller for an interconnected system subject to disturbances. Sufficient conditions for obtaining consistent estimates of controller modules in an augmented network have been given. The estimated distributed controller therefore solves the model-reference control problem asymptotically in the number of data. By a simple network




(a) DOCI (b) DVRFT

Fig. 7: Parameter estimates for 100 experiments (gray) and the true parameter (red) for controller 1.

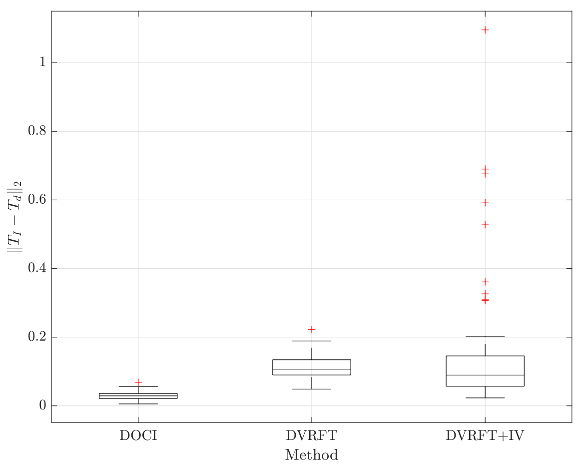



Fig. 8: Distribution of the achieved performance for DOCI, DVRFT and DVRFT with IVs, where $T_I$ and $T_d$ denote the transfers $r \to y$ and $r \to y_d$, respectively.

consisting of two interconnected systems, we have shown the effectiveness and the improvement over biased or high-variance alternative methods on the closed-loop performance.

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