Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 54-9 (2021) 66–71

Scalable distributed $\mathscr{H}_2$ controller synthesis for interconnected linear discrete-time systems

Tom R.V. Steentjes, Mircea Lazar, Paul M.J. Van den Hof

Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands (e-mails: t.r.v.steentjes@tue.nl, m.lazar@tue.nl, p.m.j.vandenhof@tue.nl)

Abstract: The current limitation in the synthesis of distributed $\mathscr{H}_2$ controllers for linear interconnected systems is scalability due to non-convex or unstructured synthesis conditions. In this paper we develop convex and structured conditions for the existence of a distributed $\mathscr{H}_2$ controller for discrete-time interconnected systems with an interconnection structure that corresponds to an arbitrary graph. Neutral interconnections and a storage function with a block-diagonal structure are utilized to attain coupling conditions that are of a considerably lower computational complexity compared to the corresponding centralized $\mathscr{H}_2$ controller synthesis problem. The effectiveness and scalability of the developed distributed $\mathscr{H}_2$ controller synthesis method is demonstrated for small- to large-scale oscillator networks on a cycle graph.

Keywords: Distributed control, H-2 controller synthesis, interconnected systems, discrete-time systems, dissipative systems

1. INTRODUCTION

Control of interconnected systems is relevant to a wide area of applications in smart grids, communication networks, irrigation networks and chemical plant networks, fueled by the digital industrial revolution, see e.g. (Lunze, 1992) and (Bullo, 2018). Distributed control is preferred for such systems due to its scalable implementation and it has been a major research topic in recent years for several control objectives, including $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance criteria.

For continuous-time systems, sufficient conditions for the existence of a controller that admits the same interconnection structure as the plant and that achieves unit $\mathscr{H}_\infty$ performance were developed by Langbort et al. (2004). The basis for these sufficient conditions is laid by dissipativity theory, introduced by Willems (1972), which is also the cornerstone for this work. Van Horssen and Weiland (2016) presented a discrete-time analogue of the work in (Langbort et al., 2004) with additional robust stability and robust $\mathscr{H}_\infty$ performance guarantees. For both the continuous- and discrete-time distributed $\mathscr{H}_\infty$ control problems, the conditions can be stated as linear matrix inequalities (LMIs) (Langbort et al., 2004), (Van Horssen and Weiland, 2016).

Eilbrecht et al. (2017) provided an approach to solve the discrete-time $\mathscr{H}_2$ output-feedback problem for interconnected systems, by minimizing a linear combination of the closed-loop system's $\mathscr{H}_2$ norm and a cost related to the sparsity of the controller matrices. However, this approach yields a non-convex problem in general. Vamsi and Elia (2016) solved the discrete-time $\mathscr{H}_2$ problem for a 'strictly causal' network, via the search for an unstructured controller and a subsequent transformation into a structured one. The structure of systems interconnected over one spatial dimension was exploited by Rice (2010) for the efficient design of $\mathscr{H}_2$ controllers interconnected in a string. The distributed $\mathscr{H}_2$ controller synthesis for continuous-time systems with arbitrary interconnection topology was recently considered by Chen et al. (2019). Unlike the $\mathscr{H}_\infty$ case, however, the feasibility problem for the distributed $\mathscr{H}_2$ controller existence in (Chen et al., 2019) is not convex, but amounts to solving a bilinear optimization problem.

The $\mathscr{H}_2$ norm has a particularly interesting interpretation in the field of data-driven modelling of interconnected systems, where stochastic assumptions on disturbance signals are key (Van den Hof et al., 2013). This is due to the fact that the $\mathscr{H}_2$ norm equals the asymptotic output variance for a white noise excitation (Scherer and Weiland, 2017). The trend for data-driven modelling of interconnected systems asks for accompanying distributed controller design methods that apply to discrete-time systems affected by stochastic disturbance signals. However, the current approaches to distributed $\mathscr{H}_2$ control, reviewed above, do not facilitate the controller synthesis for arbitrarily-structured large-scale systems, due to non-convex or unstructured synthesis conditions, or due to restrictions to systems that are spatially distributed in one dimension. Hence, it is of interest to develop scalable (convex) conditions for the synthesis of distributed $\mathscr{H}_2$ controllers for systems with a general interconnection structure.

⋆ This work has received funding from the European Research Council (ERC), Advanced Research Grant SYSDYNET, under the European Unions Horizon 2020 research and innovation programme (Grant Agreement No. 694504).

10.1016/j.ifacol.2021.06.178

In this paper, we therefore develop sufficient conditions for the existence of a distributed $\mathscr{H}_2$ controller for a discrete-time system with an arbitrary interconnection structure, by adopting the fundamental approach to distributed controller synthesis of Langbort et al. (2004). Analogous to distributed $\mathscr{H}_2$ controller synthesis for linear continuous-time systems (Chen et al., 2019), the conditions are principally not convex, which is induced by a number of scalar terms that are nonlinear w.r.t. the optimization variables, equal to the number of subsystems. However, we show that the resulting conditions are equivalent to alternative *convex* conditions stated as LMIs, with no reduction in generality or scalability.

Basic nomenclature

The integers are denoted by $\mathbb{Z}$. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$. Let $I_n \in \mathbb{R}^{n\times n}$, or simply I, denote the identity matrix. The operator $\mathrm{col}(\cdot)$ vertically concatenates its arguments. The block diagonal matrix $\mathrm{diag}(X_1, \ldots, X_m)$ has matrices X_i, $i \in \mathbb{N}_{[1:m]}$, in its block diagonal entries. For $S \subseteq \mathbb{Z}$, the block diagonal matrix $\mathrm{diag}_{i\in S} X_i$ has matrices X_i, $i \in S$, in its block diagonal entries. The image of a matrix $A \in \mathbb{R}^{m\times n}$ is $\mathrm{im}\, A := \{Ax \,|\, x \in \mathbb{R}^n\}$. For a real symmetric matrix X, $X \succ 0$ denotes that X is positive definite.

2. PRELIMINARIES

Let the structure of an interconnected system be given by a graph $G = (V, E)$, where V is the vertex set of cardinality L and $E \subseteq V \times V$ is the edge set. Each vertex $v_i \in V$, corresponds to a discrete-time system $\mathcal{P}_i$. An edge $(v_i, v_j) \in E$ exists if subsystems $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected. For ease of presentation, self-connections are excluded for all subsystems $\mathcal{P}_i$, $i \in \mathbb{Z}_{[1:L]}$.

Each subsystem $\mathcal{P}_i$ is assumed to admit a state-space representation

$$\begin{pmatrix} x_i(k+1) \\ o_i(k) \\ z_i(k) \end{pmatrix} = \begin{pmatrix} A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{T}d} & B_i^{\mathrm{T}u} \\ A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{S}d} & B_i^{\mathrm{S}u} \\ C_i^{z\mathrm{T}} & C_i^{z\mathrm{S}} & D_i^{zd} & D_i^{zu} \end{pmatrix} \begin{pmatrix} x_i(k) \\ s_i(k) \\ d_i(k) \end{pmatrix}, \quad (1)$$

where $x_i : \mathbb{Z} \to \mathbb{R}^{k_i}$ is the subsystem's state, $o_i : \mathbb{Z} \to \mathbb{R}^{n_i}$ and $s_i : \mathbb{Z} \to \mathbb{R}^{n_i}$ are the outgoing and incoming interconnection variables, and $z_i : \mathbb{Z} \to \mathbb{R}^{q_i}$ and $d_i : \mathbb{Z} \to \mathbb{R}^{f_i}$ are the performance output and disturbance input, respectively.

We write the interconnection signals s_i and o_i as $s_i = \mathrm{col}(s_{i1}, s_{i2}, \ldots, s_{iL})$ and $o_i = \mathrm{col}(o_{i1}, o_{i2}, \ldots, o_{iL})$ so that (s_{ij}, o_{ij}) denotes the interconnection channel between subsystem $\mathcal{P}_i$ and subsystem $\mathcal{P}_j$. For the ease of the interconnection definition, we assume, without loss of generality (Langbort et al., 2004), that o_{ij}, s_{ij}, o_{ji} and s_{ji} are all elements of $\mathbb{R}^{n_{ij}}$, $n_{ij} \geq 0$. The interconnection between system $\mathcal{P}_i$ and $\mathcal{P}_j$ is defined through the interconnection equation

$$\begin{pmatrix} o_{ij}(k) \\ s_{ij}(k) \end{pmatrix} = \begin{pmatrix} s_{ji}(k) \\ o_{ji}(k) \end{pmatrix}, \quad \forall k \in \mathbb{Z}. \quad (2)$$

Hence, $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected if and only if $n_{ij} > 0$, if and only if $(v_i, v_j) \in E$.

The interconnected system can be compactly represented by

$$\begin{pmatrix} x(k+1) \\ o(k) \\ z(k) \end{pmatrix} = \begin{pmatrix} A^{\mathrm{TT}} & A^{\mathrm{TS}} & B^{\mathrm{T}} \\ A^{\mathrm{ST}} & A^{\mathrm{SS}} & B^{\mathrm{S}} \\ C^{\mathrm{T}} & C^{\mathrm{S}} & D \end{pmatrix} \begin{pmatrix} x(k) \\ s(k) \\ d(k) \end{pmatrix},$$

with straightforward corresponding definitions for the signals and (block-diagonal) system matrices, and the interconnection $o = \Delta s$, with the matrix Δ defined by aggregating (2) for all corresponding index pairs. Elimination of the interconnection variables s and o yields a state-space representation

$$\mathcal{P}_{\mathcal{I}} : \quad \begin{pmatrix} x(k+1) \\ z(k) \end{pmatrix} = \begin{pmatrix} A_{\mathcal{I}} & B_{\mathcal{I}} \\ C_{\mathcal{I}} & D_{\mathcal{I}} \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix} \quad (3)$$

where

$$\begin{pmatrix} A_{\mathcal{I}} & B_{\mathcal{I}} \\ C_{\mathcal{I}} & D_{\mathcal{I}} \end{pmatrix} := \begin{pmatrix} A^{\mathrm{TT}} & B^{\mathrm{T}} \\ C^{\mathrm{T}} & D \end{pmatrix} + \begin{pmatrix} A^{\mathrm{TS}} \\ C^{\mathrm{S}} \end{pmatrix} (\Delta - A^{\mathrm{SS}})^{-1} \begin{pmatrix} A^{\mathrm{ST}} & B^{\mathrm{S}} \end{pmatrix}.$$

Consider the interconnection variable subspaces (Langbort et al., 2004)

$\mathcal{S}_{\mathcal{I}} := \{(o, s) \in \mathbb{R}^{2n} \,|\, o = \Delta s\}$ and

$\mathcal{S}_{\mathcal{B}} := \{(o, s) \in \mathbb{R}^{2n} \mid \mathrm{col}(o_i, s_i) \in \mathrm{im}\,\mathrm{col}(A_i^{\mathrm{SS}}, I), i \in \mathbb{Z}_{[1:L]}\}$.

Definition 2.1. An interconnected system described by (1) and (2) is said to be well-posed if $\mathcal{S}_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{B}} = \{0\}$.

Definition 2.2. A well-posed interconnected system is said to be asymptotically stable (AS) if the roots of $\det(zI - A_{\mathcal{I}})$ are inside the unit circle on the complex plane.

Definition 2.3. The $\mathscr{H}_2$ norm of a well-posed and AS interconnected system with a transfer function $T(z) := C_{\mathcal{I}}(zI - A_{\mathcal{I}})^{-1}B_{\mathcal{I}} + D_{\mathcal{I}}$ is defined by

$$\|\mathcal{P}_{\mathcal{I}}\|_{\mathscr{H}_2} := \left(\frac{1}{2\pi} \operatorname{trace} \int_{-\pi}^{\pi} T^*(e^{i\omega}) T(e^{i\omega}) \,\mathrm{d}\omega \right)^{\frac{1}{2}}.$$

2.1 Interconnected-system analysis

The following result provides sufficient conditions for well-posedness, stability and bounding the $\mathscr{H}_2$ norm of the interconnected system, and provides a discrete-time counterpart of the continuous-time result (Chen et al., 2019, Theorem 1). Define the matrix

$$T_i := \left(\begin{array}{ccc} I & 0 & 0 \\ A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{T}d} \\ \hline A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{S}d} \\ 0 & I & 0 \\ \hline C_i^{z\mathrm{T}} & C_i^{z\mathrm{S}} & D_i^{zd} \\ 0 & 0 & I \end{array} \right). \quad (4)$$

Proposition 2.4. The interconnected system $\mathcal{P}_{\mathcal{I}}$ is well-posed, AS and $\|\mathcal{P}_{\mathcal{I}}\|_{\mathscr{H}_2} < \gamma$, if $B_i^{\mathrm{S}d} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$ and there exist positive-definite $X_i \in \mathbb{R}^{k_i\times k_i}$, $\rho_i > 0$, symmetric $X_{ij}^{11} \in \mathbb{R}^{n_{ij}\times n_{ij}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, and $X_{ij}^{12} \in \mathbb{R}^{n_{ij}\times n_{ij}}$, $(i,j) \in \mathbb{Z}^2_{[1:L]}$, $i > j$, with

$$T_i^\top \left(\begin{array}{cc|cc|cc} -X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho_i I \end{array} \right) T_i \prec 0, \quad (5)$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) < \gamma^2, \qquad (6)$$

where

$$Z_i^{11} := -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} X_{ij}^{11},\ Z_i^{22} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} X_{ji}^{11},$$

$$Z_i^{12} := \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} X_{ij}^{12},\ \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})^\top\right). \qquad \square$$

A proof for the aforementioned proposition can be found in (Steentjes et al., 2021). We illustrate the analysis conditions by a simple example.

Example 2.5. Consider two identical scalar subsystems described by

$$x_i(k+1) = \frac{1}{2}x_i(k) + \frac{1}{10}s_i(k) + d_i(k), \quad i = 1,2,\ k\in\mathbb{Z},$$

and $z_i(k) = o_i(k) = x_i(k)$, with interconnection constraints $s_1(k) = o_2(k)$, $s_2(k) = o_1(k)$. It is easily verified that LMI (5) holds for $i = 1,2$, with $X_i = \frac{7}{4}$, $X_{12}^{11} = X_{21}^{11} = -\frac{1}{5}$, $X_{21}^{12} = 0$ and $\rho_i = 20$. By Proposition 2.4, the interconnected system is well-posed, asymptotically stable and the expression $\|\mathcal{P}_\mathcal{I}\|_{\mathscr{H}_2} < \gamma$ holds for all $\gamma > \sqrt{X_1 + X_2} = \sqrt{\frac{7}{2}} \approx 1.87$. The actual $\mathscr{H}_2$ norm of the system is $\|\mathcal{P}_\mathcal{I}\|_{\mathscr{H}_2} = 1.68$. $\square$

3. DISTRIBUTED $\mathscr{H}_2$ CONTROLLER SYNTHESIS

Consider the case where each subsystem $\mathcal{P}_i$ has a control input u_i and a measured output y_i, such that

$$\begin{pmatrix} x_i(k+1)\\ o_i(k)\\ z_i(k)\\ y_i(k)\end{pmatrix} = \begin{pmatrix} A_i^{\mathrm{TT}} & A_i^{\mathrm{TS}} & B_i^{\mathrm{T}d} & B_i^{\mathrm{T}u}\\ A_i^{\mathrm{ST}} & A_i^{\mathrm{SS}} & B_i^{\mathrm{S}d} & B_i^{\mathrm{S}u}\\ C_i^{z\mathrm{T}} & C_i^{z\mathrm{S}} & D_i^{zd} & D_i^{zu}\\ C_i^{y\mathrm{T}} & C_i^{y\mathrm{S}} & D_i^{yd} & D_i^{yu}\end{pmatrix} \begin{pmatrix} x_i(k)\\ s_i(k)\\ d_i(k)\\ u_i(k)\end{pmatrix}, \qquad (7)$$

where we assume that $D_i^{yu} = 0$, without loss of generality (Langbort et al., 2004).

The to-be-synthesized distributed controller is also an interconnected system, with subsystems $\mathcal{C}_i$, $i \in \mathbb{Z}_{[1:L]}$, described by

$$\begin{pmatrix} \xi_i(k+1)\\ o_i^{\mathcal{C}}(k)\\ u_i(k)\end{pmatrix} = \begin{pmatrix} (A_i^{\mathrm{TT}})_\mathcal{C} & (A_i^{\mathrm{TS}})_\mathcal{C} & (B_i^{\mathrm{T}})_\mathcal{C}\\ (A_i^{\mathrm{ST}})_\mathcal{C} & (A_i^{\mathrm{SS}})_\mathcal{C} & (B_i^{\mathrm{S}})_\mathcal{C}\\ (C_i^{\mathrm{T}})_\mathcal{C} & (C_i^{\mathrm{S}})_\mathcal{C} & (D_i)_\mathcal{C}\end{pmatrix} \begin{pmatrix} \xi_i(k)\\ s_i^{\mathcal{C}}(k)\\ y_i(k)\end{pmatrix}, \qquad (8)$$

where $\xi_i : \mathbb{Z} \to \mathbb{R}^{k_i}$ is the controller's state, and $o_i^{\mathcal{C}} : \mathbb{Z} \to \mathbb{R}^{n_i^{\mathcal{C}}}$, $s_i^{\mathcal{C}} : \mathbb{Z} \to \mathbb{R}^{n_i^{\mathcal{C}}}$ are the controller's interconnection (communication) variables. Controller $\mathcal{C}_i$ and $\mathcal{C}_j$ are interconnected only if $\mathcal{P}_i$ and $\mathcal{P}_j$ are interconnected and the interconnection equation is

$$\begin{pmatrix} o_{ij}^{\mathcal{C}}(k)\\ s_{ij}^{\mathcal{C}}(k)\end{pmatrix} = \begin{pmatrix} s_{ji}^{\mathcal{C}}(k)\\ o_{ji}^{\mathcal{C}}(k)\end{pmatrix}, \quad \forall k\in\mathbb{Z}. \qquad (9)$$

The local closed-loop (controlled) system, $\mathcal{K}_i$ say, can then be represented by

$$\begin{pmatrix} x_i^{\mathcal{K}}(k+1)\\ o_i^{\mathcal{K}}(k)\\ z_i(k)\end{pmatrix} = \underbrace{\begin{pmatrix} (A_i^{\mathrm{TT}})_\mathcal{K} & (A_i^{\mathrm{TS}})_\mathcal{K} & (B_i^{\mathrm{T}})_\mathcal{K}\\ (A_i^{\mathrm{ST}})_\mathcal{K} & (A_i^{\mathrm{SS}})_\mathcal{K} & (B_i^{\mathrm{S}})_\mathcal{K}\\ (C_i^{\mathrm{T}})_\mathcal{K} & (C_i^{\mathrm{S}})_\mathcal{K} & (D_i)_\mathcal{K}\end{pmatrix}}_{=:\Gamma_i} \begin{pmatrix} x_i^{\mathcal{K}}(k)\\ s_i^{\mathcal{K}}(k)\\ d_i(k)\end{pmatrix}, \qquad (10)$$

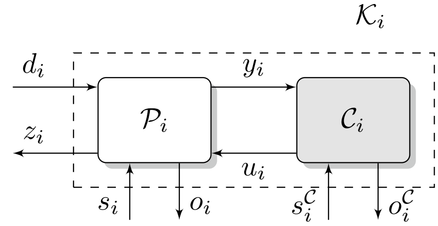

Fig. 1. Interconnection visualization of a locally controlled system $\mathcal{K}_i$, $i \in \mathbb{Z}_{[1:L]}$.

where $x_i^{\mathcal{K}} := \operatorname{col}(x_i, \xi_i)$, $o_i^{\mathcal{K}} := \operatorname{col}(o_i, o_i^{\mathcal{C}})$ and $s_i^{\mathcal{K}} := \operatorname{col}(s_i, s_i^{\mathcal{C}})$. Such a representation is obtained through elimination of the control variables y_i, u_i, as depicted in Figure 1. The state-space matrices of a closed-loop subsystem are affine with respect to the state-space matrices of the local controller:

$$\Gamma_i = U_i^\top \Theta_i V_i + W_i, \qquad (11)$$

with

$$\Theta_i := \begin{pmatrix} (A_i^{\mathrm{TT}})_\mathcal{C} & (A_i^{\mathrm{TS}})_\mathcal{C} & (B_i^{\mathrm{T}})_\mathcal{C}\\ (A_i^{\mathrm{ST}})_\mathcal{C} & (A_i^{\mathrm{SS}})_\mathcal{C} & (B_i^{\mathrm{S}})_\mathcal{C}\\ (C_i^{\mathrm{T}})_\mathcal{C} & (C_i^{\mathrm{S}})_\mathcal{C} & (D_i)_\mathcal{C}\end{pmatrix}, V_i := \begin{pmatrix} 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0\\ C_i^{y\mathrm{T}} & 0 & C_i^{y\mathrm{S}} & 0 & D_i^{yd}\end{pmatrix},$$

$$U_i^\top := \begin{pmatrix} 0 & 0 & B_i^{\mathrm{T}u}\\ I & 0 & 0\\ 0 & 0 & B_i^{\mathrm{S}u}\\ 0 & I & 0\\ 0 & 0 & D_i^{zu}\end{pmatrix},\ W_i := \begin{pmatrix} A_i^{\mathrm{TT}} & 0 & A_i^{\mathrm{TS}} & 0 & B_i^{\mathrm{T}d}\\ 0 & 0 & 0 & 0 & 0\\ A_i^{\mathrm{ST}} & 0 & A_i^{\mathrm{SS}} & 0 & B_i^{\mathrm{S}d}\\ 0 & 0 & 0 & 0 & 0\\ C_i^{z\mathrm{T}} & 0 & C_i^{z\mathrm{S}} & 0 & D_i^{zd}\end{pmatrix}.$$

The feasibility test provided by Proposition 2.4 directly induces a feasibility test for well-posedness, stability and $\mathscr{H}_2$ performance for the closed-loop system, which consists of subsystems (10), as stated in the following corollary. Define the matrix

$$T_i^{\mathcal{K}} := \left(\begin{array}{ccc} I & 0 & 0\\ (A_i^{\mathrm{TT}})_\mathcal{K} & (A_i^{\mathrm{TS}})_\mathcal{K} & (B_i^{\mathrm{T}})_\mathcal{K}\\ \hline (A_i^{\mathrm{ST}})_\mathcal{K} & (A_i^{\mathrm{SS}})_\mathcal{K} & (B_i^{\mathrm{S}})_\mathcal{K}\\ 0 & I & 0\\ \hline (C_i^{\mathrm{T}})_\mathcal{K} & (C_i^{\mathrm{S}})_\mathcal{K} & (D_i)_\mathcal{K}\\ 0 & 0 & I\end{array}\right).$$

Corollary 3.1. The interconnected system $\mathcal{K}_\mathcal{I}$ of (10) is well-posed, AS and $\|\mathcal{K}_\mathcal{I}\|_{\mathscr{H}_2} < \gamma$, if $(B_i^{\mathrm{S}})_\mathcal{K} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$ and there exist positive-definite $X_i^{\mathcal{K}} \in \mathbb{R}^{2k_i\times 2k_i}$, $\rho_i > 0$, symmetric $(X_{ij}^{11})_\mathcal{K} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_\mathcal{K} \in \mathbb{R}^{(n_{ij}+n_{ij}^{\mathcal{C}})\times(n_{ij}+n_{ij}^{\mathcal{C}})}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, with

$$(T_i^{\mathcal{K}})^\top \left(\begin{array}{cc|cc|cc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0\\ 0 & X_i^{\mathcal{K}} & 0 & 0 & 0 & 0\\ \hline 0 & 0 & (Z_i^{11})_\mathcal{K} & (Z_i^{12})_\mathcal{K} & 0 & 0\\ 0 & 0 & (Z_i^{12})_\mathcal{K}^\top & (Z_i^{22})_\mathcal{K} & 0 & 0\\ \hline 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 & -\rho_i I\end{array}\right) T_i^{\mathcal{K}} \prec 0, \qquad (12)$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}})_\mathcal{K}^\top X_i^{\mathcal{K}} (B_i^{\mathrm{T}})_\mathcal{K} + (D_i)_\mathcal{K}^\top (D_i)_\mathcal{K}\right) < \gamma^2, \qquad (13)$$

with the closed-loop scales $(Z_i^{11})_\mathcal{K} := \begin{pmatrix} (Z_i^{11})_\mathcal{P} & (Z_i^{11})_{\mathcal{PC}} \\ (Z_i^{11})_{\mathcal{PC}}^\top & (Z_i^{11})_\mathcal{C} \end{pmatrix}$, $(Z_i^{12})_\mathcal{K} := \begin{pmatrix} (Z_i^{12})_\mathcal{P} & (Z_i^{12})_{\mathcal{PC}} \\ (Z_i^{12})_{\mathcal{CP}} & (Z_i^{12})_\mathcal{C} \end{pmatrix}$, $(Z_i^{22})_\mathcal{K} := \begin{pmatrix} (Z_i^{22})_\mathcal{P} & (Z_i^{22})_{\mathcal{PC}} \\ (Z_i^{22})_{\mathcal{PC}}^\top & (Z_i^{22})_\mathcal{C} \end{pmatrix}$ and the submatrices defined in Appendix A. □

Recall the definition of T_i in (4) and define

$$S_i = \begin{pmatrix} (A_i^{\mathrm{TT}})^\top & (A_i^{\mathrm{ST}})^\top & (C_i^{z\mathrm{T}})^\top \\ -I & 0 & 0 \\ \hline 0 & -I & 0 \\ (A_i^{\mathrm{TS}})^\top & (A_i^{\mathrm{SS}})^\top & (B_i^{\mathrm{S}d})^\top \\ \hline 0 & 0 & -I \\ (B_i^{\mathrm{T}d})^\top & (B_i^{\mathrm{S}d})^\top & (D_i^{zd})^\top \end{pmatrix}.$$

3.1 Convex distributed $\mathscr{H}_2$ controller existence conditions

We are now ready to state the main result, which provides necessary and sufficient conditions for the existence of a distributed controller that satisfies the conditions in Corollary 3.1, in the form of LMIs.

Proposition 3.2. Let $B_i^{\mathrm{S}d} = 0$, $D_i^{yd} = 0$ for all $i \in \mathbb{Z}_{[1:L]}$. The following statements are equivalent:

- There exist controllers $\mathcal{C}_i$, with $n_{ij}^\mathcal{C} = 3n_{ij}$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$ so that the controlled interconnected system described by (2), (9) and (10) admits $\rho_i > 0$, matrices $X_i^\mathcal{K} \succ 0$, $i \in \mathbb{Z}_{[1:L]}$, symmetric $(X_{ij}^{11})_\mathcal{K}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_\mathcal{K}$, $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, that satisfy inequalities (12) and (13).
- There exist X_i, Y_i, symmetric $(X_{ij}^{11})_\mathcal{P}$, $(Y_{ij}^{11})_\mathcal{P}$, $\alpha_i, \beta_i > 0$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$, and $(X_{ij}^{12})_\mathcal{P}$, $(Y_{ij}^{12})_\mathcal{P}$ for all $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, that satisfy

$$\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} \succ 0, \tag{14}$$

$$\sum_{i=1}^{L} \operatorname{trace}\left((B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd}\right) < \gamma^2, \tag{15}$$

$$\Psi_i^\top T_i^\top \left(\begin{array}{cc|cc|cc} -X_i & 0 & 0 & 0 & 0 & 0 \\ 0 & X_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_\mathcal{P} & (Z_i^{12})_\mathcal{P} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_\mathcal{P}^\top & (Z_i^{22})_\mathcal{P} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_i I \end{array}\right) T_i \Psi_i \prec 0, \tag{16}$$

$$\Phi_i^\top S_i^\top \left(\begin{array}{cc|cc|cc} -Y_i & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_i & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (W_i^{11})_\mathcal{P} & (W_i^{12})_\mathcal{P} & 0 & 0 \\ 0 & 0 & (W_i^{12})_\mathcal{P}^\top & (W_i^{22})_\mathcal{P} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_i I \end{array}\right) S_i \Phi_i \succ 0, \tag{17}$$

where the columns of Ψ_i and Φ_i form a basis of $\ker(C_i^{yT}\ C_i^{yS}\ D_i^{yd})$ and $\ker((B_i^{\mathrm{T}u})^\top\ (B_i^{\mathrm{S}u})^\top\ (D_i^{zu})^\top)$, respectively, and

$(W_i^{11})_\mathcal{P} := -\operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} (Y_{ij}^{11})_\mathcal{P}$, $(W_i^{22})_\mathcal{P} := \operatorname*{diag}_{j \in \mathbb{Z}_{[1:L]}} (Y_{ji}^{11})_\mathcal{P}$,

$$(W_i^{12})_\mathcal{P} := \operatorname{diag}\left(-\operatorname*{diag}_{j \in \mathbb{Z}_{[1:i]}} (Y_{ij}^{12})_\mathcal{P}, \operatorname*{diag}_{j \in \mathbb{Z}_{[i+1,L]}} (Y_{ji}^{12})_\mathcal{P}^\top\right).$$

Proof. We first show that the existence of positive scalars α_i and β_i such that (16) and (17) hold is equivalent with the existence of a positive scalar ρ_i such that

$$\Psi_i^\top T_i^\top \Lambda_i(\rho_i) T_i \Psi_i \prec 0 \text{ and } \Phi_i^\top S_i^\top \Pi_i(\rho_i^{-1}) S_i \Phi_i \succ 0, \tag{18}$$

with

$$\Lambda_i : \xi \mapsto \operatorname{diag}(-X_i, X_i, (Z_i)_\mathcal{P}, I, -\xi I) \text{ and}$$
$$\Pi_i : \xi \mapsto \operatorname{diag}(-Y_i, Y_i, (W_i)_\mathcal{P}, I, -\xi I).$$

For sufficiency, let α_i and β_i satisfy (16) and (17). We distinguish two cases. First, if $\alpha_i\beta_i \geq 1$, then

$$\underbrace{\Phi_i^\top S_i^\top \Pi_i(\beta_i) S_i \Phi_i}_{\succ 0} + \underbrace{\Phi_i^\top S_i^\top \operatorname{diag}(0,0,0,0,(\beta_i - \alpha_i^{-1})I) S_i \Phi_i}_{\succeq 0}$$
$$= \Phi_i^\top S_i^\top \Pi_i(\alpha_i^{-1}) S_i \Phi_i \succ 0.$$

Hence, (18) holds for $\rho_i = \alpha_i$. In the other case $\alpha_i\beta_i < 1$, thus it follows that

$$\underbrace{\Psi_i^\top T_i^\top \Lambda_i(\alpha_i) T_i \Psi_i}_{\prec 0} + \underbrace{\Psi_i^\top T_i^\top \operatorname{diag}(0,0,0,0,(\alpha_i - \beta_i^{-1}) T_i \Psi_i}_{\preceq 0}$$
$$= \Psi_i^\top T_i^\top \Lambda_i(\beta_i^{-1}) T_i \Psi_i \prec 0.$$

Hence, (18) holds for $\rho_i = \beta_i^{-1}$. Necessity follows directly by taking $\alpha_i = \rho_i$ and $\beta_i = \rho_i^{-1}$.

For a proof that the existence of X_i, Y_i, $(Z_i)_\mathcal{P}$, $(W_i)_\mathcal{P}$ and ρ_i that satisfy (18) and (14) is equivalent with the existence of $X_i^\mathcal{K}$, $(Z_i)_\mathcal{K}$ and ρ_i that satisfy (12), we refer the reader to (Langbort et al., 2004) due to space limitations.

Finally, we will show that (15) is equivalent with (13). We note that for necessity X_i can be taken as the upper-left block of $X_i^\mathcal{K}$, while for sufficiency, $X_i^\mathcal{K}$ can be taken such that its upper-left block equals X_i (Langbort et al., 2004). Thus, by (11), we have that

$$(B_i^{\mathrm{T}d})^\top X_i B_i^{\mathrm{T}d} + (D_i^{zd})^\top D_i^{zd} = (B_i^{\mathrm{T}})_\mathcal{K}^\top X_i^\mathcal{K} (B_i^{\mathrm{T}})_\mathcal{K} + (D_i)_\mathcal{K}^\top (D_i)_\mathcal{K}$$

for all $i \in \mathbb{Z}_{[1:L]}$, since $D_i^{yd} = 0$. It therefore follows that (15) ⇔ (13), which concludes the proof. ■

Remark 3.3. The equivalence between the convex conditions (16), (17) and non-convex conditions (18) can be transferred to the continuous-time case (Chen et al., 2019, Theorem 2) *mutatis mutandis*. The continuous-time distributed $\mathscr{H}_2$ controller existence problem can then be solved via equivalent LMIs, instead of the equivalent bilinear optimization problem with L additional LMIs in (Chen et al., 2019), with L the cardinality of the vertex set V.

3.2 Controller construction

Algorithm 3.4. For each pair $(i,j) \in \mathbb{Z}_{[1:L]}^2$, let X_i, Y_i, ρ_i, $(X_{ij}^{11})_\mathcal{P}$, $(Y_{ij}^{11})_\mathcal{P}$, and for each pair $(i,j) \in \mathbb{Z}_{[1:L]}^2$, $i > j$, let $(X_{ij}^{12})_\mathcal{P}$, $(Y_{ij}^{12})_\mathcal{P}$, be computed to satisfy (14)-(17).

For each $i \in \mathbb{Z}_{[1:L]}$, the synthesis of controller $\mathcal{C}_i$ proceeds as follows:

(1) Let M_i and N_i be non-singular and such that $M_i N_i^\top = I - X_i Y_i$. Compute $X_i^\mathcal{K}$ as the unique solution to the linear equation

$$X_i^{\mathcal{K}} \begin{pmatrix} Y_i & I \\ N_i^\top & 0 \end{pmatrix} = \begin{pmatrix} I & X_i \\ 0 & M_i^\top \end{pmatrix}.$$

(2) Define

$$X_{ij}^{\mathcal{P}} := \begin{pmatrix} (X_{ij}^{11})_{\mathcal{P}} & (X_{ij}^{12})_{\mathcal{P}} \\ (X_{ij}^{12})_{\mathcal{P}}^\top & -(X_{ji}^{11})_{\mathcal{P}} \end{pmatrix}, Y_{ij}^{\mathcal{P}} := \begin{pmatrix} (Y_{ij}^{11})_{\mathcal{P}} & (Y_{ij}^{12})_{\mathcal{P}} \\ (Y_{ij}^{12})_{\mathcal{P}}^\top & -(Y_{ji}^{11})_{\mathcal{P}} \end{pmatrix}.$$

and compute an eigendecomposition $X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1} = V_{ij}\Lambda_{ij}V_{ij}^\top$, with Λ_{ij} a diagonal matrix with the eigenvalues on its diagonal in a descending order. Scale the eigenvectors as $\bar{V}_{ij} = V_{ij}|\Lambda_{ij}|^{\frac{1}{2}}$ such that

$$X_{ij}^{\mathcal{P}} - (Y_{ij}^{\mathcal{P}})^{-1} = (\bar{V}_{ij}^+ \ \bar{V}_{ij}^-)\,\mathrm{diag}(I, -I)(\bar{V}_{ij}^+ \ \bar{V}_{ij}^-)^\top,$$

with $\bar{V}_{ij} =: (\bar{V}_{ij}^+ \ \bar{V}_{ij}^-)$. Let $M_{ij}^{22} := \mathrm{diag}(I_{3n_{ij}}, -I_{3n_{ij}})$ and $M_{ij}^{12} := \frac{1}{\sqrt{3}}(\bar{V}_{ij}^+ \ \bar{V}_{ij}^+ \ \bar{V}_{ij}^+ \ \bar{V}_{ij}^- \ \bar{V}_{ij}^- \ \bar{V}_{ij}^-)$ and define

$$M_{ij}^{12} =: \begin{pmatrix} (X_{ij}^{11})_{\mathcal{PC}} & (X_{ij}^{12})_{\mathcal{PC}} \\ (X_{ij}^{12})_{\mathcal{CP}}^\top & -(X_{ji}^{11})_{\mathcal{PC}} \end{pmatrix},$$

$$M_{ij}^{22} =: \begin{pmatrix} (X_{ij}^{11})_{\mathcal{C}} & (X_{ij}^{12})_{\mathcal{C}} \\ (X_{ij}^{12})_{\mathcal{C}}^\top & -(X_{ji}^{11})_{\mathcal{C}} \end{pmatrix}.$$

(3) Construct the closed-loop scales defined in Appendix A and let

$$P_i := \left(\begin{array}{ccc|ccc} -X_i^{\mathcal{K}} & 0 & 0 & 0 & 0 & 0 \\ 0 & (Z_i^{22})_{\mathcal{K}} & 0 & 0 & (Z_i^{12})_{\mathcal{K}}^\top & 0 \\ 0 & 0 & -\rho_i I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & X_i^{\mathcal{K}} & 0 & 0 \\ 0 & (Z_i^{12})_{\mathcal{K}} & 0 & 0 & (Z_i^{11})_{\mathcal{K}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right).$$

Solve the following inequality for Θ_i:

$$\begin{pmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{pmatrix}^\top P_i \begin{pmatrix} I \\ U_i^\top \Theta_i V_i + W_i \end{pmatrix} \prec 0. \quad (19)$$

The quadratic matrix inequality (19) can be solved by computing an eigendecomposition and a linear equation, see e.g. (Scherer, 2001) for details. □

4. NUMERICAL EXAMPLES

To illustrate the distributed $\mathscr{H}_2$ controller synthesis method, we consider a linear coupled-oscillator network consisting of L oscillators. For each node $i \in \mathbb{Z}_{[1:L]}$, the dynamics are described by

$$m_i\ddot{\theta}_i + b_i\dot{\theta}_i = u_i - \sum_{j\in\mathcal{N}_i} k_{ij}(\theta_i - \theta_j) + d_i, \quad (20)$$

with inertia m_i, damping b_i and coupling coefficient $k_{ij} = k_{ji}$. The mechanical analogue of a linear coupled-oscillator network is a network of masses that are interconnected through linear springs and have linear damping. A typical system that is modeled as a linear oscillator network is a linearized power network, consisting of generators ($m_i \neq 0$) and loads ($m_i = 0$) (Bergen and Hill, 1981; Dörfler et al., 2013). The local measurement is assumed to be $y_i := \theta_i$ and the performance output is set equal to the state $z_i := x_i := \mathrm{col}(\theta_i, \dot{\theta}_i)$. We use a zero-order hold discretization with sampling time $T = 0.1$ seconds for each subsystem and an approximation $e^M \approx I + M$, so that each subsystem $\mathcal{P}_i$ has an input/state/output representation (1) with matrices

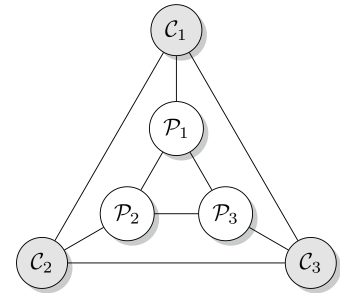

Fig. 2. Structure of the oscillator network represented by a triangle graph ($L = 3$). The synthesized distributed $\mathscr{H}_2$ controller modules are depicted in gray.

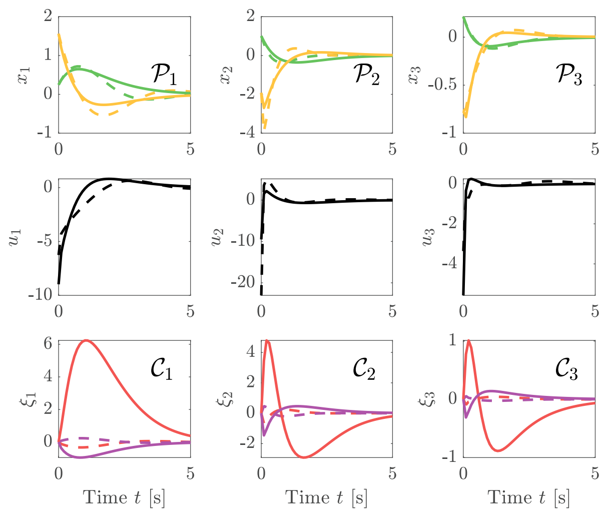

Fig. 3. Subsystem states $[x_i]_1$ (green) and $[x_i]_2$ (yellow), controller states $[\xi_i]_1$ (red) and $[\xi_i]_2$ (violet) and control inputs u_i (black), $i \in \{1, 2, 3\}$, for the distributed (solid) and central (dashed) controller.

$$A_i^{\mathrm{TT}} = \begin{pmatrix} 1 & T \\ -\sum_{j\in\mathcal{N}_i} \frac{k_{ij}}{m_i}T & 1 - \frac{b_i}{m_i}T \end{pmatrix}, A_i^{\mathrm{TS}} = \underset{j\in\mathcal{N}_i}{\mathrm{row}} \begin{pmatrix} 0 \\ \frac{k_{ij}}{m_i}T \end{pmatrix},$$

$$A_i^{\mathrm{ST}} = C_i^{y\mathrm{T}} = \underset{j\in\mathcal{N}_i}{\mathrm{col}} \begin{pmatrix} 1 & 0 \end{pmatrix}, \ A_i^{\mathrm{SS}} = 0_{n_i\times n_i},$$

$$B_i^{\mathrm{S}d} = B_i^{\mathrm{S}u} = 0_{n_i\times 1}, \ B_i^{\mathrm{T}d} = B_i^{\mathrm{T}u} = \mathrm{col}(0, \frac{T}{m_i}), \ C_i^{z\mathrm{T}} = I_2,$$

$$C_i^{z\mathrm{S}} = 0_{2\times n_i}, D_i^{zd} = D_i^{zu} = 0_{2\times 1}, D_i^{yd} = D_i^{yu} = 0.$$

Let us consider a network with a triangular structure, as depicted in Figure 2. The systems' inertia, damping and coupling coefficients are $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $b_1 = 2$, $b_2 = 1$, $b_3 = 4$ and $k_{12} = k_{23} = k_{31} = 1$. The open-loop system is not AS. We aim for disturbance attenuation via the synthesis of a distributed controller that achieves unit $\mathscr{H}_2$ performance for the controlled network. We therefore verify the feasibility of the LMIs in Proposition 3.2 for $\gamma = 1$. We find that the LMIs are feasible, hence there exists a distributed controller that achieves $\|\mathcal{K}_{\mathcal{I}}\|_{\mathscr{H}_2} < 1$. The distributed controller is constructed according to Algorithm 3.4 and results in a closed-loop $\mathscr{H}_2$ norm of 0.22. Simulation of the controlled network with zero disturbance, with the subsystems' initial

Table 1. Computation times for solving the LMIs in Proposition 3.2 for the distributed $\mathscr{H}_2$ controller and the corresponding LMIs for the central $\mathscr{H}_2$ controller for a network of L subsystems. †: No solution after 4 hours.

L	Central controller	Distributed controller
3	0.44s	0.24s
10	0.78s	0.29s
50	831.57s	0.34s
100	†	0.42s
1,000	†	1.35s
10,000	†	5.77s

conditions drawn from a normal distribution $\mathcal{N}(0,1)$ and the controllers' initial conditions set identical to zero, results in the trajectories depicted in Figure 3. We observe that the subsystems' and controllers' states asymptotically converge to zero, illustrating asymptotic stability of the closed-loop system. For validation, we also compute a central controller via the feasibility problem in (Scherer and Weiland, 2017) for an $\mathscr{H}_2$ upper-bound equal to 0.22. The resulting controller achieves an $\mathscr{H}_2$ norm of 0.18 and the trajectories are shown in Figure 3 (the central controller state $\xi \in \mathbb{R}^6$ is denoted $\xi = \mathrm{col}(\xi_1, \xi_2, \xi_3)$).

4.1 Computation times

To demonstrate the scalability of the developed synthesis method, we consider the controller construction for the oscillator network on cycle graphs with increased values of L. For each graph, the constants m_i, b_i and $k_{ij} = k_{ji}$ are drawn from uniform distributions $\mathcal{U}(1,2)$, $\mathcal{U}(2,3)$ and $\mathcal{U}(1,2)$, respectively. Table 4.1 summarizes the times required to solve the controller existence LMIs in Proposition 3.2. The performance bound is chosen as $\gamma = 10$, such that the LMIs are feasible for all values of L in Table 4.1. Computations were performed on a PC with Intel Core i5 at 2.3GHz and 16GB memory using MOSEK version 8.1. We observe that for a cycle graph of moderate size ($L = 50$), the computation time is considerably lower for the distributed controller compared to the central controller. For $L \geq 100$, no solution was obtained for the central controller after 4 hours of computation, while the distributed controller problem was solved for up to $L = 10,000$ in less than 6 seconds.

5. CONCLUSIONS

In this paper, methods have been developed to compute distributed controllers that achieve an $\mathscr{H}_2$ performance bound for interconnected linear discrete-time systems with arbitrary interconnection structure. Convex controller existence conditions have been derived in the form of LMIs, which provide a scalable approach to the construction of distributed $\mathscr{H}_2$ controllers. We have observed a considerable reduction in computation time with respect to centralized $\mathscr{H}_2$ controller synthesis for moderately-sized networks and efficient computation for large-scale networks for which the centralized $\mathscr{H}_2$ synthesis is not tractable.

Appendix A. CLOSED-LOOP SCALES

$$(Z_i^{11})_{\mathcal{P}} := -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{P}}, (Z_i^{22})_{\mathcal{P}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{P}},$$

$$(Z_i^{12})_{\mathcal{P}} := \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{P}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{P}}^\top\right),$$

$$(Z_i^{11})_{\mathcal{C}} := -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{C}}, (Z_i^{22})_{\mathcal{C}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{C}},$$

$$(Z_i^{12})_{\mathcal{C}} := \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{C}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{C}}^\top\right),$$

$$(Z_i^{11})_{\mathcal{PC}} := -\operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ij}^{11})_{\mathcal{PC}}, (Z_i^{22})_{\mathcal{PC}} := \operatorname*{diag}_{j\in\mathbb{Z}_{[1:L]}} (X_{ji}^{11})_{\mathcal{PC}},$$

$$(Z_i^{12})_{\mathcal{PC}} := \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{PC}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{CP}}^\top\right),$$

$$(Z_i^{12})_{\mathcal{CP}} := \operatorname{diag}\left(-\operatorname*{diag}_{j\in\mathbb{Z}_{[1:i]}} (X_{ij}^{12})_{\mathcal{CP}}, \operatorname*{diag}_{j\in\mathbb{Z}_{[i+1:L]}} (X_{ji}^{12})_{\mathcal{PC}}^\top\right).$$

REFERENCES

Bergen, A.R. and Hill, D.J. (1981). A structure preserving model for power system stability analysis. *IEEE Transactions on Power Apparatus and Systems*, PAS-100(1), 25–35.

Bullo, F. (2018). *Lectures on Network Systems*. CreateSpace, 1st edition. With contributions by J. Cortés, F. Dörfler, and S. Martínez.

Chen, X., Xu, H., and Feng, M. (2019). H_2 performance analysis and H_2 distributed control design for systems interconnected over an arbitrary graph. *Systems & Control Letters*, 124, 1 – 11.

Dörfler, F., Chertkov, M., and Bullo, F. (2013). Synchronization in complex oscillator networks and smart grids. *Proceedings of the National Academy of Sciences*, 110(6), 2005–2010.

Eilbrecht, J., Jilg, M., and Stursberg, O. (2017). Distributed $\mathcal{H}_2$-optimized output feedback controller design using the ADMM. *IFAC-PapersOnLine*, 50(1), 10389 – 10394. 20th IFAC World Congress.

Langbort, C., Chandra, R.S., and D'Andrea, R. (2004). Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9), 1502–1519.

Lunze, J. (1992). *Feedback Control of Large Scale Systems*. Prentice Hall PTR, Upper Saddle River, NJ, USA.

Rice, J.K. (2010). *Efficient algorithms for distributed control: a structured matrix approach*. Ph.D. thesis, Delft University of Technology.

Scherer, C. and Weiland, S. (2017). Linear matrix inequalities in control. DISC lecture notes.

Scherer, C. (2001). LPV control and full block multipliers. *Automatica*, 37(3), 361 – 375.

Steentjes, T.R.V., Lazar, M., and Van den Hof, P.M.J. (2021). Scalable distributed and decentralized $\mathscr{H}_2$ controller synthesis for interconnected linear discrete-time systems. *ArXiv e-prints*, arXiv:2001.04875.

Vamsi, A.S.M. and Elia, N. (2016). Optimal distributed controllers realizable over arbitrary networks. *IEEE Transactions on Automatic Control*, 61(1), 129–144.

Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods – Basic methods for consistent module estimates. *Automatica*, 49(10), 2994 – 3006.

Van Horssen, E.P. and Weiland, S. (2016). Synthesis of distributed robust H-infinity controllers for interconnected discrete time systems. *IEEE Transactions on Control of Network Systems*, 3(3), 286–295.

Willems, J.C. (1972). Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, 45(5), 321–351.