Handling unmeasured disturbances in data-driven distributed control with virtual reference feedback tuning ⋆

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Abstract: The data-driven synthesis of a distributed controller in the presence of noise is considered, via the distributed virtual reference feedback tuning (DVRFT) framework. The analysis is performed for a linear interconnected system on an arbitrary graph that is subject to unmeasured exogenous inputs. By solving a dynamic network identification problem with prediction-error filtering and a tailor-made noise model, we show that the distributed model-reference control problem can be solved directly from data. Sufficient conditions are provided for which the local controller estimates are consistent. Moreover, it is shown how the method can be applied in the single-input-single-output case, leading to consistent estimates with standard virtual reference feedback tuning as well. The effectiveness of the method is demonstrated via a small network example with two interconnected systems.

Keywords: System identification, data-driven control, distributed control, dynamic networks

1. INTRODUCTION

Plant models are typically not directly available for controller design. When data from the plant is available, two approaches to controller design can be followed: (i) indirect data-driven control and (ii) direct data-driven control. Indirect data-driven control is model based: first a plant model is estimated on the basis of data and consecutively a controller design is performed on the basis of the plant model. In direct data-driven control, the plant modelling step is omitted; a controller is synthesized directly from data. Typical advantages of direct-data driven controller design are that no loss of data can occur due to undermodelling of the plant and the order of the controller can be fixed. Therefore, direct data-driven control is particularly interesting for the design of distributed controllers for interconnected systems, due to their complex nature and involved data-driven modelling (Van den Hof et al., 2013).

State-of-the art methods for direct data-driven controller design are virtual reference feedback tuning (VRFT) (Campi et al., 2002), optimal controller identification (OCI) (Campestrini et al., 2017; Huff et al., 2019), correlation-based tuning (CbT) (van Heusden et al., 2011), asymptotically exact (Formentin et al., 2015) and moment-matching (Breschi et al., 2019) controller tuning. Aforementioned methods are typically applicable to (multivariable) isolated or small-scale systems. In (Steentjes et al., 2020), a data-driven *distributed* controller design method was introduced, called distributed virtual reference feedback tuning (DVRFT). Through DVRFT, a distributed model-reference control problem can be equivalently stated as a network identification problem (Van den Hof et al., 2013), by constructing a virtual reference network (Steentjes et al., 2020).

When the considered plant is affected by disturbances, VRFT inherently introduces a bias in the controller estimates (Bazanella et al., 2011), leading to a degraded closed-loop performance. For DVRFT, also biased estimates are obtained for local controllers in the case that a process noise affects the corresponding subsystem. One approach to solve this problem is the use of an instrumental variable (IV), in case the controller model is linear with respect to the parameters. Depending on the choice of IV, however, additional experiments on the system are required (Bazanella et al., 2011) and the parameter variance is increased with a negative effect on the control performance. In the general case, no method for obtaining consistent estimates for VRFT is present in the literature, to the best of the author's knowledge.

In this paper, we present a method for dealing with noise in VRFT and DVRFT. The method relies on the VRFT and DVRFT frameworks, while modifying the prediction-error identification criteria via prediction-error filtering. With the introduction of a tailor-made noise model for VRFT, we provide sufficient conditions under which consistent controller estimates are obtained. The method extends naturally to the DVRFT framework in (Steentjes et al., 2020) and solves the distributed model reference control problem via DVRFT for a class of interconnected systems with unmeasured exogenous inputs.

The remainder of this paper is organized as follows: In Section 2, we introduce the considered dynamical network

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and the distributed control problem. In Section 3, a method for consistent VRFT is developed for the scalar case. Section 4 describes sufficient conditions for solving the data-driven control problem in the network case via DVRFT. A numerical example is an analyzed in Section 5 and concluding remarks are provided in Section 6.

2. PRELIMINARIES

2.1 Dynamical network and distributed controller

Consider a simple and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality L and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbour set of vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. The graph $\mathcal{G}$ describes the structure of a network of linear discrete-time systems, where the dynamics associated with vertex $i \in \mathcal{V}$ are described by

$$y_i(t) = G_i(q)u_i(t) + \sum_{j \in \mathcal{N}_i} G_{ij}(q)y_j(t) + H_i(q)e_i(t), \quad (1)$$

with $u_i : \mathbb{Z} \to \mathbb{R}$ the control input, $y_i : \mathbb{Z} \to \mathbb{R}$ the output, e_i an unmeasured zero-mean white-noise proces such that, for all (t,s), $Ee_i(t)e_j(s) = 0$ for $(i,j) \in \mathcal{E}$ and $Ee_i(t)u_j(s) = 0$ for $(i,j) \in \mathcal{V} \times \mathcal{V}$, and q the forward shift defined as $qx(t) = x(t+1)$. The rational transfer functions G_i, G_{ij} and H_i, $(i,j) \in \mathcal{E}$, describe the local dynamics, coupling dynamics and noise dynamics, respectively. The noise filter H_i is assumed to be monic, stable and minimum phase. We omit the time and shift arguments t and q occasionally for brevity, when the context does not yield ambiguity. The network can be compactly written as

$$y = G_I y + Gu + He, \quad (2)$$

where $G = \mathrm{diag}(G_1, \ldots, G_L)$, $H = \mathrm{diag}(H_1, \ldots, H_L)$ and

$$G_I = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \cdots & G_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1} & G_{L2} & \cdots & 0 \end{bmatrix}.$$

The transfer between external inputs and outputs is described by

$$y = (I - G_I)^{-1}(Gu + He), \quad (3)$$

under the assumption that the network (2) is well posed, i.e. $\mathcal{G} := (I - G_I)^{-1}$ exists. The simplest network consists of one node ($L = 1$), so that

$$y = Gu + He, \quad (4)$$

which is a standard single-input-single-output process with a disturbance.

Considering a reference tracking problem for the network, each system is equipped with a reference signal r_i and the corresponding tracking error $z_i := r_i - y_i$:

$$\mathcal{P}_i : \begin{cases} y_i = G_i u_i + \displaystyle\sum_{j \in \mathcal{N}_i} G_{ij} y_j + H_i e_i, \\ z_i = r_i - y_i. \end{cases} \quad (5)$$

The dynamical network is operating in closed-loop with a distributed controller that consists of local controllers

$$\mathcal{C}_i(\rho_i) : \begin{cases} u_i = C_{ii}(q,\rho_i)z_i + \displaystyle\sum_{j \in \mathcal{N}_i} C_{ij}(q,\rho_i)s^c_{ij}, \\ o^c_{ij} = K_{ij}(q,\rho_i)z_i + \displaystyle\sum_{h \in \mathcal{N}_i} K_{ijh}(q,\rho_i)s^c_{ih},\ j \in \mathcal{N}_i, \end{cases}$$

where each controller is parametrized by a parameter vector ρ_i, and is interconnected with other controllers as:

$$s^c_{ij} = o^c_{ji} \text{ for } j \in \mathcal{N}_i \quad \text{and} \quad s^c_{ij} = 0 \text{ otherwise.}$$

With the definitions $s_i := \mathrm{col}_{j \in \mathcal{N}_i} s^c_{ij}$ and $o_i := \mathrm{col}_{j \in \mathcal{N}_i} o^c_{ij}$, we compactly represent $\mathcal{C}_i$ by

$$\mathcal{C}_i(\rho_i) : \begin{bmatrix} u_i \\ o^c_i \end{bmatrix} = C_i(q,\rho_i) \begin{bmatrix} z_i \\ s^c_i \end{bmatrix}. \quad (6)$$

2.2 Distributed model reference control

Model reference control in dynamic networks considers the synthesis of a structured controller such that the closed-loop network dynamics are optimal with respect to a structured reference model. In general, the structured reference model is composed of subsystems $\mathcal{K}_i$, $i \in \mathcal{V}$, that can be interconnected (Steentjes et al., 2020). In this paper, for clarity of presentation, we will consider a special, but important, case of such a reference model with subsystems that are decoupled:

$$\mathcal{K}_i : \quad y^d_i = T_i(q)r_i. \quad (7)$$

For well-posedness, we assume that the reference model is such that $y^d \neq r$ for all non-zero r, i.e., $\mathrm{diag}_{i \in \mathcal{V}} T_i \neq I$.

The distributed model reference control problem is defined for the situation where $e_i = 0$ for all $i \in \mathcal{V}$:

$$\min_{\rho_1,\ldots,\rho_L} J_{\mathrm{MR}}(\rho_1,\ldots,\rho_L) = \min_{\rho_1,\ldots,\rho_L} \sum_{i=1}^{L} \bar{E}[y^d_i(t) - y_i(t)]^2, \quad (8)$$

where $\bar{E} := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E$ and E is the expectation. A distributed controller that solves (8) was developed in (Steentjes et al., 2020). For $i \in \mathcal{V}$, define $G_{i\mathcal{N}} := \mathrm{row}_{j \in \mathcal{N}_i} G_{ij}$.

Proposition 2.1. Consider $e_i = 0$ for all $i \in \mathcal{V}$ and consider a distributed controller described by the subsystems

$$\mathcal{C}^d_i : \begin{bmatrix} u_i \\ o^c_i \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{T_i}{G_i(1-T_i)} & -\dfrac{1}{G_i}G_{i\mathcal{N}} \\ \dfrac{T_i}{1-T_i}\mathbf{1} & 0 \end{bmatrix}}_{=:C^d_i(q)} \begin{bmatrix} z_i \\ s^c_i \end{bmatrix}, \quad (9)$$

for $i \in \mathcal{V}$ and the controller interconnections described by

$$s^c_{ij} = o^c_{ji} \quad \text{and} \quad s^c_{ji} = o^c_{ij}, \quad (10)$$

for $(i,j) \in \mathcal{E}$. The network (2) in closed-loop with the distributed controller (9)-(10) satisfies

$$y_i = y^d_i, \quad i \in \mathcal{V}.$$

This result follows directly from (Steentjes et al., 2020, Theorem III.1), where a reference model with possible interconnections is considered. The controller described by (9)-(10) provides a solution to (8), by showing the existence of a minimizing argument $(\rho^d_1, \ldots, \rho^d_L)$, if the chosen controller class is 'rich' enough, i.e., for each $i \in \mathcal{V}$, there exists ρ_i such that $C_i(q,\rho_i) = C^d_i(q)$.

3. VRFT: A TAILOR-MADE NOISE MODEL FOR CONSISTENT ESTIMATION

In this section we consider the modelling of a noise filter for consistent controller estimation with VRFT for a single process. Consider the single process described by (4). The

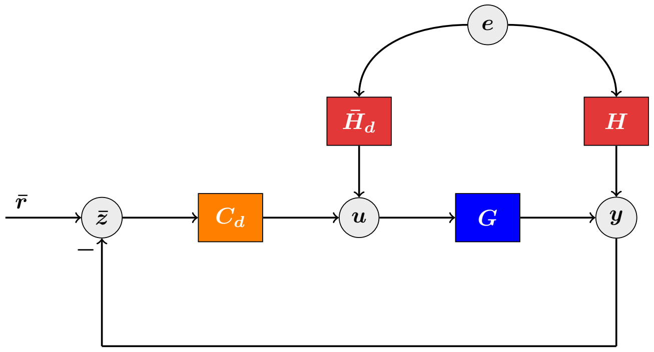

Fig. 1. Virtual control loop - noisy case.

input u and e are assumed to be independent. The tracking error for this process is denoted $z := r - y$, with r the reference. A reference model for the process is assumed to be given and described by

$$y^d = T_d r.$$

Given the reference model, it is known that the ideal controller is (Bazanella et al., 2011)

$$C_d = \frac{T_d}{G(1 - T_d)}.$$

We will now discuss the identification of C_d from the data (u, y).

The virtual reference $\bar{r}$ and tracking error $\bar{z}$ are given by

$$\bar{r} := T_d^{-1} y \quad \text{and} \quad \bar{z} := \bar{r} - y.$$

We can rewrite (4) in terms of C_d, such that

$$u = C_d \bar{z} - G^{-1} He = C_d \bar{z} + \bar{H}_d e, \qquad (11)$$

with $\bar{H}_d := -G^{-1} H$. This leads to the virtual control loop shown in Figure 1.

It is known that if we follow the standard VRFT procedure for the identification of C_d, we will obtain a biased estimate due to the noise e (Bazanella et al., 2011). We will now consider the direct identification of C_d together with the identification of $\bar{H}_d$. The question is: by including the estimation of the auxiliary noise filter, can we obtain consistent estimates of C_d?

3.1 Modelling $\bar{H}_d$ directly

Consider a model $C(q, \rho)$ for $C_d(q)$ and a model $\bar{H}(q, \rho)$ for $\bar{H}_d(q)$. Writing the predictor

$$\hat{u}(t, \rho) = \bar{H}(q, \rho)^{-1} C(q, \rho) \bar{z}(t) + (1 - \bar{H}(q, \rho)^{-1}) u(t),$$

leads to the prediction error $\varepsilon(t, \rho) = u(t) - \hat{u}(t, \rho)$:

$$\begin{aligned} \varepsilon(t, \rho) &= \bar{H}(\rho)^{-1} (u - C(\rho) \bar{z}) \\ &= \bar{H}(\rho)^{-1} (C_d \bar{z} + \bar{H}_d e - C(\rho) \bar{z}) \\ &= \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) \frac{1 - T_d}{T_d} y + \bar{H}_d e \right). \end{aligned} \qquad (12)$$

After manipulation it can be shown that

$$\varepsilon(t, \rho) = \frac{1}{\bar{H}(\rho) C_d} (C_d - C(\rho)) u + \frac{C(\rho)}{C_d} \frac{\bar{H}_d}{\bar{H}(\rho)} e. \qquad (13)$$

Consider now the asymptotic parameter estimate

$$\rho^* = \arg\min_{\rho} \bar{V}(\rho), \quad \bar{V}(\rho) := \bar{E} \varepsilon^2(t, \rho). \qquad (14)$$

The cost σ_e^2 is obtained if $C(\rho) = C_d$ and $\bar{H}(\rho) = \bar{H}_d$, but we cannot conclude that this is the minimum, since $\frac{C(\rho)}{C_d} \frac{\bar{H}_d}{\bar{H}(\rho)}$ is not necessarily monic. Hence, we cannot conclude that $C(\rho^*) = C_d$ and $\bar{H}(\rho^*) = H_d$ for the minimizing argument ρ^*.

3.2 Tailor-made noise model with prediction-error filtering

Let us return to the (virtual) data-generating system

$$u = C_d \bar{z} + \bar{H}_d e. \qquad (15)$$

We have seen in the previous subsection that by modelling the auxiliary noise filter directly, consistent estimates cannot be guaranteed. Note that if we filter the prediction error (12) with G, then a noise filter $-H$ is obtained in the prediction error $\varepsilon_G = G\varepsilon$, with H monic. The plant G is, however, assumed to be unknown.

A more attractive solution is obtained as follows. By the definition of C_d, it follows that

$$\frac{T_d}{1 - T_d} = C_d G.$$

Hence, by filtering the prediction error with $L := C_d G$ instead of G, we have

$$\varepsilon_F(t, \rho) := C_d G \varepsilon(t, \rho) = \frac{T_d}{1 - T_d} \varepsilon(t, \rho). \qquad (16)$$

The filter L depends only on the reference model T_d, which is known. Rewriting ε_F yields

$$\begin{aligned} \varepsilon_F(t, \rho) &= L \varepsilon(t, \rho) \\ &= \frac{T_d}{1 - T_d} \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) \frac{1 - T_d}{T_d} y + \bar{H}_d e \right) \\ &= \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) y + \frac{T_d}{1 - T_d} \bar{H}_d e \right) \\ &= \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) y - C_d He \right). \end{aligned}$$

Substituting the relation $y = Gu + He$ yields

$$\begin{aligned} \varepsilon_F(t, \rho) &= \bar{H}(\rho)^{-1} \left((C_d - C(\rho))(Gu + He) - C_d He \right) \\ &= \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) Gu - C(\rho) He \right). \end{aligned}$$

By selecting a tailor-made parametrization $\bar{H}(\rho) = -C(\rho) \breve{H}(\rho)$ with $\breve{H}(\rho)$ monic, we have

$$\begin{aligned} \varepsilon_F(t, \rho) &= -\bar{H}(\rho)^{-1} \bar{H}(\rho) e + e \\ &\quad + \bar{H}(\rho)^{-1} \left((C_d - C(\rho)) Gu - C(\rho) He \right) \\ &= \bar{H}(\rho)^{-1} \left(\Delta C(\rho) Gu + C(\rho) \Delta H(\rho) e \right) + e, \end{aligned}$$

with $\Delta C(\rho) := C_d - C(\rho)$ and $\Delta H(\rho) := \breve{H}(\rho) - H$. Now, since u and e are independent, $\Delta C(\rho) Gu$ and e are uncorrelated. Furthermore, since H and $\breve{H}(\rho)$ are both monic, $\Delta H(\rho)$ is strictly proper so that $C(\rho) \Delta H(\rho) e$ and e are uncorrelated. Therefore,

$$\begin{aligned} \bar{V}_F(\rho) &:= \bar{E} \varepsilon_F^2(t, \rho) \\ &= \bar{E} \left[\left(\bar{H}(\rho)^{-1} \left(\Delta C(\rho) Gu + C(\rho) \Delta H(\rho) e \right) + e \right)^2 \right] \\ &= \bar{E} \left[\left(\bar{H}(\rho)^{-1} \left(\Delta C(\rho) Gu + C(\rho) \Delta H(\rho) e \right) \right)^2 \right] + \sigma_e^2, \end{aligned}$$

which implies $\bar{V}_F(\rho) \geq \sigma_e^2$ for all ρ. The minimum of $\bar{V}_F$ is σ_e^2 and, if u is persistently exciting of sufficient order, then $\bar{V}_F(\rho^*) = \sigma_e^2$ if and only if $\Delta C(\rho^*) = 0$ and $\Delta H(\rho^*) = 0$. We conclude that consistent estimates of C_d are obtained.

Theorem 3.1. Consider the filtered prediction error ε_F and let ρ^* be a minimizing argument of $\bar{V}_F$. Let the following conditions be satisfied:

- *the spectral density of u, Φ_u, is positive definite for almost all $\omega \in [-\pi, \pi]$,*
- *there exists a ρ^d such that $C(\rho^d) = C^d$ and $\breve{H}(\rho^d) = H$.*

Then $C(\rho^*) = C^d$ and $\breve{H}(\rho^*) = H$. □

For the single-process case, a tailor-made noise model was considered in (van Heusden et al., 2011) for correlation-based tuning (CbT) of a linearly parametrized controller. To the best of the authors' knowledge, modelling the noise filter to obtain consistent estimates for VRFT is new. This approach also provides a method to deal with noise for distributed VRFT, which will be discussed in the following section.

4. DISTRIBUTED VRFT IN DYNAMIC NETWORKS: CONSISTENT ESTIMATION

In distributed VRFT with a decoupled reference model, the virtual reference signal $\bar{r}_i$ and virtual tracking error $\bar{z}_i$ are determined for each $i \in \mathcal{V}$ such that (Steentjes et al., 2020)

$$y_i = T_i \bar{r}_i \quad \text{and } \bar{z}_i = \bar{r}_i - y_i. \tag{17}$$

As was done in Section 3 for the single process, we can similarly form an inverse model of the network (1). From (1), we can write

$$\begin{aligned} u_i &= G_i^{-1} y_i - \sum_{j\in\mathcal{N}_i} G_i^{-1} G_{ij} y_j - G_i^{-1} H_i e_i \\ &= C_{ii}^d \bar{z}_i + \sum_{j\in\mathcal{N}_i} C_{ij}^d \bar{o}_{ji}^c + \bar{H}_i^d e_i, \quad i \in \mathcal{V}, \end{aligned} \tag{18}$$

where we have used the definition of the ideal controller modules, (17), and, in accordance with (9), $K_{ji}^d := T_i(1-T_i)^{-1}$ for $j \in \mathcal{N}_i$ such that

$$\begin{aligned} \bar{o}_{ji}^c &:= K_{ji} \bar{z}_j = y_j, \quad j \in \mathcal{N}_i, \\ \bar{H}_i^d &:= -G_i^{-1} H_i. \end{aligned} \tag{19}$$

In conjunction with the network dynamics described by (1), equation (18) describes a virtual network, with the transfer functions describing the ideal controller C_i^d being unknown modules in this network. Figure 2 depicts this network for $L = 2$.

Consider now the predictor

$$\begin{aligned} \hat{u}_i(t,\rho_i) = \bar{H}_i(\rho_i)^{-1} &\left(C_{ii}(\rho_i)\bar{z}_i + \sum_{j\in\mathcal{N}_i} C_{ij}(\rho_i)\bar{o}_{ji}^c \right) \\ &+ (1 - \bar{H}_i(\rho_i)^{-1}) u_i, \end{aligned}$$

which leads to the prediction error

$$\begin{aligned} \varepsilon_i(t,\rho_i) &:= u_i(t) - \hat{u}_i(t,\rho_i) \\ &= \bar{H}_i(\rho_i)^{-1} \big(u_i - C_{ii}(\rho_i)\bar{z}_i - \sum_{j\in\mathcal{N}_i} C_{ij}(\rho_i)\bar{o}_{ji}^c \big). \end{aligned}$$

Now, by filtering the prediction error ε_i with a filter $L_i := C_{ii}^d G_i = T_i(1-T_i)^{-1}$, the filtered prediction error $\varepsilon_i^F(t,\rho_i) := L_i \varepsilon_i(t,\rho_i)$ is obtained. We can now formulate conditions for the corresponding filtered network identification problem. Consider the asymptotic parameter estimate

$$\rho_i^* = \arg\min_{\rho_i} \underbrace{\bar{E}\varepsilon_i^F(t,\rho_i)^2}_{=:\bar{V}_i^F(\rho_i)}.$$

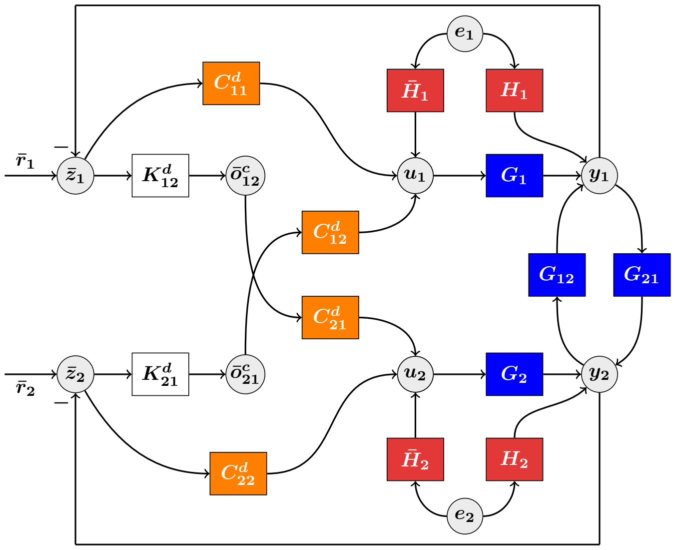

Fig. 2. Virtual network for $L = 2$, with a controller $\mathcal{C}^d$ for plant $\mathcal{P}$ with disturbance model H.

Theorem 4.1. Consider a tailor-made noise model defined by $\bar{H}_i(\rho_i) := -C_{ii}(\rho_i)\breve{H}_i(\rho_i)$ with $\breve{H}_i$ a monic transfer function and let the following conditions be satisfied:

- *the spectral density of $\zeta_i := \mathrm{col}(\bar{z}_i, u_i, \mathrm{col}_{j\in\mathcal{N}_i} \bar{o}_{ji}^c)$, Φ_{ζ_i}, is positive definite for almost all $\omega \in [-\pi, \pi]$,*
- *there exists a ρ_i^d such that $C_{ii}(\rho_i^d) = C_{ii}^d$, $C_{ij}(\rho_i^d) = C_{ij}^d$ and $\breve{H}_i(\rho_i^d) = H_i$,*
- *G_{ji} contains a delay for every $j \in \mathcal{N}_i$.*

Then it holds that $C_{ii}(\rho_i^*) = C_{ii}^d$, $C_{ij}(\rho_i^*) = C_{ij}^d$, $j \in \mathcal{N}_i$, and $\breve{H}_i(\rho_i^*) = H_i$. □

A proof for Theorem 4.1 is given in Appendix B, which is preceded by two instrumental lemmas in Appendix A.

The filter L_i is known, since T_i is a known transfer function that describes the reference model $\mathcal{K}_i$. Hence, the distributed model reference control problem (8) can be solved using data, via a DVRFT framework by (i) filtering a prediction-error with a known filter and (ii) a tailor-made noise model. Consistent estimates are guaranteed under the conditions in Theorem 4.1, but one should note that the identification criteria $\bar{V}_i^F$ are not convex, in general. 'Standard' DVRFT (Steentjes et al., 2020) does lead to convex identification criteria for linearly-parametrized controllers, but provides non-consistent estimates in the presence of noise.

5. NUMERICAL EXAMPLE

Consider a two-node network, described by

$$\begin{aligned} y_1 &= G_1 u_1 + G_{12} y_2 + H_1 e_1, \\ y_2 &= G_2 u_2 + G_{21} y_1 + H_2 e_2, \end{aligned}$$

with e_1 and e_2 Gaussian white-noise processes with variance $\sigma_{e_1}^2 = \sigma_{e_2}^2 = \sigma_e^2$, u_1 and u_2 white-noise processes with distribution $\mathcal{U}(0,1)$ and

$$G_1 = \frac{1}{q-0.8}, \quad G_{12} = \frac{0.1}{q-0.8}, \quad H_1 = \frac{q}{q-0.8},$$
$$G_2 = \frac{1}{q-0.6}, \quad G_{21} = \frac{0.1}{q-0.6}, \quad H_2 = \frac{q}{q-0.6}.$$

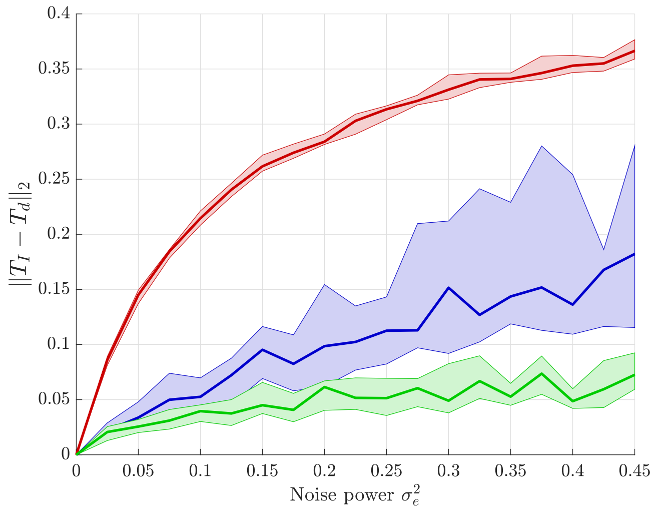

Fig. 3. Achieved performance versus noise power σ_e^2 for DVRFT (red), DVRFT with IVs (blue) and DVRFT with tailor-made noise modelling and PE filtering (green). The solid lines indicate the median performance and the shaded areas are bounded by the 25$^{\text{th}}$ and 75$^{\text{th}}$ percentiles.

In order to synthesize the data-driven controller, we consider that for each $i = 1, 2$, data $(u_i(t), y_i(t))$, $t = 1, \ldots, N$, is collected for $N = 500$ samples. We choose the reference models $T_1 = T_2 = 0.2(q - 0.8)^{-1}$ and the parametrization as

$$C_{ii}(\rho_i) = \frac{\rho_{i1}q + \rho_{i2}}{q-1},\ C_{ij}(\rho_i) = \rho_{i3},\ \bar{H}_i(\rho_i) = -\frac{C_{ii}(\rho_i)q}{q + \rho_{i4}},$$

such that the second condition in Theorem 4.1 is satisfied. Each controller $\mathcal{C}_i(\rho_i)$, $i = 1, 2$, is obtained by minimizing the identification criterion $V_i^F(\rho_i) := \sum_{t=1}^{N} \varepsilon_i^F(t, \rho_i)^2$, using the `lsqnonlin.m` function in MATLAB, with initial parameters $\rho_i^{\text{init}} = 0.1\rho_i^d$ relatively 'far' from ρ_i^d.

To illustrate the obtained closed-loop performance with respect to the influence of the disturbances, a Monte-Carlo simulation over 25 experiments is performed for each noise power level $\sigma_e^2 \in \{0, 0.05, \ldots, 0.045\}$. The obtained closed-loop transfer T_I from $r \to y$ results in a performance for DVRFT with the tailor-made noise model and filtered prediction error as depicted in Figure 3, in green. For comparison, we compute two distributed controllers from the same experiments via (i) DVRFT (Steentjes et al., 2020) and (ii) DVRFT with IVs (obtained by performing an additional experiment for each estimation, cf. (Bazanella et al., 2011)). As shown in Figure 3 in red, the results confirm the expectation that the noise-induced bias for DVRFT degrades the closed-loop performance considerably. The use of IVs in combination with DVRFT provides consistent estimates, but leads to an increased variance for the estimates. This can be observed for the corresponding closed-loop performance for DVRFT with IVs as well, as illustrated in blue in Figure 3. By Theorem 4.1, DVRFT with a tailor-made noise model and prediction-error filtering also provides consistent estimates. However, due to a decrease in estimator variance, the method described in this paper outperforms DVRFT with IVs considerably for higher noise levels.

6. CONCLUDING REMARKS

In this paper we have considered virtual reference feedback tuning in dynamics networks with noise. The standard VRFT method yields biased estimates for the controller in the single-process case. So does the corresponding DVRFT method for dynamic networks. We have shown for the single-process case that by including the direct modelling of the auxiliary noise filter, it cannot be concluded that consistent estimates are obtained. However, a filtered prediction-error identification problem can be formulated for which consistent estimates are obtained when a tailor-made noise model is used. For DVRFT applied to dynamic networks, a similar approach is obtained for estimating a local controller that is part of a distributed controller. Sufficient conditions have been given for obtaining consistent estimates in the considered framework, thereby solving the distributed model-reference control problem in the presence of noise. Through an example network consisting of two subsystems, we have shown that the developed method provides a substantial closed-loop performance improvement for increasing levels of noise power.

Appendix A. INSTRUMENTAL LEMMAS

In this appendix, we provide two lemmas that will be instrumental in the proof for Theorem 4.1. The proofs are omitted due to space limitations.

Lemma A.1. $\mathcal{G} = I + \bar{\mathcal{G}}$, *where* $\bar{\mathcal{G}} := \mathcal{G}G_I$.

Lemma A.2. *If* G_{ji} *is strictly proper for each* $j \in \mathcal{N}_i$, *then*

- $\bar{\mathcal{G}}_{ii}$ *is strictly proper,*
- $\bar{\mathcal{G}}_{ji}$ *is strictly proper for* $j \in \mathcal{N}_i$.

Appendix B. PROOF OF THEOREM 4.1

To prove Theorem 4.1, we start by writing the prediction error as

$$\varepsilon_i(t, \rho_i) = \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)\bar{z}_i + \sum_{j \in \mathcal{N}_i} \Delta C_{ij}(\rho_i)\bar{o}_{ji}^c + \bar{H}_i^d e_i\big),$$

using (18), where $\Delta C(\rho_i) := C_{ii}^d - C_{ii}(\rho_i)$, $\Delta C_{ij}(\rho_i) := C_{ij}^d - C_{ij}(\rho_i)$, $j \in \mathcal{N}_i$. The virtual signals are related to y_i and y_j by (17) and (19), leading to

$$\varepsilon_i(t, \rho_i) = \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i)\frac{1 - T_i}{T_i} y_i + \sum_{j \in \mathcal{N}_i} \Delta C_{ij}(\rho_i) y_j + \bar{H}_i^d e_i\big).$$

Now consider the filtered prediction error ε_i^F, with filter $L_i = C_{ii}^d G_i$. By definition, the filter is

$$L_i = C_{ii}^d G_i = \frac{T_i}{1 - T_i}.$$

Hence, we have that

$$\begin{aligned}
\varepsilon_i^F(t, \rho_i) &= L_i \varepsilon_i(t, \rho_i) \\
&= \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i) y_i + \frac{T_i}{1 - T_i}\sum_{j \in \mathcal{N}_i} \Delta C_{ij}(\rho_i) y_j \\
&\quad + C_{ii}^d G_i \bar{H}_i^d e_i\big) \\
&= \bar{H}_i(\rho_i)^{-1}\big(\Delta C_{ii}(\rho_i) y_i + \sum_{j \in \mathcal{N}_i} \Delta C_{ij}(\rho_i) K_{ij}^d y_j \\
&\quad - C_{ii}^d H_i e_i\big).
\end{aligned}$$

We proceed by writing the 'node' variables y_j in terms of 'external' variables. By (3) we have that

$$y = \mathcal{G}(Gu + He) = \mathcal{G}(\bar{u} + v),$$

where $\bar{u} := Gu$ and $v = He$. Hence, by Lemma A.1, it follows that

$$\mathcal{G}H = (I - G_I)^{-1}H = H + \mathcal{G}G_I H.$$

Therefore, we can write the node variables in y as

$$y = \mathcal{G}\bar{u} + \bar{\mathcal{G}}v + He, \tag{B.1}$$

where $\bar{\mathcal{G}} = \mathcal{G}G_I$, or, equivalently,

$$y_i = H_i e_i + \sum_{j\in\mathcal{V}} \mathcal{G}_{ij}\bar{u}_j + \bar{\mathcal{G}}_{ij}v_j, \quad i \in \mathcal{V}.$$

It follows that

$$\varepsilon_i^F(t,\rho_i) = \bar{H}_i(\rho_i)^{-1}\left[x_i(\rho_i) - C_{ii}(\rho_i)H_i e_i\right],$$

where

$$x_i(\rho_i) := \Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\mathcal{G}_{ij}\bar{u}_j + \sum_{j\in\mathcal{N}_i} K_{ij}^d \Delta C_{ij}(\rho_i)\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}\bar{u}_k + \Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\bar{\mathcal{G}}_{ij}v_j + \sum_{j\in\mathcal{N}_i}K_{ij}^d\Delta C_{ij}(\rho_i)\left(v_j + \sum_{k\in\mathcal{V}}\bar{\mathcal{G}}_{jk}v_k\right).$$

Now, considering the tailor-made noise model $\bar{H}_i(\rho_i) = -C_{ii}(\rho_i)\breve{H}_i(\rho_i)$, we obtain the filtered prediction error

$$\begin{aligned}\varepsilon_i^F(t,\rho_i) &= \bar{H}_i(\rho_i)^{-1}\left[x_i + C_{ii}(\rho_i)\breve{H}_i(\rho_i)e_i - C_{ii}(\rho_i)H_i e_i\right] + e_i \\ &= \bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right] + e_i,\end{aligned}$$

where $\Delta H_i(\rho_i) := \breve{H}_i(\rho_i) - H_i$.

We will now show that the noise e_i is uncorrelated with $x_i(\rho_i)$ and $C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i$:

- Since both $\breve{H}_i(\rho_i)$ and H_i are monic, $\Delta H_i(\rho_i)$ is strictly proper. Hence, $C_{ii}(\rho_i)\Delta H_i(\rho_i)$ has a delay, because $C_{ii}(\rho_i)$ is proper, which implies that $C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i$ is uncorrelated with e_i;
- $\Delta C_{ii}(\rho_i)\sum_{j\in\mathcal{V}}\mathcal{G}_{ij}\bar{u}_j$ is uncorrelated with e_i, since it is a filtered linear combination of u_j, $j \in \mathcal{V}$, which are uncorrelated with e_i by assumption;
- $\sum_{j\in\mathcal{N}_i}K_{ij}^d\Delta C_{ij}(\rho_i)\sum_{k\in\mathcal{V}}\mathcal{G}_{jk}\bar{u}_k$ is uncorrelated with e_i, since it is a filtered linear combination of u_j, $j \in \mathcal{V}$, which are uncorrelated with e_i by assumption;
- $\Delta C_{ij}(\rho_i)\sum_{j\in\mathcal{V}}\bar{\mathcal{G}}_{ij}v_j$ is uncorrelated with e_i, because (i) $\bar{G}_{ii}$ is strictly proper by Lemma A.2 and (ii) e_j, $j \in \mathcal{V}\setminus\{i\}$ is uncorrelated with e_i by assumption;
- $\sum_{j\in\mathcal{N}_i}K_{ij}^d\Delta C_{ij}(\rho_i)\left(v_j + \sum_{k\in\mathcal{V}}\bar{\mathcal{G}}_{jk}v_k\right)$ is uncorrelated with e_i, because (i) $\bar{G}_{ji}$ is strictly proper for $j \in \mathcal{N}_i$ by Lemma A.2 and (ii) e_j, $j \in \mathcal{V}\setminus\{i\}$ is uncorrelated with e_i by assumption.

Hence,

$$\begin{aligned}\bar{V}_i^F(\rho_i) &= \bar{E}\varepsilon_i^F(t,\rho_i)^2 \\ &= \bar{E}\left[\left(\bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right] + e_i\right)^2\right] \\ &= \bar{E}\left[\left(\bar{H}_i^{-1}\left[x_i(\rho_i) + C_{ii}(\rho_i)\Delta H_i(\rho_i)e_i\right]\right)^2\right] + \sigma_{e_i}^2.\end{aligned} \tag{B.2}$$

But then the minimum of $\bar{V}_i^F(\rho_i)$ must be $\sigma_{e_i}^2$ and a minimizing argument is ρ_i^d by the second condition.

Next, we will show that the minimizing argument is unique. A minimizing argument ρ_i^* must satisfy $V_i^F(\rho_i^*) = \sigma_{e_i}^2$, which, by (B.2), is equivalent with

$$\begin{aligned}0 &= \bar{E}\left[\bar{H}_i(\rho_i^*)^{-1}\left[x_i(\rho_i^*) + C_{ii}(\rho_i^*)\Delta H_i(\rho_i^*)e_i\right]\right]^2 \\ &= \bar{E}\left[\Delta x_i(\rho_i^*)\underbrace{\begin{bmatrix}\frac{T_i}{H_i(1-T_i)} & -\frac{G_i}{H_i} & -\frac{1}{H_i}G_{i\mathcal{N}} \\ 0 & G_i & G_{i\mathcal{N}} \\ 0 & 0 & K_{i\mathcal{N}}^d\end{bmatrix}}_{=:\Gamma_i}\begin{bmatrix}\bar{z}_i \\ u_i \\ \bar{o}_{\mathcal{N}i}^c\end{bmatrix}\right]^2,\end{aligned}$$

with $K_{i\mathcal{N}}^d := \operatorname{diag}_{j\in\mathcal{N}_i}K_{ij}^d$ and

$$\Delta x_i(\rho_i^*) = \frac{1}{\bar{H}_i(\rho_i^*)}\left[\Delta H_i C_{ii}\ \ \Delta C_{ii}\ \ \operatorname{row}_{j\in\mathcal{N}_i}\Delta C_{ij}\right](\rho_i^*).$$

Hence, by Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta x_i(e^{i\omega},\rho_i^*)^\top\Gamma_i\Phi_{\zeta_i}(\omega)\Gamma_i^*\Delta x_i(e^{-i\omega},\rho_i^*)\,\mathrm{d}\omega = 0.$$

Now, $\Gamma_i(e^{i\omega})$ has full rank for almost all ω and, by the first condition, $\Phi_{\zeta_i}(\omega)$ is positive definite for all ω. Hence, $[\Delta H_i C_{ii}\ \Delta C_{ii}\ \operatorname{row}_{j\in\mathcal{N}_i}\Delta C_{ij}](\rho_i^*)$ is equal to the zero row for almost all ω, which implies $C_{ii}(\rho_i^*) = C_{ii}^d$, $C_{ij}(\rho_i^*) = C_{ij}^d$, $j \in \mathcal{N}_i$, and $\breve{H}_i(\rho_i^*) = H_i$. This concludes the proof.

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