Local module identification in dynamic networks with correlated noise: the full input case

Paul M.J. Van den Hof, Karthik R. Ramaswamy, Arne G. Dankers and Giulio Bottegal

***Abstract*— The identification of local modules in dynamic networks with known topology has recently been addressed by formulating conditions for arriving at consistent estimates of the module dynamics, typically under the assumption of having disturbances that are uncorrelated over the different nodes. The conditions typically reflect the selection of a set of node signals that are taken as predictor inputs in a MISO identification setup. In this paper an extension is made to arrive at an identification setup for the situation that process noises on the different node signals can be correlated with each other. In this situation the local module may need to be embedded in a MIMO identification setup for arriving at a consistent estimate with maximum likelihood properties. This requires the proper treatment of confounding variables. The result is an algorithm that, based on the given network topology and disturbance correlation structure, selects an appropriate set of node signals as predictor inputs and outputs in a MISO or MIMO identification setup. As a first step in the analysis, we consider the situation where the selected output node signals are predicted based on all of their in-neighbor node signals in the network.**

I. INTRODUCTION

In recent years increasing attention has been given to the development of new tools for the identification of large-scale interconnected systems, also known as dynamic networks. These networks are typically thought of as a set of measurable signals (the node signals) interconnected through linear dynamic systems (the modules), possibly driven by external excitations (the reference signals). Among the literature on this topic, we can distinguish three main categories of research. The first one focuses on identifying the topology of the dynamic network [1], [2], [3], [4], [5]. The second category concerns identification of the full network dynamics including aspects of identifiability [6], [7], [8], [9] while the third one deals with identification of a specific component (module) of the network, assuming that the network topology is known (the so called local module identification, see [10], [11], [12], [13], [14]).

In this paper we will further expand the work on the local module identification problem. In [10], the classical *direct-method* [15] for closed-loop identification has been generalized to a dynamic network framework using a MISO identification setup. Consistent estimates of the target module can be obtained when the network topology is known and all the node signals in the MISO identification setup are measured. The work has been extended in [16], [17] towards the situation where some node signals might be non-measurable, leading to an additional predictor input selection problem. A similar setup has also been studied in [11], where an approach has been presented based on empirical Bayesian methods to reduce the variance of the target module estimates. In [14] and [12], dynamic networks having node measurements corrupted by sensor noise have been studied, and informative experiments for consistent local module estimates have been addressed in [13].

A standing assumption in the aforementioned works [10], [11], [13], [17] is that the process noises entering the nodes of the dynamic network are uncorrelated with each other. This assumption facilitates the analysis and the development of methods for local module identification, reaching *consistent* module estimates for either the direct or indirect methods. However, when process noises are correlated over the nodes, the consistency results for the considered MISO direct method collapse. In this situation it is seems necessary to consider also the *noise topology or disturbance correlation structure*, when selecting an appropriate identification setup. Even though the two-stage and indirect methods in [14], [12], [13] can handle the situation of correlated noise and deliver consistent estimates, the obtained estimates will not have minimum variance.

In this paper we precisely consider the situation of having dynamic networks with disturbance signals on different nodes that possibly are correlated, while our target moves from consistency only, to also minimum variance (or Maximum Likelihood (ML)) properties of the obtained estimates. While one could use techniques for full network identification (e.g., [6]), our aim is to develop a method that uses only local information. In this way, we avoid (i) the need to collect node measurements that are "far away" from the target module, and (ii) the need to identify unnecessary modules that would come with the price of higher variance in the estimates. We will assume that the topology of network is known, as well as the (Boolean) correlation structure of the noise disturbances.

Using the reasoning first introduced [18], we build a constructive procedure that, choosing a limited number of predictor inputs and predicted outputs, builds an identification setup that guarantees maximum likelihood (ML) properties (and thus asymptotic minimum variance) when

Paul Van den Hof and Karthik Ramaswamy are with the Dept. of Electrical Engineering, Eindhoven University of Technology, The Netherlands {p.m.j.vandenhof, k.r.ramaswamy}@tue.nl

Giulio Bottegal was with the same department of Eindhoven Univ. Technology, guilio.bottegal@gmail.com

Arne Dankers is with the Electrical and Computer Engineering Dept. at the University of Calgary, Canada, adankers@hifieng.com

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applying a direct prediction error identification method. In this situation we have to deal with so-called *confounding variables* (see e.g. [18], [19]), that is, unmeasured variables that directly or indirectly influence both the predicted output and the predictor inputs, and lead to lack of consistency. A direct influence, caused by correlated process noise, can be treated by adding predicted outputs to our identification setting, while an indirect influence, caused by unmeasured nodes, can be resolved by adding predictor inputs. In this paper, we restrict our attention to the situation where all the nodes that are in-neighbors of predicted outputs are measured, which we refer to as the *full input case*.

This paper is organized as follows. In section II, the dynamic network setup is defined. Section III provides a summary of available results from the existing literature. Confounding variables are discussed in, and Section V provides an algorithm for selecting the node signals in the identification setup. This setup is then analyzed in Section VI, and the main results are provided in Section VII. The paper is concluded with two illustrative examples. The proofs of all results are collected in the extended report [20].

II. Network and identification setup

Following the basic setup of [10], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1, \dots, L$, and K *external variables* r_k, $k = 1, \dots K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^{L} G_{jl}(q) w_l(t) + r_j(t) + v_j(t) \tag{1}$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl} are proper rational transfer functions, referred to as *modules*.
- r_j are *external variables* that can directly be manipulated by the user and that may or may not be present; if r_j is not present it is replaced by $r_j = 0$.
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase. The situation of correlated noise refers to the case that $\Phi_v(\omega)$ and H are non-diagonal, while we assume that we know a priori which entries of Φ_v are nonzero.

We will assume that the standard regularity conditions on the data are satisfied that are required for convergence results of prediction error identification method[1].

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1\ L} \\ G_{L1} & \cdots & G_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + r + He. \tag{2}$$

The identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of a subset of measured variables w, and possibly r.

Let us define $\mathcal{N}_j$ as the set of node indices k such that $G_{jk} \neq 0$, i.e. the node signals in $\mathcal{N}_j$ are the in-neighbors of the node signal w_j. Let $\mathcal{D}_j$ denote the set of indices of the internal variables that are chosen as predictor inputs. Let $\mathcal{Z}_j$ denote the set of indices not in $\{j\} \cup \mathcal{D}_j$, i.e. $\mathcal{Z}_j = \{1, \dots, L\} \setminus \{\{j\} \cup \mathcal{D}_j\}$. Let $w_{\mathcal{D}}$ denote the vector $[w_{k_1} \ \cdots \ w_{k_n}]^T$, where $\{k_1, \dots, k_n\} = \mathcal{D}_j$. Let $r_{\mathcal{D}}$ denote the vector $[r_{k_1} \ \cdots \ r_{k_n}]^T$, where $\{k_1, \dots, k_n\} = \mathcal{D}_j$, and where the ℓth entry is zero if r_ℓ is not present in the network. The vectors $w_{\mathcal{Z}}$, $v_{\mathcal{D}}$, $v_{\mathcal{Z}}$ and $r_{\mathcal{Z}}$ are defined analogously. The ordering of the elements of $w_{\mathcal{D}}$, $v_{\mathcal{D}}$, and $r_{\mathcal{D}}$ is not important, as long as it is the same for all vectors. The transfer function matrix between $w_{\mathcal{D}}$ and w_j is denoted $G^0_{j\mathcal{D}}$. The other transfer function matrices are defined analogously. To illustrate the notation, consider the network sketched in Figure 1, and let module G^0_{21} be the target module for identification.

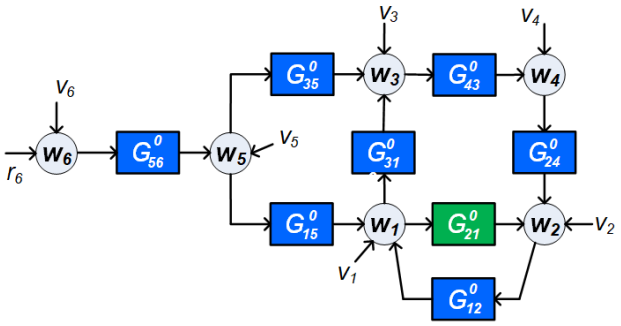

Fig. 1. Example network

Then $j = 2$, $i = 1$; $\mathcal{N}_j = \{1, 4\}$. If we choose the set of predictor inputs as $\mathcal{D}_j = \mathcal{N}_j$, then the set of remaining (nonmeasured) signals, becomes $\mathcal{Z}_j = \{3, 5, 6\}$.

By this notation, the network equation (2) is rewritten as:

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{D}} & G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} & G^0_{\mathcal{D}\mathcal{Z}} \\ G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{D}} & G^0_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \\ r_{\mathcal{Z}} \end{bmatrix}, \tag{3}$$

where $G^0_{\mathcal{D}\mathcal{D}}$ and $G^0_{\mathcal{Z}\mathcal{Z}}$ have zeros on the diagonal.

Identification of module G^0_{ji} can now be done by selecting $\mathcal{D}_j$ such that $i \in \mathcal{D}_j$, and subsequently estimating a multiple-input single output model for the transfer functions in $G_{j\mathcal{D}}$. This can be done by considering the one-step-ahead predictor[2] $\hat{w}_j(t|t-1) := \bar{\mathbb{E}}\{w_j(t) \mid w_j^{t-1}, w_{\mathcal{D}_j}^t\}$, and the resulting

[1] See [15] page 249. This includes the property that $e(t)$ has bounded moments of order higher than 4.

[2] $\bar{\mathbb{E}}$ refers to $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \mathbb{E}$, and w_j^ℓ and $w_{\mathcal{D}_j}^\ell$ refer to signal samples $w_j(\tau)$ and $w_k(\tau)$, $k \in \mathcal{D}_j$, respectively, for all $\tau \leq \ell$.

prediction error ([15]) $\varepsilon_j(t,\theta) = w_j(t) - \hat{w}_j(t|t-1,\theta)$, leading to

$$\varepsilon_j(t,\theta) = H_j(\theta)^{-1}[w_j - \sum_{k\in\mathcal{D}_j} G_{jk}(\theta)w_k - r_j] \qquad (4)$$

where arguments q and t have been dropped for notational clarity, and H_j is a scalar noise model for the disturbance $v_j(t)$. The parameterized transfer functions $G_{jk}(\theta)$, $k \in \mathcal{D}_j$ and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors: $V_j(\theta) = \frac{1}{N}\sum_{t=0}^{N-1}\varepsilon_j^2(t,\theta)$, where N is the length of the data set. We refer to this identification method as the *direct method*, [10]. Let $\hat{\theta}_N$ denote the minimizing argument of $V_j(\theta)$.

III. Available results

The following results are available from previous work:

- When $\mathcal{D}_j$ is chosen equal to $\mathcal{N}_j$ and noise v_j is uncorrelated to all v_k that have a path to w_j, then G_{ji}^0 can be consistently estimated in a MISO setup, provided that there is enough excitation in the predictor input signals, see [10].
- When $\mathcal{D}_j$ is a subset of $\mathcal{N}_j$, confounding variables[3] can occur in the estimation problem, and these have to be taken into account in the choice of $\mathcal{D}_j$ in order to arrive at consistent estimates of G_{ji}^0, see [17]. This situation has been analyzed for uncorrelated disturbances only, i.e. Φ_v being diagonal.
- In [19] relaxed conditions for the previous situation have been formulated, while still staying in the context of MISO identification with Φ_v being diagonal. This is particularly done by choosing additional predictor input signals that are not in $\mathcal{N}_j$,.i.e. that are no in-neighbors of the output w_j of the target module.
- Irrespective of noise correlations, an indirect/two-stage identification method can be used to arrive at consistent estimates of G_{ji}^0, if particular conditions on $\mathcal{D}_j$ are satisfied, [10], [17], [13]. However the drawback of indirect methods is that they do not allow for a maximum likelihood analysis, i.e. they will not lead to minimum variance results.

The step that we would like to make in this paper, is to go beyond consistency properties, and to formulate an identification setup that leads to Maximum Likelihood properties, and thus also minimum variance properties, of the estimated module, for the situation that the disturbance signals can be correlated, i.e. Φ_v not necessarily being diagonal. This requires a more careful treatment and modelling of the noise that is acting on the different node signals. In [18] a two-node example network has been studied, which has led to the following two suggestions:

- confounding variables can be dealt with by modelling correlated disturbances on the node signals, and
- this can be done by moving from a MISO identification setup to a MIMO setup.

[3]A confounding variable is an unmeasured variable that induces correlation between the input and output signal of an estimation problem. [21]. A formal definition follows in Definition 1.

These suggestions are being worked out in the current paper, and, as a first step in this analysis, we will stay in the situation of "full input modeling", meaning that for every node signal that is included as a predicted output we will include all in-neighbors in the network as predictor input. We will first present an example to explain the mechanism.

Example 1: Consider the network sketched in Figure 1, and let module G_{21} be the target module for identification. If the node signals w_1, w_2 and w_4 can be measured, then a two-input one-output model with inputs w_1, w_4 and output w_2 will (under the appropriate conditions) lead to a consistent estimate of G_{21} and G_{24}, provided that the disturbance signal v_2 is uncorrelated to the signals v_1 and v_4. However if e.g. v_4 and v_2 are dynamically correlated, then consistency is lost for this approach. A solution is then to include w_4 in the set of predicted outputs, and by adding node signal w_3 as predictor input for w_4. We then combine predicting w_2 on the basis of (w_1, w_4) with predicting w_4 on the basis of w_3. The correlation between v_2 and v_4 is then covered by modelling a 2×2 non-diagonal noise model of the joint process (v_2, v_4).

In the next sections we will formalize the procedure as sketched in Example 1 for general networks.

IV. Concepts and notation

Definition 1 (confounding variable): Consider a dynamic network defined by

$$w = Gw + He + r \qquad (5)$$

with $cov(e) = I$, and consider the graph related to this network, with node signals w and e. Let $w_{\mathcal{X}}$ and $w_{\mathcal{Y}}$ be two subsets of measured node signals in w, and let $w_{\mathcal{Z}}$ be the set of unmeasured node signals in w.

Then a noise component e_ℓ in e is a *confounding variable for the estimation problem* $w_{\mathcal{X}} \to w_{\mathcal{Y}}$, if in the graph there exist simultaneous paths[4] from e_ℓ to node signals $w_k, k \in \mathcal{X}$ and $w_n, n \in \mathcal{Y}$, while these paths are either direct[5] or only run through nodes that are in $w_{\mathcal{Z}}$. □

We will denote $w_{\mathcal{Y}}$ as the node signals in w that serve as predicted outputs, and $w_{\mathcal{D}}$ as the node signals in w that serve as predictor inputs. Next we decompose $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ in disjoint sets according to: $\mathcal{Y} = \mathcal{Q}\cup\{o\}$; $\mathcal{D} = \mathcal{Q}\cup\mathcal{A}\cup\mathcal{B}$ where $w_{\mathcal{Q}}$ are the node signals that are common in $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$; w_o is the output w_j of the target module; if $j \in \mathcal{Q}$ then $\{o\}$ is void; $\mathcal{A} \subset \mathcal{N}_{\mathcal{Y}}$ and $\mathcal{B} \not\subset \mathcal{N}_{\mathcal{Y}}$, to be specified later on. In this situation the measured nodes will be $w_{\mathcal{D}\cup\mathcal{Y}}$ and the unmeasured nodes $w_{\mathcal{Z}}$ will be determined by the set $\mathcal{Z} = \mathcal{L}\backslash\{\mathcal{D}\cup\mathcal{Y}\}$, where $\mathcal{L} = \{1, 2, \cdots L\}$. There can exist two types of confounding variable namely *direct and indirect confounding variable*. For *direct confounding variables* the simultaneous paths mentioned in the definition are both *direct paths*, while in all other cases we refer to the confounding variables as *indirect confounding variables*.

[4]A simultaneous path from e_1 to node signal w_1 and w_2 implies that there exist a path from e_1 to w_1 as well as from e_1 to w_2.

[5]A direct path from e_1 to node signal w_1 implies that there exist a path from e_1 to w_1 which do not pass through nodes in w.

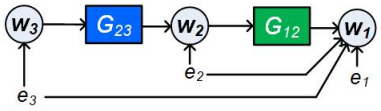

Fig. 2. A simple network with 3 nodes w_1, w_2, w_3 and unmeasured noise sources e_1, e_2 and e_3. G_{12} is the target module to be identified.

For example, in the network as shown in Figure 2 with $\mathcal{D} = \{2\}$, $\mathcal{Y} = \{1\}$ and $\mathcal{Z} = \{3\}$, for the estimation problem $w_2 \to w_1$, e_2 is a *direct confounding variable* since it has a simultaneous path to w_1 and w_2 where both the paths are *direct paths*. Meanwhile e_3 is an *indirect confounding variable* since it has a simultaneous path to w_1 and w_2 where one of the path is an unmeasured path[6].

V. Algorithm for signal selection: full input case

In order to arrive at an appropriate identification setup we will take the following strategy:

- We start by constructing sets $\mathcal{Q}$ and $\mathcal{A}$ in such a way that all w-in-neighbors of $w_{\mathcal{Y}}$ are included in $w_{\mathcal{Q}\cup\mathcal{A}}$ and that all disturbance terms v_k, $k \in \mathcal{A}$ are uncorrelated to disturbance terms v_ℓ, $\ell \in \mathcal{Y}$. In this way we handle the direct confounding variables.
- Then we choose $w_{\mathcal{B}}$ as a subset of nodes that are not in $w_{\mathcal{Y}}$ nor in $w_{\mathcal{A}}$. This set needs to be introduced to deal with the indirect confounding variables, and will be further specified in Section VII.
- Finally, we define the identification setup as the estimation problem $w_{\mathcal{D}} \to w_{\mathcal{Y}}$.

The conditions that need to be imposed on the selection of $w_{\mathcal{B}}$ in order to arrive at attractive properties of the estimation results, will be the main subject of analysis in this paper.

The following algorithm formalizes the procedure as indicated above.

Algorithm A

1) Select target module G_{ji}
2) Include j in the index set $\mathcal{Y}$ of node variables that are to be predicted.
3) For every element x of $\mathcal{Y}$:
   a) For every $k \in \mathcal{N}_x$:
      - include k in $\mathcal{D}$, and
      - if v_k is correlated with any w_ℓ, $\ell \in \mathcal{Y}$, then include k in $\mathcal{Y}$;
   b) If $\mathcal{Y}$ has changed, start step 3 from the beginning again.
4) Determine $\mathcal{Q}$ as the intersection of $\mathcal{Y}$ and $\mathcal{D}$;
5) If $j \notin \mathcal{Q}$ then set $w_o = w_j$, else w_o is void;
6) Determine $\mathcal{A} = \mathcal{D}\backslash\mathcal{Q}$;
7) Make a selection $\mathcal{B}$ of node signals that are not in $\mathcal{Y}$ and not in $\mathcal{A}$.

When this algorithm finishes, then the set $\mathcal{Y}$ contains the index set of to be predicted node variables, while for each predicted node variable x in this set, the set of predictor inputs is $\mathcal{N}_x$.

[6]An unmeasured path is a path that runs through nodes in $w_{\mathcal{Z}}$ only. Analogously, we can define unmeasured loop through a node w_i.

VI. MIMO identification setup

On the basis of the decomposition of node signals as defined in the previous section we are going to rewrite the system's equations (5) in the following structured form:

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{Q}\mathcal{Q}} & G_{\mathcal{Q}o} & G_{\mathcal{Q}\mathcal{B}} & G_{\mathcal{Q}\mathcal{A}} & G_{\mathcal{Q}\mathcal{Z}} \\ G_{o\mathcal{Q}} & G_{oo} & G_{o\mathcal{B}} & G_{o\mathcal{A}} & G_{o\mathcal{Z}} \\ G_{\mathcal{B}\mathcal{Q}} & G_{\mathcal{B}o} & G_{\mathcal{B}\mathcal{B}} & G_{\mathcal{B}\mathcal{A}} & G_{\mathcal{B}\mathcal{Z}} \\ G_{\mathcal{A}\mathcal{Q}} & G_{\mathcal{A}o} & G_{\mathcal{A}\mathcal{B}} & G_{\mathcal{A}\mathcal{A}} & G_{\mathcal{A}\mathcal{Z}} \\ G_{\mathcal{Z}\mathcal{Q}} & G_{\mathcal{Z}o} & G_{\mathcal{Z}\mathcal{B}} & G_{\mathcal{Z}\mathcal{A}} & G_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} + He \quad (6)$$

where we make the notation agreement that the matrix H is not necessarily monic, and the scaling of the white noise process e is such that $cov(e) = I$. Without loss of generality, we can assume $r = 0$ for the sake of brevity.

If we follow Algorithm A for the signal selection then we satisfy the following assumption.

Assumption 1: All w-in-neighbours of $w_{\mathcal{Y}}$ are collected in $w_{\mathcal{Q}\cup\mathcal{A}}$, and all disturbance signals $v_{\mathcal{A}}$ are uncorrelated to $v_{\mathcal{Y}}$.

Proposition 1: Under the conditions of Assumption 1 it follows that in (6), (a) $G_{\mathcal{Q}\mathcal{Z}} = G_{o\mathcal{Z}} = G_{\mathcal{Q}\mathcal{B}} = G_{o\mathcal{B}} = 0$; (b) $G_{oo} = 0$; (c) If w_o is present then $G_{\mathcal{Q}o} = 0$. □

Proposition 2: Under the conditions of Assumption 1, the system equations for the measured variables $w_{\mathcal{D}} \cup w_{\mathcal{Y}}$ can be written as

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{Q}\mathcal{Q}} & 0 & 0 & G_{\mathcal{Q}\mathcal{A}} \\ G_{o\mathcal{Q}} & 0 & 0 & G_{o\mathcal{A}} \\ \breve{G}_{\mathcal{B}\mathcal{Q}} & \breve{G}_{\mathcal{B}o} & \breve{G}_{\mathcal{B}\mathcal{B}} & \breve{G}_{\mathcal{B}\mathcal{A}} \\ \breve{G}_{\mathcal{A}\mathcal{Q}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{A}\mathcal{B}} & \breve{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \breve{H}e. \quad (7)$$

In the sequel we are going to formulate conditions on the choice of node variables in $w_{\mathcal{B}}$, such that the systems equations for the output variables in $w_{\mathcal{Y}}$ can be written as

$$\underbrace{\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \end{bmatrix}}_{w_{\mathcal{Y}}} = \underbrace{\begin{bmatrix} \bar{G}^0_{\mathcal{Q}\mathcal{Q}} & \bar{G}^0_{\mathcal{Q}\mathcal{B}} & \bar{G}^0_{\mathcal{Q}\mathcal{A}} \\ \bar{G}^0_{o\mathcal{Q}} & \bar{G}^0_{o\mathcal{B}} & \bar{G}^0_{o\mathcal{A}} \end{bmatrix}}_{\bar{G}^0} \underbrace{\begin{bmatrix} w_{\mathcal{Q}} \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix}}_{w_{\mathcal{D}}} + \underbrace{\begin{bmatrix} \bar{H}^0_{\mathcal{Q}\mathcal{Q}} & \bar{H}^0_{\mathcal{Q}o} \\ \bar{H}^0_{o\mathcal{Q}} & \bar{H}^0_{oo} \end{bmatrix}}_{\bar{H}^0} \underbrace{\begin{bmatrix} \xi_{\mathcal{Q}} \\ \xi_o \end{bmatrix}}_{\xi_{\mathcal{Y}}} \quad (8)$$

with $\xi_{\mathcal{Q}}$ and ξ_o white noise processes with dimensions conforming to $w_{\mathcal{Q}}$ and w_o, respectively, with $cov(\xi_{\mathcal{Y}}) = \bar{\Lambda}$ and with $\bar{H}^0$ being monic, stable and stably invertible. In the situation of a network system with the system's equations as in (8) we can set up a predictor model based on a parametrized model set determined by $\mathcal{M} := \{(\bar{G}(\theta), \bar{H}(\theta), \bar{\Lambda}(\theta)), \theta \in \Theta\}$, while the actual data generating system is represented by $\mathcal{S} = (\bar{G}(\theta_o), \bar{H}(\theta_o), \bar{\Lambda}(\theta_0))$. The corresponding identification problem is defined by considering the one-step-ahead prediction of $w_{\mathcal{Y}}$, according to $\hat{w}_{\mathcal{Y}}(t|t-1) := \mathbb{E}\{w_{\mathcal{Y}}(t) \mid w_{\mathcal{Y}}^{t-1}, w_{\mathcal{D}}^t\}$, where $w_{\mathcal{D}}^t$ denotes the past of $w_{\mathcal{D}}$, i.e. $\{w_{\mathcal{D}}(k), k \le t\}$. The resulting prediction error $\varepsilon(t,\theta) := w_{\mathcal{Y}}(t) - \hat{w}_{\mathcal{Y}}(t|t-1;\theta)$ then becomes:

$$\varepsilon(t,\theta) = \bar{H}(q,\theta)^{-1}\left[w_{\mathcal{Y}}(t) - \bar{G}(q,\theta)w_{\mathcal{D}}(t)\right], \quad (9)$$

and the weighted least squares identification criterion

$$\hat{\theta}_N = \arg\min_\theta \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon^T(t,\theta) W \varepsilon(t,\theta), \quad (10)$$

with W any positive definite weighting matrix. This parameter estimate then leads to an estimated subnetwork $G_{\mathcal{YD}}(q,\hat{\theta}_N)$, with the estimated target module $G_{ji}(q,\hat{\theta}_N)$ as a component of this.

VII. Main results

First we will formulate conditions for the selection of the blocking node variables $w_{\mathcal{B}}$, that will allow to derive consistent identification results next.

Property 1: Let the node signals $w_{\mathcal{B}}$ be chosen to satisfy the following properties:

1) If, in the original network (6), there are no confounding variables for the estimation problem $w_{\mathcal{A}} \rightarrow (w_{\mathcal{Q}}, w_o)$, then $\mathcal{B}$ is void implying that $w_{\mathcal{B}}$ is not present;
2) If, in the original network (6), there are confounding variables for the estimation problem $w_{\mathcal{A}} \rightarrow (w_{\mathcal{Q}}, w_o)$, then all of the following conditions are satisfied:
   a. For any confounding variable for the estimation problem $w_{\mathcal{A}} \rightarrow (w_{\mathcal{Q}}, w_o)$, the paths from the confounding variable to a node signal $w_{\mathcal{A}}$ is blocked by a node signal in $w_{\mathcal{B}}$, where the paths are either direct or unmeasured;
   b. For every simultaneous path from any e_k in e to node signals in $w_{\mathcal{B}}$ and $w_{\mathcal{A}}$, at least one of the paths should pass through nodes in $w_{\mathcal{L}\backslash\mathcal{Z}}$. Alternatively formulated: the nonmodelled disturbances on $w_{\mathcal{B}}$ and $w_{\mathcal{A}}$ are uncorrelated;
   c. There are no direct or unmeasured paths from w_i to node variables in $w_{\mathcal{B}}$;
   d. There are no direct or unmeasured paths from w_j to node variables in $w_{\mathcal{B}}$. □

Next we can formulate the main consistency result of this paper.

Theorem 1: Consider a (MIMO) network identification setup with predictor inputs $w_{\mathcal{D}}$ and predicted outputs $w_{\mathcal{Y}}$, satisfying the conditions of Assumption 1 (full input case). Then a prediction error identification method according to (9)-(10), applied to a parametrized model set $\mathcal{M}$ will provide a consistent estimate of the target module G^0_{ji}, if

1) $\mathcal{M}$ is chosen to satisfy $\mathcal{S} \in \mathcal{M}$;
2) The blocking node signals $w_{\mathcal{B}}$ are chosen to satisfy Property 1;
3) $\Phi_\kappa(\omega) > 0$ for a sufficiently high number of frequencies, where $\kappa(t) := \begin{bmatrix} w_{\mathcal{D}}^\top & \xi_{\mathcal{Q}}^\top & w_o \end{bmatrix}^\top$;
4) All the elements in $G_{\mathcal{QQ}}, G_{\mathcal{QA}}, G_{o\mathcal{Q}}, G_{o\mathcal{A}}$ are strictly proper (or) all existing paths/loops from $w_{\mathcal{Q}}, w_o, w_{\mathcal{B}}$ to $w_{\mathcal{Q}}$ and from $w_{\mathcal{Q}}, w_o, w_{\mathcal{B}}$ to w_o have at least a delay. □

Conditions 1,2,4 are (generalizations of) the typical consistency conditions for closed-loop and networked systems [10]. There are typically two major conditions for arriving at consistency of the target module G_{ji}: one needs to be able to deal with the confounding variables through the selection of an appropriate set of (blocking) node variables $w_{\mathcal{B}}$ that is included as predictor input, and there should be enough excitation present in the node signals. Note that this excitation condition may require that there are external excitation signals present at some locations, see also [22]. Note that since we are using a direct method for identification, the signals r are not directly used in the predictor model, although they serve the purpose of providing excitation in the network.

Since in the result of Theorem 1 we arrive at white innovation signals, the result can be extended to formulate Maximum Likelihood properties.

Theorem 2: Consider the situation of Theorem 1, and let the conditions for consistency be satisfied. Let $\xi_{\mathcal{Y}}$ be normally distributed, and let $\bar{\Lambda}(\theta)$ be parametrized independently from $\bar{G}(\theta)$ and $\bar{H}(\theta)$. Then, under zero initial conditions, the Maximum Likelihood estimate of θ^0 is

$$\hat{\theta}_N^{ML} = \arg\min_\theta \det\left(\frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\theta)\varepsilon^T(t,\theta)\right) \quad (11)$$

$$\Lambda(\hat{\theta}_N^{ML}) = \frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\hat{\theta}_N^{ML})\varepsilon^T(t,\hat{\theta}_N^{ML}). \quad (12)$$

Proof: Can be shown by following a similar reasoning as in Theorem 1 of [6]. □

VIII. Examples

In this section we will apply the developed local module identification methodology to two examples of dynamic networks. First we will consider the dynamic network in example 1 where v_2 and v_4 are mutually correlated while the other disturbance signals are uncorrelated with these and with each other. The target of identification is module G_{21}, and all node signals are available for measurements. Using the identification method developed in this paper, we first select the signals $w_{\mathcal{Q}}, w_o, w_{\mathcal{A}}$ using the algorithm A. Since v_2 and v_4 are correlated we choose them both as outputs. Consequently, w_1, w_3 and w_4 are chosen as inputs, so that

$$\mathcal{Y} = \{2,4\} \quad ; \quad \mathcal{D} = \{1,4,3\} \quad (13)$$

$$\mathcal{Q} = \mathcal{Y} \cap \mathcal{D} = \{4\} \quad ; \quad \mathcal{A} = \mathcal{D}\backslash\mathcal{Q} = \{1,3\} \quad (14)$$

$$w_o = w_2. \quad (15)$$

For the selection of $w_{\mathcal{B}}$, according to Property 1, we need to check the presence of confounding variables. Since all disturbance terms $v_k, k \in \mathcal{Z} \cup \mathcal{A}$ are uncorrelated to all disturbance terms $v_l, l \in \mathcal{Y}$, there are no confounding variables for the estimation problem $w_{\mathcal{A}} \rightarrow (w_{\mathcal{Q}}, w_o)$. Therefore $w_{\mathcal{B}}$ is void. Now we have the predictor inputs $w_{\mathcal{D}}$ and the predicted outputs $w_{\mathcal{Y}}$ for the MIMO identification setup that will satisfy the essential conditions of Theorems 1 and 2.

Example 2: Consider the network sketched in Figure 3, and let module G_{12} be the target module for identification. The disturbance correlation structure in the network is presented in Figure 3 with modules in red indicating the noise dynamics.

The direct method using a MISO predictor, as addressed in [10], does not provide a consistent estimate of G_{12} since the disturbance term v_1 is correlated with v_2 as well as v_3 and therefore we resort to the identification framework developed in this paper. Similar to the previous example, the first step will be selection of $w_{\mathcal{Q}}, w_o, w_{\mathcal{A}}$ using the algorithm A.

First we select w_1 as output and w_2 and w_3 as inputs. Since v_2 and v_3 are correlated with v_1, both w_2 and w_3

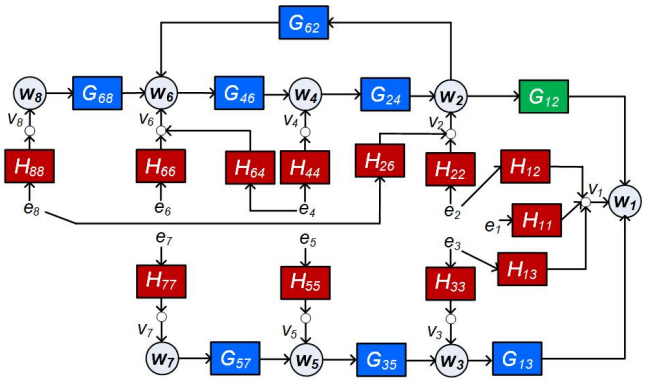

Fig. 3. Example network

need to be added as outputs too. Then w_4 and w_5 need to be added as inputs. As a result of the first six steps in algorithm A we get,

$$\mathcal{Y} = \{1,2,3\} \quad ; \quad \mathcal{D} = \{2,3,4,5\} \tag{16}$$

$$\mathcal{Q} = \mathcal{Y} \cap \mathcal{D} \ = \{2,3\} \quad ; \quad \mathcal{A} = \mathcal{D} \backslash \mathcal{Q} \ = \{4,5\} \tag{17}$$

$$w_o \ = \ w_1. \tag{18}$$

In the resulting situation e_8 acts as a confounding variable that affects both input w_4 and output w_2. As per condition 2a of Property 1, the path from $e_8 \rightarrow w_4$ should be blocked by a node signal in $w_{\mathcal{B}}$, which can be either w_8 or w_6. In order to choose the node signals $w_{\mathcal{B}}$, we also need the conditions 2b, 2c and 2d in Property 1 to be satisfied. w_6 cannot be chosen in $w_{\mathcal{B}}$ since it does not satisfy conditions 2b and 2c in Property 1. The former condition is not satisfied due to the simultaneous path from e_4 in $e_{\mathcal{A}}$ to w_6 and w_4 and the latter condition is not satisfied due to the path from w_2 in $w_i \rightarrow w_6$. When w_8 is chosen in $w_{\mathcal{B}}$, the conditions in Property 1 are satisfied and hence $\mathcal{B} = \{8\}$. Now we have the predictor inputs $w_{\mathcal{D}}$ and the predicted outputs $w_{\mathcal{Y}}$ for the MIMO identification setup that provide the consistent and maximum likelihood estimation results of G_{12}.

IX. CONCLUSIONS

A new local module identification approach has been presented to identify local modules in a dynamic network with given topology, addressing the situation that process noise on different nodes can be correlated with each other. For this case, it is shown that the problem can be solved by moving from a MISO to a MIMO identification setup. In this setup the target module is embedded in a MIMO problem with appropriately chosen inputs and outputs, that warrant the consistent estimation of the target module with maximum likelihood properties. A key part of the procedure is the handling of direct and indirect confounding variables, through the introduction of appropriately chosen additional predictor input node signals (blocking nodes) and predicted output node signals respectively. We have considered the "full input" case, implying that all in-neighbours of an output node are included as input. A further relaxation of this condition and a generalized theory is presented in [23]. The presented approach has been illustrated by two examples.

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