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Criterion Based Equivalence for Equation Error Models

PAUL M. J. VAN DEN HOF

Abstract—**An equivalence relation is investigated that is used when parametrizing a set of linear time-invariant models for applying equation error system identification methods. In order to obtain identified models that are not dependent on the specific parametrization of the model set, a notion of model equivalence is introduced that is related to the identification criterion, leading to an unconventional set of canonical forms.**

I. Introduction

When applying methods of system identification in order to construct linear, time-invariant, and finite-dimensional models based on a given sequence of measurement data, two aspects play a central role in determining the properties of the identified model: the set of models that is considered, and the identification criterion. The identification procedure should select that (those) model(s) from the model set that is (are) optimal with respect to the criterion. Using a notion of model equivalence a model set $\mathfrak{M}$ can be parametrized by considering only a subset $\mathfrak{M}^* \subset \mathfrak{M}$ such that every model within $\mathfrak{M}$ has an equivalent model in $\mathfrak{M}^*$, e.g., by using a set of (pseudo-) canonical forms [1]-[3]. Since it is undesirable that the identification result is essentially dependent on the specific choice of $\mathfrak{M}^*$, a relevent question is: are the optimal model(s) within the parametrized subset $\mathfrak{M}^*$ of $\mathfrak{M}$ equivalent to the optimal model(s) within $\mathfrak{M}$ itself?

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This problem will be analyzed for the well-known class of equation error identification methods because of the nontriviality of its solution in this case. After some preliminary definitions, it will be shown that the above question has to be answered negatively for the commonly used model equivalence, based on input-output behavior. A criterion based notion of model equivalence will be introduced, leading to a new corresponding set of canonical forms.

The problem of system identification will be considered as a (deterministic) problem of approximate modeling of a sequence of measurement data, making no statistical assumptions about the data (see [4]).

Consider a given sequence of measurement data of a multivariable process, denoted by $w^N = \{w(t),\ t = 0, \cdots, N\}$, where $w(t) = \text{col}\,(y(t),\ u(t)) \in \mathbb{R}^{p+m}$, with the output signal $y(t) \in \mathbb{R}^p$ and the input signal $u(t) \in \mathbb{R}^m$.

We will consider the class of equation error models, represented by

$$M(z)\,w(t) = \epsilon(t) \tag{1}$$

with $M(z) = [P(z) | -Q(z)]$; $P(z)$ and $Q(z)$ are $[p \times p]$ and $[p \times m]$ polynomial matrices in the advance shift operator z: $z\,w(t) = w(t+1)$, and $\epsilon(t) \in \mathbb{R}^p$ is the equation error residual.

A model M of the data sequence w^N is any polynomial operator $M(z)$ that transforms the data $w(t)$ into a residual signal $\epsilon(t)$ according to (1). Notation (1) is quite natural since identification criteria in general are based on operations on residuals. In order to guarantee the existence of the equation (1) for a value of $t \geq 0$, it will be assumed that $N > \rho$ with ρ the degree of the polynomial matrix $M(z)$, throughout this note.

Since we are only interested in models that describe a causal and finite-dimensional relationship between input and output signals, we will specify the following sets of models.

Definition 1: Denote by:

$\mathcal{S}$: the set of all models $M(z)$, with $\det\{P(z)\} \not\equiv 0$, and $P(z)^{-1}Q(z)$ a proper rational matrix;

$\mathcal{S}_n$: the subset of $\mathcal{S}$ of all models having $\text{degr}\{\det(P(z))\} = n$;

$\mathcal{S}_{\bar{n}}$: the subset of $\mathcal{S}_n$ of all models with $(P(z), Q(z))$ left coprime. □

Two models $M_1, M_2 \in \mathcal{S}$ will be considered identical ($M_1 = M_2$) if and only if their corresponding representations $M_1(z)$ and $M_2(z)$ are equal.

Through the notion of strict system equivalence [5] the model set $\mathcal{S}_n$ has a one-to-one correspondence with the set of completely observable state-space models with input $u(t)$ and output $y(t)$, having a state-space dimension n. A similar statement can be made for $\mathcal{S}_{\bar{n}}$, corresponding to state-space models with a *minimal* dimension $\bar{n}$. Next, we will define an identification criterion in our context.

Definition 2a: An identification criterion J on the model set $\mathcal{S}$ is an operator J: $\mathbb{R}^{(p+m)^{N+1}} \times 2^{\mathcal{S}} \to 2^{\mathcal{S}}$, with $J(w^N, \mathfrak{M}) \subset \mathfrak{M}$, where $2^{\mathcal{S}}$ is the set of all subsets of $\mathcal{S}$. □

Definition 2b: For a given data sequence w^N a model M is called optimal within $\mathfrak{M}$ with respect to J if $M \in J(w^N, \mathfrak{M})$. □

In fact, the identification criterion is defined as a selection rule. Given a data sequence w^N and a model set $\mathfrak{M}$, the criterion J selects one or more optimal models. This definition of a criterion is very general. It encompasses both criterion minimization techniques (e.g., least-squares methods) and correlation techniques (e.g., instrumental variable methods.) In this note we will focus on identification criteria based on criterion minimization, which can be denoted by

$$J(w^N, \mathfrak{M}) = \arg\min_{M \in \mathfrak{M}} F(w^N, M)$$

$$\text{with } F \text{ an operator } F: \mathbb{R}^{(p+m)^{N+1}} \times \mathfrak{M} \to \mathbb{R}. \tag{2}$$

A specific choice for the operator F will be discussed dependent on the model set $\mathfrak{M}$ that is taken into account.

In our analysis we will further need the following two definitions.

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Definition 3: Two models $M_1, M_2 \in \mathcal{S}_n$ are called I/O equivalent, denoted $M_1 \overset{IO}{\sim} M_2$, if $P_1(z)^{-1}Q_1(z) \equiv P_2(z)^{-1}Q_2(z)$. □

Definition 4: Two model sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are called I/O equivalent, denoted $\mathfrak{M}_1 \overset{IO}{\sim} \mathfrak{M}_2$ if for all models in either of the two model sets there exists an I/O equivalent model in the other set. □

We will make use of the following notations: col $(y(t), u(t)) = [y(t)^T \; u(t)^T]^T$; $\Gamma_c(M)$ is the leading column coefficient matrix of $P(z)$ of a given model $M(z)$; $\mathcal{S}_n(\Gamma)$ is the subset of $\mathcal{S}_n$ of all models $M(z)$ with $\Gamma_c(M) = \Gamma$; $\mathcal{S}_n^c$ is the union of sets $\mathcal{S}_n(\Gamma)$ with nonsingular Γ.

II. Parametrizations Based on I/O Equivalence

Constructing uniquely identifiable parametrizations for the set of models $\mathcal{S}_n$ has received considerable attention in literature, both in terms of state-space models and input-output models; see, e.g., [1]-[3]. A number of parametrizations based on canonical or so-called overlapping forms, using the notion of I/O equivalence, has been proposed. All of these parametrizations select—in different ways—model sets $\mathfrak{J}_{\bar{n}} \subset \mathcal{S}_{\bar{n}}(I)$ such that $\mathfrak{J}_{\bar{n}} \overset{IO}{\sim} \mathcal{S}_{\bar{n}}$. In the accompanying identification algorithms $\mathfrak{J}_{\bar{n}}$ is then viewed as a representative for the model set $\mathcal{S}_{\bar{n}}$. See [6] and [7] for a comparative overview of these parametrizations.

Now let us consider the following problem. Given a data sequence w^N, an identification criterion J, and two I/O equivalent model sets $\mathfrak{M}_1 \overset{IO}{\sim} \mathfrak{M}_2$, will $J(w^N, \mathfrak{M}_1)$ and $J(w^N, \mathfrak{M}_2)$ be I/O equivalent?

In other words, will the solution of an identification problem be invariant under I/O equivalence transformations on the model set? We must first specify an identification criterion J before answering this question.

The well-known least-squares (LS) criterion leads to a criterion J according to (2) and a criterion function

$$F(w^N, M) = \frac{1}{N+1-\rho} \sum_{t=0}^{N-\rho} \epsilon^T(t)\epsilon(t). \tag{3}$$

Notice that this criterion, which is commonly applied on $\mathcal{S}_n(I)$, is not appropriate for the model set $\mathcal{S}_n$; it would lead to a situation in which $J(w^N, \mathfrak{M})$ is empty for any generic data sequence w^N, since $F(w^N, M) \to 0$ for $M(z) \to 0$. A natural generalization of this LS criterion function (3) can be defined by

$$F(w^N, M) = \frac{1}{N+1-\rho} \sum_{t=0}^{N-\rho} \epsilon^T(t)\Gamma_c(M)^{-T}\Gamma_c(M)^{-1}\epsilon(t). \tag{4}$$

This criterion (4) equals the standard LS criterion (3) on the model set $\mathcal{S}_n(I)$ and extends it in a straightforward way to the model set $\mathcal{S}_n^c$. In fact, application of (4) to a model $M \in \mathcal{S}_n^c$ amounts to first premultiplying $M(z)$ by $\Gamma_c(M)^{-1}$ and then calculating (3).

We can now state the following conjecture with respect to the question raised above.

Conjecture 1: Given a data sequence w^N, the least-squares identification criterion J, as defined by (2) and (4), and two model sets $\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathcal{S}_{\bar{n}}^c$, then $\mathfrak{M}_1 \overset{IO}{\sim} \mathfrak{M}_2$ *does not imply* that $J(w^N, \mathfrak{M}_1) \overset{IO}{\sim} J(w^N, \mathfrak{M}_2)$. □

The conjecture can be made plausible by considering the following. I/O equivalence on $\mathcal{S}_{\bar{n}}^c$ is known to be represented by a premultiplication of a model $M(z)$ by a unimodular matrix. This does not change the input-output transfer function $P(z)^{-1}Q(z)$ nor the value of degr { det $(P(z))$}. However, if we consider two I/O equivalent models M_1 and M_2, with $M_2(z) = U(z)M_1(z)$ and $U(z)$ unimodular, then consequently $\epsilon_2(t) = U(z)\epsilon_1(t)$. Since premultiplication by a unimodular polynomial matrix $U(z)$ influences the dynamics of the residual, application of the criterion J will generally lead to different optimal models, depending on the residual used. In situations in which $\epsilon_1(t)$ is an independent white noise sequence—often denoted by the expression the system is in the model set—the residual $\epsilon_2(t)$ of the I/O equivalent model will generally not be white.

The conjecture has far-reaching consequences for the application of a least-squares equation error identification method to the unique identifiable parametrizations, as mentioned in the beginning of this section. When using a set of canonical forms generating a model set $\mathfrak{J}_{\bar{n}} = \cup \mathfrak{J}_{\bar{n},i}$ with the subsets $\mathfrak{J}_{\bar{n},i}$ disjoint and $\mathfrak{J}_{\bar{n}} \subset \mathcal{S}_{\bar{n}}(I)$, as a representative for the model set $\mathcal{S}_{\bar{n}}$ (see, e.g., [1]), the identification result will be dependent on the specific choice of the set $\mathfrak{J}_{\bar{n}}$, thus the use of different sets of canonical forms for one model set will lead to models with different dynamic properties! In [8] an illustrative simulation example is presented that shows the consequence of this phenomenon. When using so-called overlapping forms $\mathfrak{J}_{\bar{n}}^* = \cup_i \mathfrak{J}_{\bar{n},i}^*$ with $\mathfrak{J}_{\bar{n}}^* \subset \mathcal{S}_{\bar{n}}(I)$ (e.g., see [2], [3], and [9]), the identification results will be dependent on the specific set of structure indexes chosen (the specific $\mathfrak{J}_{\bar{n},i}^*$), although each $\mathfrak{J}_{\bar{n},i}^*$ is generic in $\mathcal{S}_{\bar{n}}(I)$ in terms of I/O equivalence. Results in [7] and [10] present evidence for this effect by showing that properties of identified equation error models are dependent on the set of structure indexes chosen.

III. Criterion Based Equivalence

In order to create a situation where equivalence of two model sets guarantees equivalence of the corresponding identified models, a notion of model equivalence will have to be introduced that is related to the identification criterion J. Before we come to a formal definition, consider a data sequence w^N, a model set $\mathfrak{M}$, a criterion J, and two elements $M_1, M_2 \in J(w^N, \mathfrak{M})$. Now the following two situations can be distinguished.

1) M_1 and M_2 are both optimal models for *this specific data sequence* w^N with respect to J, but they are essentially different models, e.g., with different transfer function matrices $P_1(z)^{-1}Q_1(z)$ and $P_2(z)^{-1}Q_2(z)$.

2) M_1 and M_2 are optimal models with respect to J, not only for this w^N but for *any data sequence* for which one of the two is optimal.

Note that situation 1) is caused by a property of the data sequence, while situation 2) is caused by a property of the models. In the latter situation the criterion J cannot distinguish between one of the models M_1 and M_2, irrespective of the data sequence. This leads to the introduction of a criterion based model equivalence.

Definition 5: Two models M_1 and M_2 of a model set $\mathfrak{M}$ are J-equivalent, denoted $M_1 \overset{J}{\sim} M_2$, if $M_1 \in J(w^N, \mathfrak{M}) \leftrightarrow M_2 \in J(w^N, \mathfrak{M})$, for all possible data sequences w^N. □

This definition states that if one of two equivalent models is optimal for a data sequence w^N, then automatically the other model is optimal too. Since the relation as defined above is reflexive, symmetric and transitive, it is an equivalence relation on $\mathfrak{M}$.

Similar to the situation of I/O equivalence we define J-equivalent model sets.

Definition 6: Two model sets $\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$ are called J-equivalent, denoted $\mathfrak{M}_1 \overset{J}{\sim} M_2$ if for all models in either of the two model sets there exists a J-equivalent model in the other set. □

We can immediately state the following result.

Proposition 1: Given a data sequence w^N, an identification criterion $J(w^N, \mathfrak{M})$ and two model sets $\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$, then $\mathfrak{M}_1 \overset{J}{\sim} \mathfrak{M}_2 \Rightarrow J(w^N, \mathfrak{M}_1) \overset{J}{\sim} J(w^N, \mathfrak{M}_2)$. □

Proof: By directly applying the definitions: $M_1 \in J(w^N, \mathfrak{M}_1) \Rightarrow M_1 \in J(w^N, \mathfrak{M}_1 \cup \mathfrak{M}_2) \Rightarrow (\exists M_2 \in \mathfrak{M}_2 | M_1 \overset{J}{\sim} M_2$ and $M_2 \in J(w^N, \mathfrak{M}_1 \cup \mathfrak{M}_2)) \Rightarrow M_2 \in J(w^N, \mathfrak{M}_2)$ and vice versa. □

Consequently, when using the notion of J-equivalence for the parametrization of model sets, the identification result will be determined in terms of J-equivalence classes, and will not be dependent on the specific parametrization chosen. We will now specify this criterion based equivalence for the LS identification criterion discussed in Section II.

Theorem 1: Consider the identification criterion

$$J_{LS}(w^N, \mathcal{S}_n^c) = \arg \min_{M \in \mathcal{S}_n^c} F_{LS}(w^N, M) \tag{5}$$

with the least-squares criterion function defined on $\mathcal{S}_n^c$

$$F_{LS}(w^N, M) = \frac{1}{N+1-\rho} \sum_{t=0}^{N-\rho} \epsilon^T(t)\Gamma_c(M)^{-T}\Gamma_c(M)^{-1}\epsilon(t). \tag{6}$$

Then two models $M_1, M_2 \in \mathcal{S}_n^c$ are J_{LS}-equivalent if and only if there exists a nonsingular constant matrix U such that $M_1(z) = U M_2(z)$. □

Proof: The proof is added in the Appendix. □

Two more or less immediate results of this theorem are reflected in the following corollary.

Corollary 1: For the LS identification criterion (5), (6), defined on $\mathcal{S}_n^c$, the following expressions for two models $M_1, M_2 \in \mathcal{S}_n^c$ are equivalent.

a) $M_1 \overset{J}{\sim} M_2$.

b) $F_{LS}(w^N, M_1) = F_{LS}(w^N, M_2)$ for arbitrary data sequences w^N.

c) $P_1(z)^{-1}Q_1(z) = P_2(z)^{-1}Q_2(z)$ and $P_1(z) = U\,P_2(z)$ with U a constant nonsingular matrix. □

Proof: Equivalence of c) and a) follows directly from Theorem 1; b) → a) is straightforward because of Definition 5. a) → b): Since $M_1(z) = U\,M_2(z)$ and $\Gamma_c(M_1) = U\,\Gamma_c(M_2)$, substitution in (6) leads directly to the result. □

The corollary shows a natural interpretation of model equivalence: equivalent models cannot be distinguished by their criterion function regardless of the data. Result c) shows that the criterion based equivalence implies I/O equivalence, and in fact constitutes a further specification of this.

The criterion based model equivalence will now be used to construct a set of canonical forms for $\mathcal{S}_n^c$.

Theorem 2: The set of models $\mathcal{S}_n(I) \subset \mathcal{S}_n^c$ constitutes a set of canonical forms with respect to $J_{\rm LS}$-equivalence, for the model set $\mathcal{S}_n^c$. □

Proof: For all $M \in \mathcal{S}_n^c$ there exists a $J_{\rm LS}$-equivalent model in $\mathcal{S}_n(I)$, by premultiplication of $M(z)$ by $\Gamma_c(M)^{-1}$. Since $\mathcal{S}_n(I)$ does not contain $J_{\rm LS}$-equivalent models (see the proof of Theorem 1), this element is unique. □

This set of canonical forms $\mathcal{S}_n(I)$ can be interpreted as

$$\mathcal{S}_n(I) = \bigcup_{\Sigma\gamma_i = n} (\mathcal{S}_n(\gamma_1, \gamma_2, \cdots, \gamma_p, I)),$$

with $\mathcal{S}_n(\gamma_1, \gamma_2, \cdots, \gamma_p, I)$ the subset of $\mathcal{S}_n(I)$ of all models $M(z)$ with $P(z)$ having column degrees $\gamma_i \geq 0$. These subsets $\mathcal{S}_n(\gamma_1, \gamma_2, \cdots, \gamma_p, I)$ with $\Sigma\gamma_i = n$ are disjoint with respect to $J_{\rm LS}$-equivalence and each one of them can be parametrized in a form which is generally known as the pseudocanonical or overlapping parametrization, reflecting its properties in connection with I/O-equivalence [2], [3]. In these circumstances the overlapping property of this form is lost and the parametrization becomes canonical! Consequently, the choice of structure indexes γ_i will surely have its influence on the properties of the identified model, as reported in [7] and [10].

Note that the overlapping parametrization formally is defined as $\mathcal{S}_n(\gamma_1, \cdots, \gamma_p, I)$. The extension to $\mathcal{S}_n(\gamma_1, \cdots, \gamma_p, I)$ is straightforward and eliminates the need to check on the left comprimeness of $P(z)$, $Q(z)$ in an estimated model. Note also that special care has to be taken with respect to the properness of the transfer matrices $P(z)^{-1}Q(z)$: see, e.g., [3].

The results presented in this section can easily be applied to the situation of ARX models, defined in the delay operator z^{-1}, as well as to a basic instrumental variable identification criterion, which in our context takes on the form

$$M \in J_{IV}(w^N, \mathfrak{M}) \leftrightarrow F_{IV}(w^N, M, \zeta^N) = 0$$

with ζ reflecting the instrumental variables and F_{IV} the set of (correlation) functions to be tuned to 0.

IV. Conclusions

The problem of (approximate) modeling an input–output data sequence is considered when applying equation error (LS) identification methods to matrix fraction description (MDF) models. In the situation that a set of models has been parametrized using the notion of I/O equivalence, it is shown that the identified model will be essentially dependent on the specific parametrization chosen. As an alternative, a notion of model equivalence is introduced that is related to the identification criterion. The pseudocanonical or overlapping parametrization of MFD models appears to constitute a set of canonical forms for this least-squares based model equivalence.

Appendix

Proof of Theorem 1

(⇐) Suppose $M_1(z) = U\,M_2(z)$, with U constant nonsingular, then $P_1(z) = U\,P_2(z)$ and $\Gamma_c(M_1) = U\,\Gamma_c(M_2)$ (see, e.g., [11]). Substitution of this into $F_{\rm LS}$ gives $F_{\rm LS}(w^N, M_1) = F_{\rm LS}(w^N, M_2)$ for any w^N. It follows that $M_1 \overset{J}{\sim} M_2$.

(⇒) *Step 1:* Show that for $M_1, M_2 \in \mathcal{S}_n(I)$: $M_1 \overset{J}{\sim} M_2 \Rightarrow M_1(z) = M_2(z)$. Consider any element $M^* \in \mathcal{S}_n(I)$; it will be shown that there exists a data sequence w^N, such that M^* is the only element of $J(w^N, \mathcal{S}_n(I))$. Construct a data sequence w^N with $N \to \infty$, defined by this same model M^*: $M^*(z)\,w(t) = \xi(t)$ or $P^*(z)\,y(t) = Q^*(z)\,u(t) + \xi(t)$ with $\xi(t)$ an independent Gaussian white noise sequence, and $u(t)$ a persistently exciting input signal. It is a well-known result in system identification with equation error methods, that if the system that has generated the data is in the model set, the identified model will asymptotically generate white residuals. Consequently, for all models $M \in J(w^N, \mathcal{S}_n(I))$ it follows that: $\epsilon(t) = P(z)\,y(t) - Q(z)\,u(t) = [P(z)P^*(z)^{-1}Q^*(z) - Q(z)]u(t) + P(z)P^*(z)^{-1}\xi(t)$ where $\epsilon(t)$ has to be a white noise sequence. Since $M, M^* \in \mathcal{S}_n(I)$ and $M^* \in J(w^N, \mathcal{S}_n(I))$, it can be shown that for all $M \in J(w^N, \mathcal{S}_n(I))$, $P(z)P^*(z)^{-1} = I$ and $Q(z) = Q^*(z)$, leading to $M(z) = M^*(z)$.

Step 2: For any $M_1 \in \mathcal{S}_n^c$ there exists a unique J-equivalent model in $\mathcal{S}_n(I)$, given by $M(z) = \Gamma_c(M_1)^{-1}M_1(z)$. Since there are no distinct models in $\mathcal{S}_n(I)$ that are J-equivalent, two models M_1, M_2 in $\mathcal{S}_n^c$ can only be J-equivalent if they have the same equivalent element in $\mathcal{S}_n(I)$. So $M_1 \overset{J}{\sim} M_2 \Rightarrow \Gamma_c(M_1)^{-1}M_1(z) = \Gamma_c(M_2)^{-1}M_2(z) \Rightarrow M_1(z) = UM_2(z)$ with $U = \Gamma_c(M_1)\Gamma_c(M_2)^{-1}$, nonsingular. □

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Vibrational Control of an Exothermic CSTR: Productivity Improvement by Multiple Input Oscillations

XIANSHU SHU, KONSTANTINOS RIGOPOULOS, AND ALI CINAR

Abstract—**Previous studies have shown that vibrational control of an exothermic continuous stirred tank reactor (CSTR) by vibrating the total flow rate modifies its behavior and leads to the stabilized operation in its**

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