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Path-based data-informativity conditions for single module identification in dynamic networks

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***Abstract*— For identification of a single module in a dynamic network several conditions need to be fulfilled for arriving at consistent or minimum variance model estimates. Conditions for single module identifiability prescribe the presence of external signals, either measured or unmeasured, allowing different models in an appropriate model set to be distinguishable from each other, on the basis of a set of measured node signals. While conditions for identifiability are independent of the chosen identification method, the particular method, or estimation setup, will require additional data-informativity conditions to be satisfied. The setup of the estimation problem is not unique, as there are different methods for arriving at local module estimates. In a prediction error setting there are e.g., direct and indirect approaches, being distinguished by the choice of predictor inputs and outputs in the estimation setting. In this paper we will formalize the notion of data-informativity for a general estimation setup that covers both direct and indirect methods, and we will specify path-based conditions on the network that guarantee data-informativity in a generic setting, i.e. independent on numerical values of the network transfer functions concerned.**

I. INTRODUCTION

Based on the motivating reasoning that many systems in our current days scientific and technological environment are interconnected dynamic subsystems, the research community involved in data-driven modeling of systems has shown an increasing interest in the problem of identifying a local (single) module in an interconnected dynamic network of which the interconnection structure (topology) is given. In [1] this problem has been formalized in a prediction error identification setting where local subsystems are described by linear dynamic systems in the form of transfer functions, based on the work of [2]. Classical methods for closed-loop identification have been generalized to be applicable in this network situation, typically leading to multiple-input single output type of estimation problems where the target module is embedded in a larger predictor model, and where consistency of the target module estimate is obtained. Reducing the number of to be measured node signals has been addressed in [3], [4]. For direct methods of identification, further specification of these results has been established by handling confounding variables and correlated disturbances and by deriving minimum variance results [5], [6]. This has led to a so-called joint-direct method of single module identification [7], that provides different scenarios for the selection of node signals to be measured. For indirect methods of identification consistency results for local module estimates are provided in [8]. For all of these methods data-informativity conditions need to be satisfied for arriving at consistent module estimates. While for indirect methods these conditions can typically be phrased in terms of conditions on external excitation signals, see e.g. [8], for direct methods they are typically formulated in terms of a spectral condition on node signals in the network, and thereby harder to interpret for the user who has to set up an experiment. This has also been addressed in [9] where it has been highlighted that the typical spectral conditions will often be conservative in case of finite modules with finite model order.

Separate from the introduction of identification methods to identify single modules, the question of single module identifiability has been considered in [10], [11], [12], [13], [14], showing that identifiablity, being independent of the identification method chosen, puts conditions on the presence and location of external signals in the network, and the measurability of node signals in the network. The external signals that need to take care for sufficient excitation in the network can be either external excitation signals that are under control of the user, or disturbance signals that are not under control. If sufficient excitation is provided by measurable external excitation signals, then an indirect identification method can be applied [13]. However if (non-measured) disturbance signals are required for satisfying the identifiability conditions, then one has to resort to a direct method [7] or a generalized/combined direct/indirect method [15].

In this paper we are going to address the situation of the direct method [7], and we are going to reformulate the data-informativity conditions for this method in terms of excitation conditions of the external excitation signals, together with path-based conditions on the topology of the network model set. In this way the data-informativity conditions become verifiable by the user, rather than remaining implicit. After recalling the dynamic network setup in Section II, we will highlight the different options for selecting predictor models in Section III. In Section IV data-informativity conditions are specified, for which path-based conditions are being derived in Section V. The result are being illustrated with examples.

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II. Dynamic networks

Following the setup of [1], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1, \ldots, L$, and K *external variables* r_k, $k = 1, \ldots K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^{L} G_{jl}(q) w_l(t) + u_j(t) + v_j(t) \tag{1}$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl} are proper rational transfer functions, referred to as *modules*. In order to avoid technicalities in this paper and without loss of generality we will assume that the modules are strictly proper.
- u_j is an input signal, $u_j(t) = \sum_{k=1}^{K} R_{jk}(q) r_k(t)$ with r_k *external variables* that can directly be manipulated by the user.
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase.

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1\ L} \\ G_{L1} & \cdots & G_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + Rr + He, \tag{2}$$

where by construction the matrix G is hollow, i.e. it has diagonal entries 0. The single module identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of measured time-series of a subset of variables in w, and possibly r.

III. Network estimation setup

We can distinguish two main different approaches for addressing the single module identification problem, where the target module is indicated by G_{ji}.

1) A *direct method*, that is based on selecting a particular set of predictor input signals w_k, $k \in \mathcal{D}$, and a set of predicted output signals w_ℓ, $\ell \in \mathcal{Y}$, with $i \in \mathcal{D}$, $j \in \mathcal{Y}$, and estimating a dynamic model based on a prediction error:

$$\varepsilon(t, \theta) = \bar{H}(q, \theta)^{-1}[w_{\mathcal{Y}}(t) - \bar{G}(q, \theta) w_{\mathcal{D}}(t)], \tag{3}$$

where $\bar{G}(q, \theta)$ and $\bar{H}(q, \theta)$ are parametrized transfer function matrices. The target module is then embedded in the model $\bar{G}(q, \theta)$, and the objective is to estimate the target module consistently and possibly with minimum variance.

2) An *indirect method*, that is based on selecting a particular set of external excitation signals r_k, $k \in \mathcal{P}$, and a set of predicted node signals w_ℓ, $\ell \in \mathcal{Y}$, that are used in a predictor model, leading to

$$\varepsilon_{\mathcal{Y}}(t, \theta) \quad = \quad w_{\mathcal{Y}}(t) - \bar{T}_{\mathcal{Y}}(q, \theta) r_{\mathcal{P}}(t) \tag{4}$$

Since $\bar{T}_{\mathcal{Y}}$ reflects a mapping from external signals (r) to internal signals ($w_{\mathcal{Y}}$), a processing step is necessary to recover the target module G_{ji} from an estimated $\bar{T}_{\mathcal{Y}}$. In the most straightforward version, the node set $w_{\mathcal{Y}}$ is composed of the output of the target module (w_j) and all of its in-neighbours in the parametrized model graph. Consistency of the target module estimate is the typical objective. Different variations of indirect methods exist, including two-stage and instrumental variable (IV) methods.

3) As introduced in [15], the above two methods can be combined to a generalized approach where both external signals $r_{\mathcal{P}}$ and node signals $w_{\mathcal{D}}$ are used as predictor inputs.

The choice for any of the methods above relies on the availability of measured external excitation signals that can guarantee the network identifiability of the target module in the considered model set. Network identifiability typically requires the presence of external signals (r,e) for making sure that different network models can be distinguished, see e.g. [10], [11] for the situation of only considering r-signals, and [12], [14] for the handling of noise signals e too. If network identifiability conditions are satisfied for r signals only, then the above indirect method can be applied. However, if the presence of noise signals e is required for satisfying the identifiability conditions, then an indirect method will fail and a direct or generalized approach needs to be followed. Note that all of these considerations are essentially dependent on the set of node signals that can be measured.

Having a generalized method that allows both r-signals and node signals w to serve as predictor inputs, allows for the highest level of flexibility to deal with situations where a limited set of node signals is available for measurements. However in order to reduce the complexity of our expositions, in this paper we will only use this flexibility in a limited setting.

In order to arrive at a consistent estimate of our target module, while relying on prediction error type of identification methods, there are three sets of conditions that jointly can lead to a consistent estimate of the target module.

1) The target module G_{ji} is *network identifiable* in the considered model set. This aspect can be treated in the context of global network identifiability [16], [12] or can be considered in a generic setting [11], [13], [14] for which attractive path-based conditions are available. The results are typically dependent on the selection of node signals that can be measured.
2) An identification method and estimation setup needs to be chosen, on the basis of which it is possible to reconstruct the target module from the estimated

objects $\bar{G}$ or $\bar{T}$. The estimation setup typically comes down to the choice of a predictor model (3) or (4), with a specification of signals that appear in $w_{\mathcal{Y}}$, $w_{\mathcal{D}}$ and posibly $r_{\mathcal{P}}$.

3) In the chosen estimation setup, the data appearing in this setup should be sufficiently informative so as to guarantee that consistent estimates of the objects $\bar{G}$, $\bar{H}$ and possibly $\bar{T}$ are obtained.

The network identifiability question has been extensively studied, and in particular in the generic setting leads to path-based conditions on the presence of external excitation and disturbance signals. Both types of signals can contribute to satisfaction of the identifiability conditions.

IV. DATA-INFORMATIVITY

We consider an estimation setup on the basis of the network equations

$$w_{\mathcal{Y}}(t) = \bar{G}(q)w_{\mathcal{D}}(t) + \bar{R}(q)r_{\mathcal{P}}(t) + \bar{H}(q)\xi(t) \qquad (5)$$

with $w_{\mathcal{Y}}$, $w_{\mathcal{D}}$, $r_{\mathcal{P}}$ selected node and excitation signals and ξ a stationary white noise process.

The one-step ahead predictor for (5) is uniquely defined through[1]

$$\hat{w}_{\mathcal{Y}}(t) \quad := \quad \bar{\mathbb{E}}\{w_{\mathcal{Y}}(t)|w_{\mathcal{Y}}^{t-1}, w_{\mathcal{D}}^{t}, r_{\mathcal{P}}^{t}\} = W(q)z(t) \qquad (6)$$

with the predictor filter given by

$$W(q) := \begin{bmatrix}(1-\bar{H}(q)^{-1}) & \bar{H}(q)^{-1}\bar{G}(q) & \bar{H}(q)^{-1}\bar{R}(q)\end{bmatrix} \qquad (7)$$

and

$$z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ w_{\mathcal{D}}(t) \\ r_{\mathcal{P}(t)} \end{bmatrix}. \qquad (8)$$

In line with the corresponding definitions in the prediction error literature ([17], Definition 8.1), we can now define the notion of data-informativity for the related network predictor model.

Definition 1: Consider a set of network signals contained in z and a network predictor model

$$\hat{w}_{\mathcal{Y}}(t,\theta) = W(q,\theta)z(t)$$

for a parametrized set of models

$$\mathcal{M} := (\bar{G}(q,\theta), \bar{R}(q,\theta), \bar{H}(q,\theta))_{\theta\in\Theta}.$$

Then a quasi-stationary data set $Z^{\infty} := \{z(t)\}_{t=0,\cdots}$ with $z(t)$ defined in (8) is *informative enough with respect to the model set* $\mathcal{M}$ if, for any two predictor models $W_1(q)$ and $W_2(q)$ in the model set,

$$\bar{\mathbb{E}}[(W_1(q) - W_2(q))z(t)]^2 = 0$$

implies that $W_1(e^{i\omega}) \equiv W_2(e^{i\omega})$ for almost all ω. □

In line with ([17], Definition 8.2), we formulate:

Definition 2: A quasi-stationary data set Z^{∞} is *informative* if it is informative enough with respect to the model set $\mathcal{L}^*$, consisting of all linear time-invariant models.

[1]The notation w^t refers to the past information $\{w(k)\}_{k=-\infty,\cdots,t-1,t}$.

And in line with ([17], Definition 13.2):

Definition 3: A quasi-stationary signal z is said to be *persistently exciting* if $\Phi_z(\omega) > 0$ for almost all ω.

The essential difference with the classical definitions in [17] is in the composition of the signal vector $z(t)$, being composed according to (8).

A. Classical open-loop case

The classical open-loop case can be represented by the situation that in the predictor model, the predictor input is r, implying that $w_{\mathcal{D}}$ is void.

In this case

$$z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ r(t) \end{bmatrix}.$$

The well known condition for data-informativity is now [17]:

$$\Phi_z(\omega) > 0 \quad \text{for almost all } \omega. \qquad (9)$$

Note that the condition includes both predictor input and predictor outputs. Since there are output disturbances on $w_{\mathcal{Y}}$ that are uncorrelated to r, the informativity condition simplifies to the condition that r should be persistently exciting. This "open-loop" situation, applies to the indirect identification method in closed-loop or in networks, since in this method the predictor input signals are all external excitation signals that are uncorrelated to disturbance signals present in the network.

B. Classical closed-loop case: direct method

The direct method for closed-loop systems is characterized by the situation that in the predictor model

- $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ are distinct signals;
- $r_{\mathcal{P}}$ is not included in the predictor;
- $w_{\mathcal{D}}$ may depend of the present and past samples of $w_{\mathcal{Y}}$ (feedback).

It follows that $z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ w_{\mathcal{D}}(t) \end{bmatrix}$ and the "open-loop" results of [17] still apply, i.e. the informativity condition of the data is represented by the condition (9).

C. The network case: local direct method

When applying a direct identification method in the network case, the resulting predictor model will not include any external excitation signals r, but will have predictor inputs $w_{\mathcal{D}}$ and predicted outputs $w_{\mathcal{Y}}$ that may have common signals, due to the handling of correlated disturbances, as discussed in [6], [7]. These correlated disturbances can appear as so-called confounding variables, i.e. non-measured signals that affect both the input and output of an estimation problem, and they can be properly handled by including the related input signals as output too, and by exploiting a multivariate noise model to cover the correlated disturbances. In this situation the predictor model is represented as depicted in Figure 1. In the multi-input multi-output predictor model, we distinguish the following terms:

- $w_{\mathcal{Y}} = \begin{bmatrix} w_o \\ w_{\mathcal{Q}} \end{bmatrix}$; $w_{\mathcal{D}} = \begin{bmatrix} w_{\mathcal{U}} \\ w_{\mathcal{Q}} \end{bmatrix}$;
- $w_o = w_j$ or w_o is void if w_j is present in $w_{\mathcal{Q}}$;

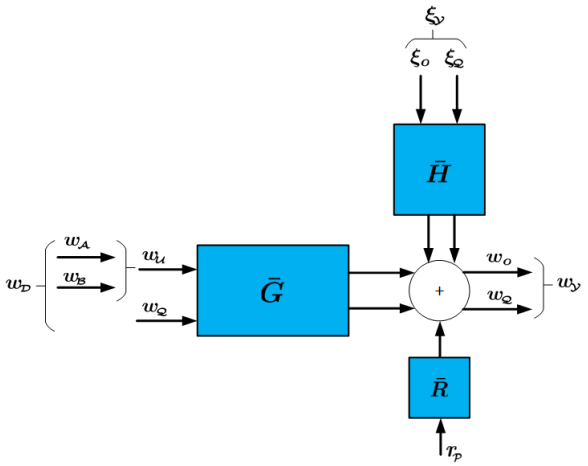

Fig. 1. Predictor model for local direct identification; the set of node signals $w_{\mathcal{Q}}$ appears both at the input and at the output of the predictor model.

- $r_{\mathcal{P}}$ contains those measured external excitation signals in r that add directly to measured outputs w_k, $k \in \mathcal{Y}$, for which every loop around w_k passes through a node in $w_{\mathcal{Y}\cup\mathcal{U}}$. Therefore $R(q)$ is a binary (selection) matrix with known elements, indicating which output signals are excited by signals $r_{\mathcal{P}}$.

Inputs and outputs are allowed to share some common signals, while all node signals are allowed to depend on each other's (present and) past. According to the consistency results in [7] the data-informativity conditions now become:

$$\Phi_\kappa(\omega) > 0 \quad \text{for almost all } \omega, \tag{10}$$

with

$$\kappa(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ u_{\mathcal{U}}(t) \\ \xi_{\mathcal{Q}}(t) \end{bmatrix}$$

and $\xi_{\mathcal{Q}}(t)$ the white noise innovation process that relates to output $w_{\mathcal{Q}}$ in (5). In the vector signal κ we collect all the measured node signals $(w_{\mathcal{Y}}, u_{\mathcal{U}})$, but on top of that we need to include the noise terms $\xi_{\mathcal{Q}}$, which appear "extra". Since these white noise terms $\xi_{\mathcal{Q}}$ also appear in a filtered version in $w_{\mathcal{Y}}$, this implies that the related output $w_{\mathcal{Q}}$ that is a subset of $w_{\mathcal{Y}}$ needs to contribute to the positive definite spectrum of κ through excitation signals that are different from $\xi_{\mathcal{Q}}$. In other words, $\xi_{\mathcal{Q}}$ can not be used for the "excitation" of the signals $w_{\mathcal{Q}}$, but this excitation has to come from other external signals in the network. This mechanism is going to be further elaborated upon in the next Section.

V. Path-based conditions for data-informativity in the direct method

A. General results

The condition (10) for data-informativity in the direct method is compactly formulated, but it is actually implicit and hard to check for the situation of a dynamic network with given topology and unknown dynamics. It would be very attractive to formulate this condition in terms of properties and locations of the external signals in the network (i.e. r and e) together with topological conditions on the interconnections structure in the network models that we consider. In order to achieve this objective, we consider the following Lemma:

Lemma 1: Let $x(t) \in \mathbb{R}^m$ be a quasi-stationary signal that is persistently exciting, and let $F(z) \in \mathbb{R}(z)^{p\times m}$ be the proper rational transfer function of a stable filter. Then the signal $y(t) = F(q)x(t)$ is persistently exciting if and only if filter $F(z)$ has rank p over the field of rational functions. □

Proof: Collected in the Appendix. ■

If we apply this Lemma with x-signals being the external signals r, e, and y signals being selected node signals w in the network, then the row rank of the considered transfer function $(r, e) \to y$ would need to be evaluated in order to make a statement about data-informativity. In line with the idea of introducing a generic form of identifiability [11], i.e. independent of particular numerical values of coefficients, we can use the same generic type of result for data-informativity, based on the results of [18].

Proposition 1: Consider the situation of Lemma 1. The property that $y(t)$ is persistently exciting holds generically[2] if in the dynamic network there are p vertex-disjoint paths between the nodes x and y. This is denoted by $b_{x\to y} = p$.

So, a persistently exciting "input" signal x and a sufficient number of vertex-disjoint paths betwen x and y, will generically provide a persistently exciting "output" signal y. This result can be used to translate persistence of excitation conditions on node signals, to persistence of excitation conditions on external network signals.

In order to further specify the data-informativity conditions that apply to the direct method, we need some formal results from [7] that concern the conditions under which the chosen predictor model will lead to results that leave the target module G_{ji} invariant in our estimation setup. This step actually refers to the second set of conditions as formulated in Section III, i.e. the choice of predictor model, and specifies conditions under which $\bar{G}_{ji}(q) = G^0_{ji}(q)$. Satisfying these conditions helps to further simplify the topological conditions for data-informativity.

Theorem 1 (Module invariance result [7]): Let G_{ji} be the target network module. In the system's equation (5) conformable to the network model depicted in Figure 1, it holds that $\bar{G}_{ji} = G^0_{ji}$ under the following conditions:

a. Every parallel path from w_i to w_j[3] and every loop around w_j passes through a measured node in $w_{\mathcal{Y}\cup\mathcal{U}}$, and
b. $\mathcal{U}$ is decomposed into two disjunct sets, $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$, such that there are no confounding variables[4] for the estimation problems $w_{\mathcal{A}} \to w_{\mathcal{Y}}$ and $w_{\mathcal{A}} \to w_{\mathcal{B}}$, and
c. $i \in \{\mathcal{A} \cup \mathcal{Q}\}$, and
d. Every path from $\{w_i, w_j\}$ to $w_{\mathcal{B}}$ passes through a measured node in $w_{\mathcal{Y}\cup\mathcal{U}}$. □

[2]Generically has to be considered here in terms of a Lebesgue measure 0 of the vector of coefficient values of the rational transfer functions in all modules of the network.

[3]A parallel path is a path from w_i to w_j that does not pass through G_{ji}.

[4]A confounding variable for the estimation problem $w_{\mathcal{A}}$ to $w_{\mathcal{Y}}$ is an unmeasured variable in the network that has paths to both $w_{\mathcal{A}}$ and $w_{\mathcal{Y}}$, that do not pass through a measured node in $w_{\mathcal{Y}\cup\mathcal{U}}$.

The interpretation of the decomposition of $\mathcal{U}$ into $\mathcal{A} \cup \mathcal{B}$ is that the signals in $w_{\mathcal{B}}$ can be used to block the effect of confounding variables in the estimation problem $w_{\mathcal{A}} \to w_{\mathcal{Y}}$, while confounding variables in the estimation problem $w_{\mathcal{B}} \to w_{\mathcal{Y}}$ are allowed. As a result the transfer functions $\bar{G}_{jk}$ with $k \in \{\mathcal{Q} \cup \mathcal{A}\}$ are invariant, i.e. $\bar{G}_{jk} = G^0_{jk}$.

B. Path-based conditions

The result on vertex-disjoint paths, as formulated in Proposition 1 can now be applied to the particular situation of condition (10). In this step the consequence of having the white noise signal ξ in the condition (10) needs to be translated to conditions on signals in the original network (2).

Theorem 2: Consider a dynamic network with external signals r and e, and let $r_{\mathcal{P}}$ be the r-signals that appear as predictor input in the setting of the local direct method. Define e_{sel^*} (resp. r_{sel^*}) as

- void, if $w_{\mathcal{Q}}$ is void, (else)
- the subset of signals in e (resp. $r \backslash r_{\mathcal{P}}$)[5] for which there is an unmeasured path[6] to a node signal in $w_{\mathcal{Y} \cup \mathcal{B}}$,

and define $e_{sel} := e \backslash e_{sel^*}$ and $r_{sel} := r \backslash r_{sel^*}$.
Then the transfer function from (r_{sel}, e_{sel}) to κ generically has full row rank if there are $n_{\mathcal{Y}} + n_{\mathcal{U}}$ vertex disjoint paths between external signals (r_{sel}, e_{sel}) and $w_{\mathcal{Y} \cup \mathcal{U}}$. □

Proof: The proof is added in the Appendix. ■

As a direct result of Proposition 1 we can now formulate the following Corollary:

Corollary 1: The data-informativity condition (10) for the local direct method is satisfied if the path-based conditions of Theorem 2 are satisfied and r_{sel} (if present) is persistently exciting.

The Corollary shows that in the case where there are common signals in the input and output of the predictor model, except for signals $r_{\mathcal{P}}$, only those external signals that do not have unmeasured paths to $w_{\mathcal{Y} \cup \mathcal{B}}$ can serve as signals that provide sufficient excitation for data-informativity. In other words: the more signals appear in $w_{\mathcal{Q}}$, the stronger the requirements on the presence of external signals. This is specified in the next Corollary.

Corollary 2: If $\mathcal{Q}$ is not void, we need at least $n_{\mathcal{Y}} + n_{\mathcal{B}}$ external signals r_{sel} in the network to satisfy the conditions of Corollary 1.

Proof: The proof is added in the Appendix. ■

We will illustrate the results of this Section in two examples.

Example 1: Consider a classical closed loop system represented by a two-node network as depicted in Figure 2 with v_1 and v_2 being process noises that are correlated. First we consider the situation of having no external excitation signals, $r_1 = r_2 = 0$. The objective is to identify the target module G_{21}. Due to the correlation between v_1 and v_2, we need to choose a predictor model where both node signals w_1 and w_2 are predicted, and therefore serve as outputs, according to [6], [7]. As a result $w_{\mathcal{Y}} = w_{\mathcal{Q}} = \{w_1, w_2\}$, and $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are void. In order to satisfy the data informativity condition according to Theorem 2, we need two vertex disjoint paths between (r_{sel}, e_{sel}) and $w_{\mathcal{Y} \cup \mathcal{U}}$. However since both e_1 and e_2 have unmeasured paths to $w_{\mathcal{Y}}$, e_{sel} will be void. Hence, we do not have data-informativity unless we have 2 external excitation signals, one on each of the two nodes, and that are included in $r_{\mathcal{P}}$ as they directly add to node signals in the output $w_{\mathcal{Y}}$. The requirement of two external excitations is attributed to the need to model both G_{21} and G_{12} since the two noise signals have been used to model the (2×2) noise model. This result is in agreement with the observations in [19], that adding one excitation signal only does not suffice to consistently identify G_{21}.

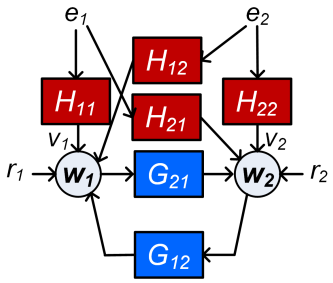

Fig. 2. Classical closed loop example with two node signals and disturbances v_1 and v_2 being correlated.

Example 2: Consider the three node network depicted in Figure 3 with v_1 and v_3 being disturbance signals that are correlated. First we consider the situation of having no external excitation signals, $r_1 = r_2 = r_3 = 0$. The objective is to identify the target module G_{12}. According to the local direct method [7], we have multiple ways to choose the identification setup in terms of the predictor model. Following the full input case [6], [7], we choose $w_{\mathcal{Y}} = w_1$, $w_{\mathcal{A}} = w_2$, and then we choose $w_{\mathcal{B}} = w_3$ in order to block the effect of the confounding variable e_3 for the estimation problem $w_2 \to w_1$. In this setup $w_{\mathcal{Q}}$ is void. The data-informativity conditions of Theorem 2 now require three vertex disjoint paths between (r_{sel}, e_{sel}) and $w_{\mathcal{Y} \cup \mathcal{A} \cup \mathcal{B}}$ with r_{sel} absent and $e_{sel} = e$ since $w_{\mathcal{Q}}$ is void. Hence, we can include all the three noise signals in e_{sel}. It can simply be verified that we satisfy the vertex disjoint path conditions without any need for external excitation signals.

When choosing an alternative predictor model, e.g. according to the minimum input case algorithm in [7], we choose $w_{\mathcal{Y}} = w_{\mathcal{Q}} = \{w_1, w_2\}$, i.e. we model w_2 as output also, in order to deal with the confounding variable e_3 for the estimation problem $w_2 \to w_1$. In this setup $w_{\mathcal{U}} = w_{\mathcal{A} \cup \mathcal{B}}$ is void. The data-informativity conditions of Theorem 2 now require two vertex disjoint paths between (r_{sel}, e_{sel}) and (w_2, w_1), but now none of the e-signals is allowed to be incorporated in e_{sel} because $w_{\mathcal{Q}} = \{w_1, w_2\}$, and all e-signals have unmeasured paths to $w_{\mathcal{Y}}$. Hence, we do not have data-informativity unless we have 2 external excitation

[5] With slight abuse of notation, we denote $r \backslash r_{\mathcal{P}}$ as those components in r that are not present in $r_{\mathcal{P}}$.

[6] An unmeasured path is a path that does not pass through a node in $w_{\mathcal{Y} \cup \mathcal{U}}$.

signals, a result which is in accordance with the statement in Corollary 2. In the current setting of the local direct method having excitation signals r_1, r_2 constituting $r_{\mathcal{P}}$ would be sufficient to guarantee data-informativity. An excitation signals r_3 would not help as this would simply be considered as an additional disturbance input in the considered local direct method.

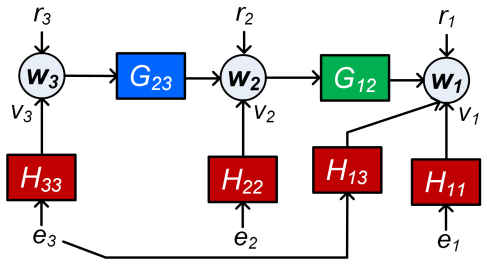

Fig. 3. A three node network example.

VI. CONCLUSIONS

For consistent identification of a single module that is embedded in a dynamic network it is necessary that the signals that constitute the chosen predictor model satisfy data-informativity conditions. We have formalized the concept of data-informativity for a generalized predictor model that is suited for dynamic network modeling, and that allows for signals to appear both as input and as output in a MIMO predictor model. It generalizes all known situations of indirect and direct methods in closed-loop systems and dynamic networks. The conditions for data-informativity have been specified for a particular identification method, the local direct method, showing that the conditions can be satisfied generically by requiring persistence of excitation of external signals, together with path-based conditions on the topology of the network model set.

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REFERENCES

[1] P. M. J. Van den Hof, A. G. Dankers, P. S. C. Heuberger, and X. Bombois, "Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates," *Automatica*, vol. 49, no. 10, pp. 2994–3006, 2013.

[2] J. Gonçalves and S. Warnick, "Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks," *IEEE Trans. Automatic Control*, vol. 53, no. 7, pp. 1670–1674, Aug. 2008.

[3] A. G. Dankers, P. M. J. Van den Hof, P. S. C. Heuberger, and X. Bombois, "Identification of dynamic models in complex networks with prediction error methods: Predictor input selection," *IEEE Trans. on Automatic Control*, vol. 61, no. 4, pp. 937–952, 2016.

[4] D. Materassi and M. V. Salapaka, "Signal selection for estimation and identification in networks of dynamic systems: a graphical model approach," *ArXiv preprint arXiv:1905.12132*, 2019.

[5] A. G. Dankers, P. M. J. Van den Hof, D. Materassi, and H. H. M. Weerts, "Relaxed predictor input selection rules for handling confounding variables in dynamic networks," *IFAC-PapersOnLine*, vol. 50-1, pp. 3983–3988, 2017, proc. 20th IFAC World Congress.

[6] P. M. J. Van den Hof, K. R. Ramaswamy, A. G. Dankers, and G. Bottegal, "Local module identification in dynamic networks with correlated noise: the full input case," in *Proc. 58th IEEE Conf. on Decision and Control (CDC)*, Nice, France, 2019, pp. 5494–5499.

[7] K. R. Ramaswamy and P. M. J. Van den Hof, "A local direct method for module identification in dynamic networks with correlated noise," Tech. Rep., 2019, arXiv:1908.00976. Provisionally accepted by IEEE Trans. Automatic Control.

[8] M. Gevers, A. Bazanella, and G. Vian da Silva, "A practical method for the consistent identification of a module in a dynamical network," *IFAC-PapersOnLine*, vol. 51-15, pp. 862–867, 2018, proc. 18th IFAC Symp. System Identif. (SYSID2018).

[9] M. Gevers and A. S. Bazanella, "Identification in dynamic networks: identifiability and experiment design," in *Proc. 2015 IEEE 54th Conf. Decision and Control, Osaka, Japan*, 2015, pp. 4006–4011.

[10] A. S. Bazanella, M. Gevers, J. M. Hendrickx, and A. Parraga, "Identifiability of dynamical networks: which nodes need to be measured?" in *Proc. 56th IEEE Conference on Decision and Control (CDC)*, 2017, pp. 5870–5875.

[11] J. Hendrickx, M. Gevers, and A. Bazanella, "Identifiability of dynamical networks with partial node measurements," *IEEE Trans. Autom. Control*, vol. 64, no. 6, pp. 2240–2253, 2019.

[12] H. H. M. Weerts, P. M. J. Van den Hof, and A. G. Dankers, "Single module identifiability in linear dynamic networks," in *Proc. 57th IEEE Conf. on Decision and Control (CDC)*. Miami Beach, FL: IEEE, 2018, pp. 4725–4730.

[13] A. S. Bazanella, M. Gevers, and J. M. Hendrickx, "Network identification with partial excitation and measurement," in *Proc. 58th IEEE Conference on Decision and Control (CDC)*, 2019, pp. 5500–5506.

[14] S. Shi, X. Cheng, and P. M. J. Van den Hof, "Excitation allocation for generic identifiability of a single module in dynamic networks: A graphic approach," Tech. Rep., 2020, to appear in Proc. IFAC World Congress, 2020.

[15] K. R. Ramaswamy, P. M. J. Van den Hof, and A. G. Dankers, "Generalized sensing and actuation schemes for local module identification in dynamic networks," in *Proc. 58th IEEE Conf. on Decision and Control (CDC)*, Nice, France, 2019, pp. 5519–5524.

[16] H. H. M. Weerts, P. M. J. Van den Hof, and A. G. Dankers, "Identifiability of linear dynamic networks," *Automatica*, vol. 89, pp. 247–258, March 2018.

[17] L. Ljung, *System Identification: Theory for the User*. Englewood Cliffs, NJ: Prentice-Hall, 1999.

[18] J. van der Woude, "A graph-theoreric characterization for the rank of the transfer matrix of a structured system," *Mathematics of Control, Signals, and Systems*, vol. 4, pp. 33–40, 1991.

[19] P. M. J. Van den Hof, A. G. Dankers, and H. H. M. Weerts, "From closed-loop identification to dynamic networks: generalization of the direct method," in *Proc. 56nd IEEE Conf. on Decision and Control (CDC)*. Melbourne, Australia: IEEE, 2017, pp. 5845–5850.

APPENDIX

A. Proof of Lemma 1

The spectral density of the output signal is given by $\Phi_y(\omega) = F(e^{i\omega})\Phi_x(\omega)F(e^{i\omega})^\star$, with $(\cdot)^\star$ the complex conjugate. For each value of ω in $-\pi \leq \omega \leq \pi$, this is a matrix multiplication for which it holds that $\Phi_y(\omega) > 0$ only if $rank_{\mathbb{R}}(F(e^{i\omega})) = p$ and $\Phi_x(\omega) > 0$. If $rank_{\mathbb{R}(z)}(F(z)) = p$ then $rank_{\mathbb{R}}(F(e^{i\omega})) = p$ for almost all ω. Since $\Phi_x(\omega) > 0$ for almost all ω this implies that $\Phi_y(\omega) > 0$ for almost all ω. If $rank_{\mathbb{R}(z)}(F(z)) < p$ then $rank_{\mathbb{R}}(F(e^{i\omega})) < p$ for all ω and there will be no value of ω for which $\Phi_y(\omega) > 0$. □

B. Proof of Theorem 2

The proof of this Theorem is based on an extensive proof in [7], as a result of which we will only provide the main line of reasoning here.

Resulting from Lemma 1 and Proposition 1, the transfer function from (r, ξ) to κ generically has full row rank if there are $n_y + n_u + n_q$ vertex disjoint paths between external

signals (r, ξ) and κ. However, since $\xi_{\mathcal{Q}}$ is a component of κ as well as of ξ, the condition can be reformulated that, for the transfer function from (r, ξ) to κ to be generically full row rank, we need $n_{\mathcal{Y}} + n_{\mathcal{U}}$ vertex disjoint paths between external signals $(r, \xi \backslash \xi_{\mathcal{Q}})$, and $w_{\mathcal{Y} \cup \mathcal{U}}$. So, in the input of the considered transfer function we need to exclude the signals in $\xi_{\mathcal{Q}}$. Since ξ is the white noise innovation signal related to the predictor model (5), we need to reformulate the condition in terms of the signals e and r in the full network equation (2). We are going to do this by, instead of excluding $\xi_{\mathcal{Q}}$, we are going to exclude all signals (r, e) that do not directly appear in the predictor model, and that contribute to (i.e. have a path to) $\xi_{\mathcal{Q}}$.

Referring to the expression in [7] for $\xi_{\mathcal{Y}}$, we have

$$\xi_{\mathcal{Y}} \quad = \quad \bar{H}_s^{\infty} D \tilde{\Gamma}_r^{-1} \tilde{\xi}_{\mathcal{Y}} = F \tilde{\xi}_{\mathcal{Y}} \tag{11}$$

where $\bar{H}_s^{\infty} := \lim_{z \to \infty} \bar{H}_s$ where $\bar{H}_s$ is a stable and minimum phase rational matrix, and D an "all pass" stable rational matrix satisfying $DD^{\star} = I$, and $\tilde{\Lambda}_{\mathcal{Y}}$ is the covariance matrix of $\tilde{\xi}_{\mathcal{Y}}$, that can be decomposed as $\tilde{\Lambda}_{\mathcal{Y}} = \tilde{\Gamma}_r \tilde{\Gamma}_r^T$. Directly resulting from (11), we have $\xi_{\mathcal{Q}} = F_{\mathcal{Q}} \tilde{\xi}_{\mathcal{Y}}$ with $F_{\mathcal{Q}}$ representing the matrix with the rows belonging to $w_{\mathcal{Q}}$. The next step is to represent the signal $\tilde{\xi}_{\mathcal{Q}}$ in terms of noise signals in e. From [7], we have

$$\tilde{\xi} = \tilde{H}^{-1} \breve{H} e = \tilde{H}^{-1} \breve{v}. \tag{12}$$

Here $\tilde{H}$ a monic, stable and minimum phase rational matrix and $\breve{v}$ is the process noise on the nodes in the immersed network, i.e. the network that results after removing the unmeasured node signals. Following Lemma 3 in [7], if condition b in Theorem 1 is satisfied, then $\tilde{H}$ is block diagonal and of the form

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & 0 \\ 0 & \tilde{H}_{22} \end{bmatrix}; \quad \breve{v} = \begin{bmatrix} \breve{v}_{\mathcal{Y} \cup \mathcal{B}} \\ \breve{v}_{\mathcal{A}} \end{bmatrix}.$$

As a result $\tilde{H}^{-1}$ will also be block-diagonal, and $\tilde{\xi}_{\mathcal{Y} \cup \mathcal{B}}$ will be a filtered version of $\breve{v}_{\mathcal{Y} \cup \mathcal{B}}$. And since $\mathcal{Q} \subset \mathcal{Y}$, also $\tilde{\xi}_{\mathcal{Q}}$ will be a filtered version of $\breve{v}_{\mathcal{Y} \cup \mathcal{B}}$, and because of $\xi_{\mathcal{Q}} = F_{\mathcal{Q}} \tilde{\xi}_{\mathcal{Y}}$ this implies that $\xi_{\mathcal{Q}}$ will also be a filtered version of $\breve{v}_{\mathcal{Y} \cup \mathcal{B}}$. As explained in [7], $\breve{v}_{\mathcal{Y} \cup \mathcal{B}}$ is a filtered version of all signals in e and $r \backslash r_{\mathcal{P}}$ that have a path to $w_{\mathcal{Y} \cup \mathcal{B}}$ which does not pass through nodes in $w_{\mathcal{A}}$. Excluding these signals, excludes the contribution of $\xi_{\mathcal{Q}}$.

When $w_{\mathcal{Q}}$ is void, the term $\xi_{\mathcal{Q}}$ does not appear in κ and the transfer function from (r, ξ) to κ generically has full row rank if there are $n_{\mathcal{Y}} + n_{\mathcal{U}}$ vertex disjoint paths between external signals (r, ξ) and κ. Since ξ is a filtered version of e, we require $n_{\mathcal{Y}} + n_{\mathcal{U}}$ vertex disjoint paths between external signals (r, e) and κ, making e_{sel} to be e, and r_{sel} to be r.

C. Proof of Corollary 2

Resulting from Theorem 2, we need $n_{\mathcal{Y}} + n_{\mathcal{B}} + n_{\mathcal{A}}$ vertex disjoint paths between external signals (r_{sel}, e_{sel}) and $w_{\mathcal{Y} \cup \mathcal{B} \cup \mathcal{A}}$. Owing to the fact that there are no confounding variables for the estimation problems $w_{\mathcal{A}} \to w_{\mathcal{Y}}$ and $w_{\mathcal{A}} \to w_{\mathcal{B}}$ (from Theorem 1), all signals in e that have unmeasured paths to $w_{\mathcal{A}}$ (let us call $e_{sel}^{\mathcal{A}}$) will not have unmeasured paths to $w_{\mathcal{Y} \cup \mathcal{B}}$, and thus they belong to e_{sel} and contribute to $n_{\mathcal{A}}$ vertex disjoint paths between (r_{sel}, e_{sel}) and $w_{\mathcal{Y} \cup \mathcal{B} \cup \mathcal{A}}$ due to its path to $w_{\mathcal{A}}$. The signals in e excluding $e_{sel}^{\mathcal{A}}$, either have unmeasured paths to $w_{\mathcal{Y} \cup \mathcal{B}}$ or do not have any path to $w_{\mathcal{Y} \cup \mathcal{U}}$. The former can not be included in e_{sel} and the latter belong to e_{sel} but can not have vertex disjoint paths to $w_{\mathcal{Y} \cup \mathcal{U}}$. Therefore, the remaining $n_{\mathcal{Y}} + n_{\mathcal{B}}$ vertex disjoint paths should come from r_{sel} and it is necessary to have at least $n_{\mathcal{Y}} + n_{\mathcal{B}}$ external signals r_{sel} in the network to satisfy the conditions of Corollary 1.