# Integrating data-informativity conditions in predictor models for single module identification in dynamic networks ⋆

**Paul M.J. Van den Hof**, **Karthik R. Ramaswamy** and
**Stefanie J.M. Fonken**

*Control Systems Group, Department of Electrical Engineering,*
*Eindhoven University of Technology, The Netherlands.*

**Abstract:** For consistent identification of a target module in a dynamic network with the local direct method, basically two prime conditions need to be satisfied: (a) a set of structural conditions on the choice of the predictor model, i.e. a set of input and output node variables, and (b) conditions on data-informativity. While for conditions (a) constructive algorithms for node selection have been presented that appropriately guarantee that the identified object can indeed reveal the target module, the requirements for satisfying (b) have not yet been integrated fully. While there exists a preliminary analysis on path-based conditions for generic data-informativity, integration of these conditions into the constructive algorithms has not yet been realized. In this paper, we will present simplified path-based results for generic data-informativity, and show how they can be integrated in constructive algorithms for predictor model selection that provide consistent target module estimates. It is shown that data-informativity not only requires a sufficient number of external excitation signals to be present in the network, but also puts restrictions on the structure of the predictor model, i.e. the selection of input and output node variables. Some examples are presented to illustrate the new results.

*Keywords:* System identification, identifiability, dynamic networks, interconnected systems.

## 1. INTRODUCTION

Large-scale data analytics is currently present in a growing number of technology areas, where data-driven diagnostics and model-based control require the handling of dynamic systems that are composed of interconnected subsystems, while the interconnections are characterized by a topology, that is often specified up front. Performing data-driven dynamic modeling in these systems has been given attention in the literature over the last fifteen years, see e.g. Gonçalves and Warnick (2008); Materassi and Innocenti (2010); Haber and Verhaegen (2014); Van den Hof et al. (2013); Hendrickx et al. (2019).

The particular problem of identifying a single module in such networks, of which the topology is known, has been addressed in e.g. Dankers et al. (2016); Gevers et al. (2018); Bazanella et al. (2019); Materassi and Salapaka (2020); Jahandari and Materassi (2022); Shi et al. (2023). In this problem, selecting the set of node variables that should be available from measurements, together with an appropriate identification method are the key issues.

In Ramaswamy and Van den Hof (2021) a local direct method has been presented in a prediction error context, that provides the conditions under which a target module can be identified with maximum likelihood properties, i.e.,

consistent and with minimum variance asymptotically in the number of data. Relative to indirect methods, as e.g. Gevers et al. (2018), it can exploit the excitation properties of nonmeasured disturbance signals for satisfying the required data-informativity conditions, and therefore can handle less expensive experiments (with less external excitation signals).

The core conditions in Ramaswamy and Van den Hof (2021) for arriving at a consistent estimate of the target module are covered by two sets: (a) a set of graph-based conditions for constructing an appropriate predictor model, specified by inputs and outputs; and (b) a spectral condition on data-informativity of the concerned signals in the predictor model. For satisfying the conditions of type (a), several algorithms were presented. While preliminary work has shown that also the conditions for (b) can be phrased as graph-based conditions, provided that we accept that the conditions hold generically (Van den Hof and Ramaswamy (2020)), they were not in a form that could be easily integrated into the conditions of type (a), and thus into the algorithms for constructing the predictor model.

In this paper we are going to show that the data-informativity conditions can be phrased in a graph-based setting, and can be integrated into the predictor model construction algorithms. While one could have the impression that any predictor model satisfying the type (a) conditions could be made data-informative by adding a sufficient number of external excitation signals, it will be shown that this is not the case, and that including the type (b) data-informativity conditions can lead to differently

structured predictor models.

After presenting the network setup and the prime result of Ramaswamy and Van den Hof (2021) in Sections 2 and 3, new results on the data-informativity conditions will be presented in Sections 4 and 5. Next the available full input and minimum input algorithms for predictor model construction are extended to include the data-informativity conditions, and the consequences are illustrated with a 4-node example.

## 2. NETWORK SETUP

We consider a dynamic network in the so-called module framework where node variables $w$ are interconnected through transfer functions. A compact representation of the network is given by

$$w(t) = G(q)w(t) + H(q)e(t) + u(t) \tag{1}$$

with $w$ an $L$-dimensional vector, $G(q)$ a hollow rational transfer function matrix, i.e. with zeros on the diagonal entries, in the the time shift operator $q$, i.e. $q^{-1}w(t) = w(t-1)$, $e$ an $L$-dimensional vector of white noise processes, $H(q)$ the rational disturbance model, and $u(t) = R(q)r(t)$, accounting for the effect of measured external excitation signals $r$ on the network, with $R(q)$ a rational transfer function.

It is further assumed that the network is well-posed in the sense that all principle minors of $\lim_{z\to\infty}(1 - G(z))$ are non-zero. In this paper it is also assumed for simplicity that all elements in $G(q)$ are strictly proper.

After selecting a target module $G_{ji}(q)$ with input $w_i$ and output $w_j$, the single module identification problem comes down to selecting a set of measured node variables $w$, that compose a predictor model with predictor inputs and predicted outputs, on the basis of which a (consistent) estimate of $G_{ji}(q)$ can be obtained through a prediction error identification method.

## 3. CONSTRUCTING A PREDICTOR MODEL WITH THE LOCAL DIRECT METHOD

On the basis of a selected set of node variables $w$, a predictor model is constructed with inputs $w_\mathcal{D}$ and outputs $w_\mathcal{Y}$, where $w_i \in w_\mathcal{D}$ and $w_j \in w_\mathcal{Y}$, that is formalized as follows:

$$w_\mathcal{Y}(t) = \bar{G}(q)w_\mathcal{D}(t) + \bar{J}(q)u_\mathcal{K}(t) + \bar{S}u_\mathcal{P}(t) + \bar{H}(q)\xi_\mathcal{Y}(t) \tag{2}$$

where $\bar{J}(q)$ is a dynamic transfer function, $\bar{S}$ is a binary matrix, $\xi_\mathcal{Y}$ a vector white noise process, and $u_\mathcal{P}$ and $u_\mathcal{K}$ are distinct sets of components of the excitation signal $u$.

This predictor model is schematically depicted in Figure 1. There can be node signals $w_\mathcal{Q}$ with $\mathcal{Q} = \mathcal{D} \cap \mathcal{Y}$ that appear in both input and output. This will be used to handle confounding variables[1] in the estimation problem, by modelling their effect through a multivariate disturbance model at the output, to be explained later on. Additional node sets include $u_\mathcal{U}$ with $\mathcal{U} = \mathcal{D}\backslash\mathcal{Q}$, while $w_o = w_j$ if

[1] A confounding variable is an unmeasured variable that has paths to both the input and output of an estimation problem Pearl (2000). The resulting paths are referred to as "'confounding paths". If one of the confounding paths passes through a node in $w$ the confounding variable is called "indirect", else it is called "direct" (Ramaswamy and Van den Hof (2021)).

$j \notin \mathcal{Q}$; else $w_o$ is void. The node signals that do not appear in the predictor model are referred to as the unmeasured nodes and are collected in $w_\mathcal{Z}$, i.e. $\mathcal{Z} := \{1, \cdots, L\}\backslash\{\mathcal{Y} \cup \mathcal{U}\}$.
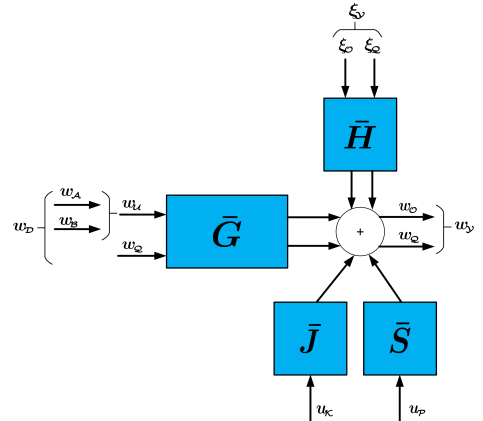


Fig. 1. Predictor model for the local direct method (Ramaswamy (2022)).

Dependent on the choices of $w_\mathcal{Y}$ and $w_\mathcal{D}$, the signals $u_\mathcal{K}$ enter the equation (2) either with a fixed term (equal to 1), or with a dynamic term.

The predictor model (2) is the result of writing the network equation for the collected set of selected node signals in the structured form:

$$\underbrace{\begin{bmatrix} w_\mathcal{Y} \\ u_\mathcal{U} \end{bmatrix}}_{w_m} = \underbrace{\left[\begin{array}{c|c} \bar{G} & 0 \\ \hline \bar{G}_{\mathcal{U}\mathcal{D}} & \bar{G}_{\mathcal{U}o} \end{array}\right]}_{\bar{G}_m}\begin{bmatrix} w_\mathcal{D} \\ w_o \end{bmatrix} + \underbrace{\left[\begin{array}{c|c} \bar{H} & 0 \\ \hline 0 & \bar{H}_{\mathcal{U}\mathcal{U}} \end{array}\right]}_{\bar{H}_m}\underbrace{\begin{bmatrix} \xi_\mathcal{Y} \\ \xi_\mathcal{U} \end{bmatrix}}_{\xi_m} + \begin{bmatrix} u_\mathcal{Y} \\ u_\mathcal{U} \end{bmatrix} \tag{3}$$

with $\xi_m$ a white noise process, while $\bar{H}$ is monic, stable and stably invertible, the components in $\bar{G}$ are zero if it concerns a mapping between identical signals, and $u_\mathcal{Y}$ and $u_\mathcal{U}$ are appropriate components of the excitation signal $u$.

A network transformation to the form (3) can always be made (Proposition 1 in Ramaswamy and Van den Hof (2021)), and is attractive because it decorrelates the disturbances on the variables $w_\mathcal{Y}$ and $u_\mathcal{U}$, reflected in the block-diagonal structure of $\bar{H}_m$. The first block row of (3) then serves as a basis for the predictor model (2).

The predictor model leads to an identification of the target module $G_{ji}(q)$ through the one-step-ahead predictor (Ljung (1999)):

$$\hat{w}_\mathcal{Y}(t, \theta) = \bar{H}(q, \theta)^{-1}[\bar{G}(q, \theta)w_\mathcal{D}(t) + \bar{J}(q, \theta)u_\mathcal{K}(t) + $$
$$+ (I - \bar{H}(q, \theta)^{-1})[w_\mathcal{Y}(t) - \bar{S}u_\mathcal{P}(t)] \tag{4}$$

where $\theta$ represent the parameters of a parametrized model. Applying a quadratic identification criterion, according to

$$\theta^* = \arg\min_\theta \bar{\mathbb{E}}(w(t) - \hat{w}_\mathcal{Y}(t, \theta))^T P(w(t) - \hat{w}_\mathcal{Y}(t, \theta)) \tag{5}$$

with $P$ a positive definite weighting matrix, then leads to an (asymptotically) estimated model $\bar{G}(q, \theta^*)$.

For recalling the consistency result of this local direct method for network identification, we first need to present the structural conditions for module invariance in (3).

2

*Proposition 1.* (Ramaswamy and Van den Hof (2021)).
*Let $G_{ji}^o$ be the target module. In the transformed system's equation (3) it holds that $\bar{G}_{ji} = G_{ji}^o$ under the following conditions*

(a) *In the network (1), every path from $w_i$ to $w_j$ that does not pass through $G_{ji}$, and every loop around $w_j$ need to pass through a node in $w_{\mathcal{D}}$ (Parallel path and loop (PPL) condition);*
(b) *$\mathcal{U}$ is decomposed in disjunct sets $\mathcal{A}$ and $\mathcal{B}$, such that in (1) there are no confounding variables for the estimation problem $w_{\mathcal{A}} \to w_{\mathcal{B} \cup \mathcal{Y}}$;*
(c) *$i \in \{\mathcal{A} \cup \mathcal{Q}\}$, and*
(d) *Every path from $\{w_i, w_j\}$ to $w_{\mathcal{B}}$ passes through a node in $u_{\mathcal{Y} \cup \mathcal{U}}$.*

The interpretation of this result is that we can allow for input signals $w_{\mathcal{B}}$ that have confounding variables with the outputs $w_{\mathcal{Y}}$, as a result of which their corresponding modules in $\bar{G}_{\mathcal{Y}\mathcal{B}}$ will typically be estimated incorrectly. However since $i \notin \mathcal{B}$ this is not a problem, while at the same time the nodes in $w_{\mathcal{B}}$ can help to satisfy the other conditions.

*Theorem 2.* (Ramaswamy and Van den Hof (2021)).
*Consider a dynamic network represented by (1) and a predictor model according to (4). Then the local direct identification method according to (5), leads to a consistent estimation of $G_{ji}^o$, i.e. $\bar{G}_{ji}(q, \theta^*) = G_{ji}^o$ if*

(1) *$\mathcal{M}$ is chosen to satisfy $\mathcal{S} \in \mathcal{M}$ (system in the model set);*
(2) *The structural conditions for module variance of Proposition 1 are satisfied;*
(3) *$\Phi_\kappa(\omega) \succ 0$ for almost all $\omega$, with*

$$\kappa(t) = \begin{bmatrix} w_{\mathcal{D}}(t) \\ \xi_{\mathcal{Y}}(t) \\ u_{\mathcal{K}}(t) \end{bmatrix} \qquad (6)$$

*(data-informativity condition).*

The interpretation of the data-informativity condition is that actually $\xi_{\mathcal{Y}}$ and $u_{\mathcal{K}}$ do not contribute to the excitation of $w_{\mathcal{D}}$. For $\xi_{\mathcal{Y}}$ this is due to the fact that this vector serves as excitation source for the noise model $\bar{H}(q, \theta)$, while $u_{\mathcal{K}}$ excites the parametrized transfer $\bar{J}(q, \theta)$ in (4).

In the next section we will particularly focus on this data-informativity condition, and analyze how the conditions for data-informativity can be turned into graphical conditions that can be merged with the structural conditions for module invariance in Proposition 1.

## 4. EXCITATION SIGNALS IN THE PREDICTOR MODEL

For further analysis of the data-informativity conditions in Theorem 2, we need to specify how the signals $u_{\mathcal{K}}$ and $u_{\mathcal{P}}$ are composed on the basis of excitation signals in the original network (1). To this end we formulate the following result.

*Proposition 3. Consider a predictor model specified by the signal sets $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{D}, \mathcal{Y}$ as defined in Section 3. Then*

(1) *The set of excitation signals $u_{\mathcal{P}}$ are a subset of $u_{\mathcal{Y}}$ and composed as follows:*

- *For $\ell \in \mathcal{Q}$, $u_\ell \in u_{\mathcal{P}}$ if all loops around $w_\ell$ and all paths from $w_\ell$ to $\mathcal{B}$ pass through a node in $\mathcal{Q} \cup \mathcal{A}$.*
- *$u_o \in u_{\mathcal{P}}$ if all loops around $w_o$ pass through a node in $\mathcal{Q} \cup \mathcal{A}$ and all paths from $w_o$ to $\mathcal{Q}$ pass through a node in $\mathcal{A}$.*

(2) *The set of excitation signals $u_{\mathcal{K}}$ is composed of*
- *All signals $u_{\mathcal{B}}$, and*
- *All signals $u_{\mathcal{Y}}$ that are not in $u_{\mathcal{P}}$, and*
- *All signals $u_{\mathcal{Z}}$ of which the effect on $w_{\mathcal{Y}}$ in the predictor model is not covered by the predictor inputs $w_{\mathcal{D}}$.*

**Proof**: See Appendix.

In Figure 2 it is illustrated how excitation signals on nodes in the original network, map to excitation signals in the predictor model. Here $\bar{\mathcal{Z}}$ denotes the subset of immersed nodes ($\mathcal{Z}$) whose excitations do not explicitly appear in (2), but whose effects are incorporated in $w_{\mathcal{D}}$. It follows
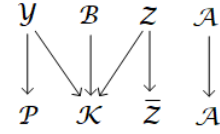


Fig. 2. Mapping of excitation signals in the original network (1) to their role in the predictor model (2).

that $u_{\mathcal{K}}$ is composed of the excitation signals $u$ that enter the network through (a selection of) node signals $w_\ell$, with $\ell \in \mathcal{Y} \cup \mathcal{B} \cup \mathcal{Z}$, while $u_{\mathcal{P}}$ is composed of the excitation signals on a subset of the node signals $w_{\mathcal{Y}}$. In order to use these results for further specifying the data-informativity condition in (6), we first need to convert the spectrum condition into a path-based condition.

## 5. A PATH-BASED RESULT FOR GENERIC DATA-INFORMATIVITY

We follow the reasoning introduced in Van den Hof and Ramaswamy (2020), based on the graph-based results in Hendrickx et al. (2019), that the spectrum condition (6) can be rephrased as a full row rank condition on the mapping from all external signals in the network to the vector $\kappa$, together with a persistence of excitation condition on the external signals in the network. This can be turned into a graph-based condition as formulated next.

*Proposition 4.* (Van den Hof and Ramaswamy (2020)).
*The spectrum condition (6) holds generically if in the graph of the network model there exist $dim(\kappa)$ vertex disjoint paths between the node sets $(u, \xi)$ and $(w_{\mathcal{D}}, \xi_{\mathcal{Y}}, u_{\mathcal{K}})$, with $u$ all measured external excitation signals in the network, and the signals in $u$ are mutually independent and persistently exciting.*

If the structural conditions for generic data-informativity can be turned into graph-based conditions, we can incorporate them during the construction of an appropriate predictor model, and thus guaranteeing that generic data-informativity can be realized by simply choosing excitation signals $u$ that are persistently exciting.

*Proposition 5. For the local direct method and predictor model (4), the structural conditions for data-informativity*

3

are satisfied if for all signals $w_k, k \in \mathcal{Q}$, every path from $w_k$ to $\mathcal{B}$, and every loop around $w_k$ passes through a node in $w_{\mathcal{A} \cup \mathcal{Q}}$.

**Proof:** The required condition for generic data informativity entails that there should be $dim(\kappa)$ vertex disjoint paths in the mapping

$$(u, \xi) \to (w_{\mathcal{D}}, \xi_{\mathcal{Y}}, u_{\mathcal{K}})$$

or equivalently

$$(u_{\mathcal{K}}, u_{\mathcal{P}}, u_{\mathcal{A}}, u_{\bar{z}}, \xi_{\mathcal{U}}, \xi_{\mathcal{Y}}) \to (u_{\mathcal{U}}, w_{\mathcal{Q}}, \xi_{\mathcal{Y}}, u_{\mathcal{K}}).$$

In the condition on vertex disjoint paths similar components on the left hand side and the right hand side can be removed, provided that we condition the remaining vertex disjoint paths not to pass through the removed components. This allows us to remove $u_{\mathcal{K}}$ and $\xi_{\mathcal{Y}}$, while we also know that, due to the assumption of full rank disturbances in the network, $\xi_{\mathcal{U}}$ has $dim(u_{\mathcal{U}})$ vertex disjoint paths to $u_{\mathcal{U}}$. As a result the vertex disjoint path condition can be reformulated as the existence of $dim(w_{\mathcal{Q}})$ vertex disjoint paths in the mapping

$$(u_{\mathcal{P}}, u_{\mathcal{A}}, u_{\bar{z}}) \to w_{\mathcal{Q}}$$

where paths are not allowed to pass through $\xi_{\mathcal{Y}}$, $u_{\mathcal{U}}$ or $u_{\mathcal{K}}$.

Since any path from $u_{\mathcal{A}}$ to $w_{\mathcal{Q}}$ passes through $w_{\mathcal{A}} \subset u_{\mathcal{U}}$, and all paths from $u_{\bar{z}}$ to $w_{\mathcal{Y}}$ pass through a node in $u_{\mathcal{U}}$, no paths from $u_{\mathcal{A}}$ and $u_{\bar{z}}$ can be used, and therefore we consider the vertex disjoint paths between

$$u_{\mathcal{P}} \to w_{\mathcal{Q}},$$

where paths are not allowed to pass through $u_{\mathcal{U}}$. By the definition of the set $\mathcal{P}$, an excitation on the output $w_o$ can only be an element of $u_{\mathcal{P}}$ if all paths from $w_o$ to $w_{\mathcal{Q}}$ pass through a node in $u_{\mathcal{U}}$, which in the above statement is not allowed. Therefore the result is that there should be $dim(\mathcal{Q})$ vertex disjoint paths between

$$u_{\mathcal{P} \cap \mathcal{Q}} \to w_{\mathcal{Q}},$$

in other words: only excitation signals that directly excite $w_{\mathcal{Q}}$ can be used for satisfying the structural data-informativity conditions. With the definition of the set $\mathcal{P}$ the result then follows directly. $\square$

The immediate implication of this result is that for satisfying the structural data-informativity conditions:

(1) All nodes in $w_{\mathcal{Q}}$ need (independent) external excitation, and
(2) All those excitations should satisfy: $u_{\mathcal{Q}} \in u_{\mathcal{P}}$.

The first condition can typically be satisfied by adding external excitations $r$ to all nodes in $w_{\mathcal{Q}}$. However the second condition is a structural condition on the graph of the network model, and is principally limited by the network topology. Whereas the construction of predictor models in Ramaswamy and Van den Hof (2021) was based on satisfying condition 2 of Theorem 2 only, we now have to extend these graphical conditions in condition 2 of Theorem 2 with the structural conditions for data-informativity, as formulated in Proposition 5. This is further explored and illustrated in the next Sections for two different algorithms presented in Ramaswamy and Van den Hof (2021), the full input algorithm and the minimum input algorithm.

# 6. FULL INPUT ALGORITHM

The original full input algorithm for predictor model construction according to Condition 2 of Theorem 2 is outlined below.

(1) We start by selecting $w_{\mathcal{D}} = w_{\mathcal{A}} = w_i$ and $w_o = w_{\mathcal{Y}} = w_j$;
(2) Extend $\mathcal{A}$ in such a way that all $w$-inneighbors of $w_{\mathcal{Y}}$ are included in $w_{\mathcal{A}}$;
(3) For every node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with $w_{\mathcal{Y}}$, move $w_k$ to $w_{\mathcal{B}}$ if this is allowed by the conditions on $\mathcal{B}$;
(4) For every remaining node signal $w_k, k \in \mathcal{A}$ that has a direct confounding variable with $w_{\mathcal{Y}}$, copy $w_k$ to the output $w_{\mathcal{Y}}$, move it to $w_{\mathcal{Q}}$ and return to Step (2).
(5) For every remaining node signal $w_k, k \in \mathcal{A}$ that has a indirect confounding variable with $w_{\mathcal{Y}}$, find a new node signal to be added to $w_{\mathcal{B}}$ that satisfies the conditions on $\mathcal{B}$ and blocks[2] the concerned confounding variable.
(6) For every remaining node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with $w_{\mathcal{Y}}$, copy $w_k$ to $w_{\mathcal{Y}}$, move it to $w_{\mathcal{Q}}$ and return to Step (2).
(8) Finally the predictor model is defined by the mapping $w_{\mathcal{D}} \to w_{\mathcal{Y}}$, with $\mathcal{D} = \mathcal{Q} \cup \mathcal{A} \cup \mathcal{B}$ and $\mathcal{Y} = \mathcal{Q} \cup \{o\}$.

This algorithm can now be extended to also include the structural data-informativity condition as formulated in Proposition 5, as follows:

- In Steps (3) and (5), add a node to $w_{\mathcal{B}}$ only if the data-informativity conditions of Proposition 5 (DI conditions) are satisfied, and
- At the end of the Algorithm, add an additional Step (7), as follows.

(7) For every $w_k \in w_{\mathcal{Q}}$ that does not satisfy the DI conditions:
  (a) Verify if moving $w_o$ to $w_{\mathcal{Q}}$ removes a violated condition for $w_k$;
  (b) Verify if moving one or more nodes in $w_{\mathcal{B}}$ to $w_{\mathcal{Q}}$ removes a violated condition for $w_k$;
  (c) Find an additional node $w_\ell$, not present yet in $w_{\mathcal{Y} \cup \mathcal{U}}$, to be added to either $w_{\mathcal{A}}$ or $w_{\mathcal{Q}}$ that solves the problem for $w_k$;
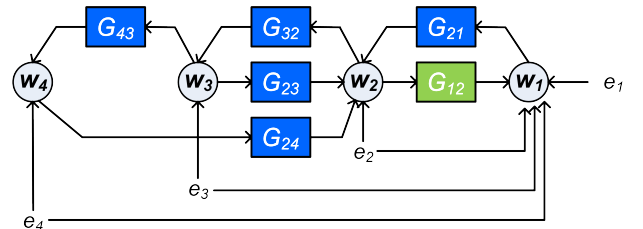  (d) If in the previous steps a new node is added to $w_{\mathcal{Q}}$, return to Step (2).



Fig. 3. Four node example with $G_{21}$ as target module.

In the following example we illustrate how the original algorithm and its extension to include the structural

---

[2] "Blocking" the confounding variable means that a measured node $w_\ell$ is added to $w_{\mathcal{B}}$ that is on the path from the confounding variable to $w_k$.

data-informativity conditions differ when applied to the example sketched in Figure 3.

*Example 1. (Full input algorithm).* Consider the network in Figure 3 with $G_{12}$ the target module we want to identify. The original full input algorithm, as presented in this section, starts with Step 1, including the input and output of the target module, i.e. $w_2 \rightarrow w_1$, with $w_{\mathcal{D}} = w_{\mathcal{A}} = w_2$; $w_{\mathcal{Y}} = w_o = w_1$. Since $w_2$ is the only in-neighbor of the output $w_1$, no further inputs need to be added in Step 2. Since $w_2$ has a direct confounding variable ($e_2$) and indirect confounding variables ($e_3$ and $e_4$) with $w_1$ these need to be blocked. In Step 3 we evaluated whether $w_2$ can be moved to $w_{\mathcal{B}}$. This is not allowed because of the rule that $i \notin \mathcal{B}$. The direct confounding variable can only be blocked by copying $w_2$ to the output $w_{\mathcal{Y}}$ (Step 4), such that it becomes an element of $w_{\mathcal{Q}}$, resulting in the predictor model $w_2 \rightarrow (w_1, w_2)$ with $w_o = w_1$; $w_{\mathcal{Q}} = w_2$. Since the new output $w_2$ has been added we need to return to Step 2 and add all its inneighbors to $w_{\mathcal{D}}$. As a result $w_3$ and $w_4$ are added to $w_{\mathcal{A}}$ and $w_1$ (already present in $w_{\mathcal{Y}}$) is added to $w_{\mathcal{Q}}$. We are back in Step 3 now with $\mathcal{A} = \{3, 4\}$ and noticing that both nodes have confounding variables with output $w_1$. We first consider node $w_3 \in w_{\mathcal{A}}$, that has a direct confounding variable ($e_3$) and an indirect confounding variable ($e_4$) with $w_1$. Moving $w_3$ to $w_{\mathcal{B}}$ is not allowed since it does not satisfy Condition 4 of Proposition 1. Therefore, (Step 4) we copy $w_3$ to the output $w_{\mathcal{Y}}$, such that it becomes an element of $w_{\mathcal{Q}}$. The direct confounding variable between $w_4$ and $w_1$ is addressed (Step 3) by moving $w_4$ to $w_{\mathcal{B}}$. This results in the predictor model

$$(w_1, w_2, w_3, w_4) \rightarrow (w_1, w_2, w_3) \qquad (7)$$

with $w_{\mathcal{A}} = \emptyset$; $w_{\mathcal{B}} = w_4$; $w_o = \emptyset$; $w_{\mathcal{Q}} = \{w_1, w_2, w_3\}$

and this concludes the original algorithm that satisfies Condition 2 of Theorem 2.

If we evaluate the DI conditions for this predictor model, it appears from the results of Section 5 that all nodes in $w_{\mathcal{Q}}$ (i.e. $w_1, w_2, w_3$) should have independent external excitations $r$, but at the same time also that the excitation on $w_3$ cannot be effectively used for data informativity, as it does not satisfy the Conditions of Proposition 5. $w_3$ has a path to $w_4 \in w_{\mathcal{B}}$ that is not blocked by a node in $w_{\mathcal{A} \cup \mathcal{Q}}$. As a result the selected predictor model can never satisfy the DI conditions. Even if we excite all nodes $(w_1, w_2, w_3)$, the excitation on $w_3$ will appear in the predictor model as input to a parametrized transfer function, and therefore cannot be used for excitation of $G_{12}$.

Following the full-input algorithm now with the additional Step 7, shows that $w_4$ is not allowed to go to $w_{\mathcal{B}}$; it cannot be moved to $w_{\mathcal{A}}$ since there is a confounding variable then between $w_{\mathcal{A}}$ and $w_{\mathcal{Y}}$. As a result $w_4$ should go to $w_{\mathcal{Q}}$, leading to the predictor model:

$$(w_1, w_2, w_3, w_4) \rightarrow (w_1, w_2, w_3, w_4) \qquad (8)$$

with all nodes in $w_{\mathcal{Q}} = \{w_1, w_2, w_3, w_4\}$. For the final predictor model it holds that all external signals $u_1, u_2, u_3, u_4$ belong to $u_{\mathcal{P} \cap \mathcal{Q}}$, and are required for satisfying the data-informativity conditions. □

## 7. MINIMUM INPUT ALGORITHM

The minimum input algorithm for predictor model selection focuses on using a minimum number of measured node signals as predictor inputs, and, in its original form, follows the following strategy

(1) We start by selecting $w_{\mathcal{D}} = w_{\mathcal{A}} = w_i$ and $w_o = w_{\mathcal{Y}} = w_j$;
(2) Extend $\mathcal{A}$ with a minimum number of $w$-inneighbors of $w_{\mathcal{Y}}$ such that the PPL condition (Condition(a) of Proposition 1) is satisfied;
(3) For every node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with $w_{\mathcal{Y}}$, copy $w_k$ to $w_{\mathcal{Y}}$, move it to $w_{\mathcal{Q}}$;
(5) Finally the predictor model is defined by the mapping $w_{\mathcal{D}} \rightarrow w_{\mathcal{Y}}$, with $\mathcal{D} = \mathcal{Q} \cup \mathcal{A}$, $\mathcal{Y} = \mathcal{Q} \cup \{o\}$ and $\mathcal{B} = \emptyset$.

This algorithm can now be extended to also include the structural data-informativity condition as formulated in Proposition 5, as follows:

- At the end of the Algorithm, add an additional Step (4), as follows.

(4) For every $w_k \in w_{\mathcal{Q}}$ that does not satisfy the DI conditions:
   (a) Verify if moving $w_o$ to $w_{\mathcal{Q}}$ removes a violated condition for $w_k$;
   (b) Find an additional node $w_\ell$, not present yet in $w_{\mathcal{Y} \cup \mathcal{U}}$, to be added to either $w_{\mathcal{A}}$ or $w_{\mathcal{Q}}$ that solves the problem for $w_k$;

We will illustrate this algorithm on the Example presented in the previous Section.

*Example 2. (Minimum input algorithm).* Consider the same network as in Example 1 in Figure 3. The predictor model based on the original algorithm starts with the target module: $w_2 \rightarrow w_1$, with $w_2 \in w_{\mathcal{A}}$ and $w_1 = w_o$, which in line with Step 2, already satisfies the PPL condition (a) of Proposition 1. To deal with the confounding variable $e_2$ between $w_2$ and $w_1$, we copy $w_2$ to the output and move it to $w_{\mathcal{Q}}$ (Step 3)

$$w_2 \rightarrow (w_1, w_2) \qquad (9)$$

with $w_{\mathcal{Q}} = w_2, w_o = w_1$, after which the original algorithm finishes.

It can be observed that with this algorithm the predictor model is considerably smaller sized than with the full input algorithm. Applying the data-informativity analysis shows that $w_2 = w_{\mathcal{Q}}$ would need external excitation, but at the same time it also shows that any excitation on $w_2$ will not become a part of the excitation signals $u_{\mathcal{P}}$, as there is a non-blocked loop $w_2 \rightarrow w_3 \rightarrow w_2$ that is violating the conditions of Proposition 5. As a result the selected predictor model can never satisfy the DI conditions.

Continuing the minimum-input algorithm with the addition of Step 4, shows that with $w_2 \in w_{\mathcal{Q}}$, there are multiple loops around $w_2$ that need to be blocked. The loop $w_2 \rightarrow w_1 \rightarrow w_2$ can be blocked by moving the output $w_1$ to $w_{\mathcal{Q}}$ (Step 4(a)). Moreover, the remaining loops around $w_2$ pass through node $w_3$, therefore we add $w_3$ to the predictor model (Step 4(b)) to block them. Because there is a confounding variable between $w_3$ and $w_{\mathcal{Y}}$, $w_3$ cannot go into $w_{\mathcal{A}}$ and so it has to go to $w_{\mathcal{Q}}$. The final solution of the extended algorithm now provides the predictor model:

$$(w_1, w_2, w_3) \rightarrow (w_1, w_2, w_3) \qquad (10)$$

with all elements in $w_{\mathcal{Q}} = \{w_1, w_2, w_3\}$, and thus requires external excitation signals $u_1, u_2, u_3$ to guarantee data-informativity, where the required excitation signals automatically belong to $u_{\mathcal{P} \cap \mathcal{Q}}$ due to the additional step 4 in the algorithm. $\qquad \square$

*Remark 1.* Note that the example in the previous Sections has been selected to emphasize the role of data-informativity conditions for the local direct method in the presence of confounding variables. Alternatively, for the particular example network, an indirect identification method as in e.g. Gevers et al. (2018) could provide a consistent target module estimate on the basis of a single external excitation signal only, while refraining from extensive noise modeling.

## 8. CONCLUSIONS

We have addressed the construction of predictor models for consistent identification of a single module in a dynamic network, with the local direct prediction error method. The original structural conditions have been extended with appropriate graph-based data-informativity conditions that are shown to have a serious impact on the predictor models. These graph-based data-informativity conditions can be integrated in constructive algorithm for arriving at predictor models that guarantee consistent estimation of a local target module.

## Appendix A. PROOF OF PROPOSITION 3.

Given size constraints we will only provide a sketch of proof, and for the details refer to Ramaswamy (2022).
In the network transformation that leads to the structured form (3), the term with external excitation signals $u_{\mathcal{Y}}$ is denoted as $u_{\mathcal{Y}}(t) = \bar{R}_{\mathcal{Y}\star}(q)u(t)$. Since the first block row of (3) provides the predictor model (3), it follows that $\bar{R}_{\mathcal{Y}\star}(q)u(t) = \bar{J}(q)u_{\mathcal{K}}(t) + \bar{S}u_{\mathcal{P}}(t)$. So, for analyzing which elements in $u$ appear in $u_{\mathcal{K}}$, which in $u_{\mathcal{P}}$, and which do not appear at all, we study the structure of $\bar{R}_{\mathcal{Y}\star}(q)$.

Under conditions (2) and (4) of Proposition 1, and the fact that $w_o$ is a scalar signal, Lemma 4.1. in Ramaswamy (2022) provides the following results:

(a) $\bar{R}_{k\ell} = 0$ with $k, \ell \in \mathcal{Q}$ and $k \neq \ell$;
(b) $\bar{R}_{o\ell} = 0$ with $\ell \in \mathcal{Q}$;
(c) $\bar{R}_{k\ell} = 0$ with $k \in \mathcal{Y}$ and $\ell \in \mathcal{A}$;
(d) $\bar{R}_{kk} = 1, k \in \mathcal{Y}$, if:
 • all loops through $w_k$ pass through a node in $w_{\mathcal{Q} \cup \mathcal{U}}$;
 • there exist no direct or unmeasured paths from $w_k$ to any $w_m, m \in \mathcal{B}$ when $k \in \mathcal{Q}$;
(e) $\bar{R}_{ko} = 0, k \in \mathcal{Q}$, if:
 (a) all paths from $w_o$ to $w_k$ pass through a node in $w_{\mathcal{Q} \cup \mathcal{U}}$;
(f) $\bar{R}_{k\ell} = 0, k \in \mathcal{Y}, \ell \in \mathcal{Z}$, if:
 (a) all paths from $w_\ell$ to $w_k$ pass through a node in $w_{\mathcal{Q} \cup \mathcal{U}}$; and
 (b) there exist no direct or unmeasured paths from $w_\ell$ to any $w_m, m \in \mathcal{B}$.

The signals that are in $u_{\mathcal{P}}$ are characterized by a binary matrix $\bar{S}$, and thus are determined by the entries in $\bar{R}$ that are equal to 1. The combined statements under result (d),

then lead to the specification of $u_{\mathcal{P}}$ in Proposition 3, see also Proposition 4.3 in Ramaswamy (2022).

In order to specify $u_{\mathcal{K}}$, we will follow a reasoning of exclusion. The above results (a)-(f) specify columns in $\bar{R}_{\mathcal{Y}\star}$ that are zero, i.e. they determine the nodes of which the excitation is not explicitly present in (3). All remaining nodes, for which the corresponding terms in $\bar{R}_{\mathcal{Y}\star}$ cannot be shown to be 0 or 1, will typically have a parametrized transfer to $w_{\mathcal{Y}}$ and therefore will constitute $u_{\mathcal{K}}$. With $\bar{\mathcal{P}}$ being defined as the set of node indices that correspond to 0-columns in $\bar{R}_{\mathcal{Y}\star}$, it follows that $\mathcal{L} = \mathcal{P} \cup \bar{\mathcal{P}} \cup \mathcal{K}$, leading to $\mathcal{K} = \mathcal{L} \backslash \{\mathcal{P} \cup \bar{\mathcal{P}}\}$.

For specification of the set $\bar{\mathcal{P}}$, it is observed that from result (c) it follows that $\mathcal{A} \in \bar{\mathcal{P}}$, and from results (a),(b) and (e) that $\mathcal{Y} \cap \bar{\mathcal{P}} = \emptyset$. Moreover result (f) specifies the subset $(\bar{\mathcal{Z}})$ of unmeasured nodes that corresponds to zero columns in $\bar{R}_{\mathcal{Y}\star}$ and thus belongs to $\bar{\mathcal{P}}$. This leads to $\mathcal{K} = \mathcal{L} \backslash \{\mathcal{P} \cup \mathcal{A} \cup \bar{\mathcal{Z}}\}$ which proves the second part of Proposition 3.

## REFERENCES

Bazanella, A.S., Gevers, M., and Hendrickx, J.M. (2019). Network identification with partial excitation and measurement. In *Proc. 58th IEEE Conference on Decision and Control (CDC)*, 5500–5506.

Dankers, A.G., Van den Hof, P.M.J., Heuberger, P.S.C., and Bombois, X. (2016). Identification of dynamic models in complex networks with prediction error methods: Predictor input selection. *IEEE Trans. on Automatic Control*, 61(4), 937–952.

Gevers, M., Bazanella, A., and Vian da Silva, G. (2018). A practical method for the consistent identification of a module in a dynamical network. *IFAC-PapersOnLine*, 51-15, 862–867. Proc. 18th IFAC Symp. System Identif. (SYSID2018).

Gonçalves, J. and Warnick, S. (2008). Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks. *IEEE Trans. Automatic Control*, 53(7), 1670–1674.

Haber, A. and Verhaegen, M. (2014). Subspace identication of large-scale interconnected systems. *IEEE Transactions on Automatic Control*, 59(10), 2754–2759.

Hendrickx, J., Gevers, M., and Bazanella, A. (2019). Identifiability of dynamical networks with partial node measurements. *IEEE Trans. Autom. Control*, 64(6), 2240–2253.

Jahandari, S. and Materassi, D. (2022). Optimal selection of observations for identification of multiple modules in dynamic networks. *IEEE Trans. Automatic Control*, 67(9), 4703–4716.

Ljung, L. (1999). *System Identification: Theory for the User*. Prentice-Hall, Englewood Cliffs, NJ.

Materassi, D. and Innocenti, G. (2010). Topological identification in networks of dynamical systems. *IEEE Trans. Automatic Control*, 55(8), 1860–1871.

Materassi, D. and Salapaka, M.V. (2020). Signal selection for estimation and identification in networks of dynamic systems: a graphical model approach. *IEEE Trans. Automatic Control*, 65(10), 4138–4153.

Pearl, J. (2000). *Causality: Models, Reasoning, and Inference*. Cambridge University Press, New York.

Ramaswamy, K.R. (2022). *A guide to learning modules in a dynamic network*. PhD dissertation, Eindhoven University of Technology.

Ramaswamy, K.R. and Van den Hof, P.M.J. (2021). A local direct method for module identification in dynamic networks with correlated noise. *IEEE Trans. Automatic Control*, 66, 5237–5252.

Shi, S., Cheng, X., and Van den Hof, P.M.J. (2023). Single module identifiability in linear dynamic networks with partial excitation and measurement. *IEEE Trans. Automatic Control*, 68(1). To appear.

Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates. *Automatica*, 49(10), 2994–3006.

Van den Hof, P.M.J. and Ramaswamy, K.R. (2020). Path-based data-informativity conditions for single module identification in dynamic networks. In *Proc. 59th IEEE Conf. on Decision and Control (CDC)*, 4354–4359. IEEE, Jeju Island, Republic of Korea.