On Residual-Based Parametrization and Identification of Multivariable Systems

Paul Van den Hof

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PROEFSCHRIFT

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VOORWOORD

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Waddinxveen, 31 december 1988
Paul Van den Hof

CONTENTS

SUMMARY

In this thesis the problem of system identification is considered as a problem of deterministic approximate modelling on the basis of measurement data. In relation with classical methods of system identification, the approach presented shows two important aspects:

- the models obtained are accepted to be only an approximation of the dynamical process, underlying the data; and
- in the methodology of constructing models from the available time series, no use is made of any statistical assumptions on the data.

As a consequence of this approach, system identification methods are required to yield models that are well–defined, in the sense that the obtained models proceed from the available data sequence and from specified users' choices, and not from implicit (statistical) and hardly verifiable assumptions on the data and the underlying process.

Based on system theoretic concepts that recently have been introduced in the literature, a framework is presented in which the identification problem as considered above, can be formulated properly. In this framework the different components of an identification method: model set $\mathscr{M}$, parametrization $\tilde{M}$, and identification criterion J, are defined in a fundamental and natural way.
The model set $\mathscr{M}$ contains all models among which one is looking for "best" or "optimal" models for the given time series; based on this time series, the identification criterion selects optimal models from this model set; the parametrization represents the models in the model set with (real valued) parameters in order to simplify the selection procedure mentioned above.
In this thesis the question is discussed which requirements have to be laid upon these three concepts in order to arrive at optimal models that are well–defined.

Models for identification are defined in terms of behaviours, i.e. spaces of admissible signal trajectories. This notion of model behaviour appears to present a unifying approach to all existing definitions of linear, time–invariant and finite–dimensional systems, such as transfer functions, state space representations and difference equations. It gives the opportunity to clearly distinguish model sets from parametrizations, and to clearly distinguish the problems of identification and parametrization.
In the identification framework presented, models are used that contain three

types of external signals: inputs, outputs and residuals. The residual signals are artificially added to the (residual–based) models, in order to represent the modelling errors, and to act as a measuring tool for evaluation of the "measure of fit" between a model and a given time series.

The common identification methods are characterized within this framework, with special attention being paid to the types of residual signals that are considered (prediction error, output error and equation error type of residuals). The specific users' choices that underlie these methods are investigated. It is shown that for the popular class of equation error identification methods, the results of identification can become rather arbitrary, being influenced by accidental situations, instead of by well–defined users' choices. This phenomenon is due to the fact that the aspects of identification and parametrization have not properly been adjusted to each other.

An introduction to the identification problem as considered in this thesis, is given in chapter one.
In chapter two the framework is presented for formulating the problem of system identification on the basis of residuals. Notions generally used, such as model set, parametrization, identification criterion and identifiability, are reconsidered and clearly defined. An important notion of discriminability of model sets is introduced.
In chapter three, the common identification methods are characterized within the framework presented. The aspect of model complexity is discussed, leading to a theory on the order and structure indices of multivariable systems. By exploiting the advantages of polynomial matrices in two indeterminates, this theory unifies the current theories for models represented in either forward or backward difference equations.
Chapter four shows an elaboration on the problem of constructing identifiable parametrizations for a least squares–based identification criterion. It is shown that the identifiability of parametrizations has to be considered in relation to the identification criterion applied, in order to arrive at well–defined optimal models.
In chapters five and six some additional remarks are collected with respect to the identification methods considered, and the conclusions are summarized.

1. INTRODUCTION

1.1 SYSTEM IDENTIFICATION – A DEVELOPING FIELD OF RESEARCH

The problem of system identification can be described as the problem of creating (mathematical) models of dynamical processes on the basis of measurement data of the processes. Since the urge of creating (simple) models for phenomena that one meets in real life is present in many areas of science, the kind of processes to be considered can be varying over a large range of specific applications. Restricting attention to mathematical models, i.e. models that present mathematical relations between the signals involved, it can be stated that the application of these kind of models is widespread in science and technology, varying from the fields of physics, mechanics, and control engineering to the areas of econometrics and social sciences. Focussing on the application of mathematical models in technical surroundings, the following goals for models to be constructed can be recognized:

- to gain knowledge of the underlying process for purpose of diagnosis or detection;
- to predict future behaviour of the process;
- to simulate the process for purpose of planning, optimization, training of operators etc;
- to design automatic control systems for the process in an off–line or on–line procedure (adaptive control).

For creating models of physical phenomena several ways can be followed. In very many situations fruitful use can be made of physical laws that are supposed to underlie the process to be modelled, in order to arrive at a theoretical model of the process at hand. However one might not be satisfied with using these kinds of theoretical models only, e.g.

- when there is a lack of theoretical knowledge, e.g. in econometric problems where one is trying to find relations between economic variables;
- when the process is too complex and the corresponding theoretical models would become too complicated to be manageable;
- when there is a need for verification of the theoretical models;
- when the essential properties of the process are varying in time in an undefined way;

In these situations, modelling on the basis of measurement data of the process variables should be an additional tool for arriving at reliable models.

Every procedure for identification of a process on the basis of measurement data contains a specific set of components, no matter what type of process is being considered. The following items are fundamental for the problem of system identification (see also Ljung, 1987, for a similar but slightly different formulation).

a. Data collection.
   One has to deal with the question how to obtain measurement data that is best suited for identification purposes.
b. Choice of a model set.
   One has to determine what kind of models are going to be considered, and what kind of restrictions are going to be made: linearity, time–invariance, which signals are going to be considered as input and output variables, size of the models in terms of model order, etc.
c. Choice of a parametrization.
   The models have to be represented by a set of parameters that constitute a compact description of the different models, and that facilitate search procedures over the models in the model set.
d. Choice of the identification method and the identification algorithm.
   How to arrive at models determined on the basis of the measurement data and the previous choices. We will briefly refer to this item as the identification criterion.
e. Validation.
   Evaluation of the model(s) finally obtained in order to determine if it is suited for the application that one had in mind for the model. If this evaluation turns out to be negative, one has to reconsider the previous stages a. through d.

Note that the various choices in the items mentioned above can influence each other. In all these steps a. through e. one should take into account that the experimenter might have a priori knowledge on the process, from which he should benefit when considering the choices described above. Apart from this, the experimenter might impose additional restrictions at the several stages, having in mind the kind of application that the model is being created for; e.g. restrictions on the complexity of the models in order to obtain models that are manageable.

The different stages in an identification procedure are schematically depicted in figure 1.1, which resembles a similar picture in Ljung (1987).

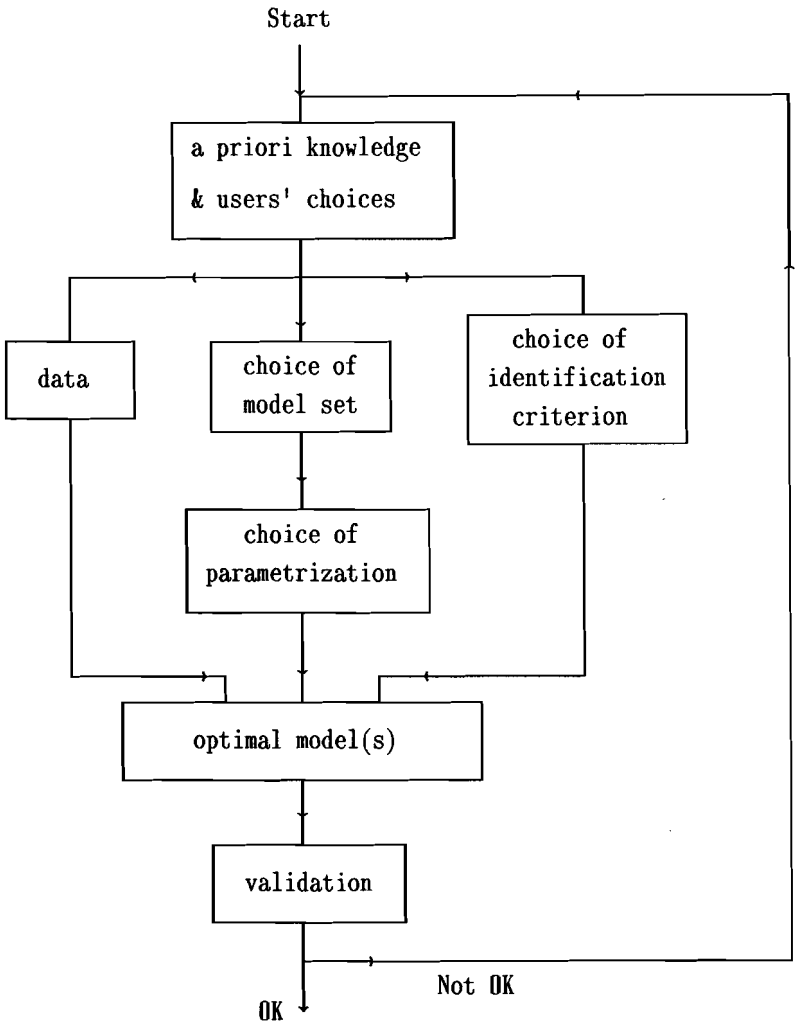

FIGURE 1.1 Schematic representation of an identification procedure

Despite these fixed stages in every system identification procedure, there are hardly any proper objective criteria for making specific choices in most situations listed above. Consequently system identification very often comes down to finding an ad–hoc solution to the problem at hand, using a lot of "engineering insight". Although the field of system identification already has been explored for a number of decades, and in spite of the fact that at the moment software for system identification becomes available on a commercial basis (like e.g. the identification

toolbox in MATLAB[1]), it is the opinion of the author that applying system identification techniques still seems to be an art rather than a science. Especially in the case of multivariable (multiple input multiple output) processes, the problems that arise when applying identification methods are numerous, and objective criteria for making proper decisions at the different stages in the identification procedure are lacking.

For this moment we will return to the basic formulation of the system identification problem and we will illustrate the philosophy based on which current methods of system identification have been developed, see e.g. Åström and Eykhoff (1971), Eykhoff (1974), Goodwin and Payne (1977), and Ljung (1987).
Let us consider a multivariable system having inputs u and outputs y, as schematically denoted in figure 1.2.

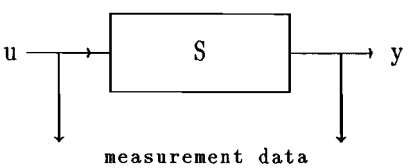

FIGURE 1.2 Dynamical system S with inputs u and outputs y

A model for this process can be considered as a set of equations that impose restrictions on the signals u and y, e.g.

$$A_0y(t)+A_1y(t-1)+\ .\ +A_ny(t-n) = B_0u(t)+B_1u(t-1)+\ .\ +B_mu(t-m) \qquad (1.1)$$

for all t=0,1,2,.., and the real matrices $A_0,.,A_n,B_0,.,B_m$ of proper dimensions.

Now considering the model, as represented by this set of equations, and considering a sequence of measured data $\{\tilde{y}(t),\tilde{u}(t)\}_{t=0,.,N}$, there are two situations possible:

1. the model is true, i.e. the set of equations (1.1) is satisfied for the measured data $\{\tilde{y}(t),\tilde{u}(t)\}_{t=0,.,N}$; or
2. the model is false, i.e. the set of equations (1.1) is not satisfied.

One can imagine that the formulation of an identification problem in this way is not very practical. Measurements that have been taken from the process will in general not satisfy the set of equations (1.1) exactly, due to all kinds of distur-

[1]MATLAB is a trademark of The Mathworks Inc.

bances. As a general philosophy to get out of this impasse, statistics has been employed in the following way.
If equation (1.1) is formulated with an error term e(t), according to:

$$A_0y(t)+A_1y(t-1)+ . +A_ny(t-n) = B_0u(t)+B_1u(t-1)+ . +B_mu(t-m)+e(t) \qquad (1.1)$$

and we assume that this error term e(t) has specific statistical properties, then the question whether a model is true or false, changes into a situation that a model is true with a certain probability, at least if we consider a finite observation interval. This probability can be optimized and an estimated model can be obtained.
Note that in this situation the *addition of statistical assumptions* creates the possibility to solve the problem.

In this philosophy of adding statistical assumptions it is incorporated that one is trying to find exact representations of the process that underlies the measured data, assuming that one indeed is able to describe this process exactly with a model of restricted complexity.
One of the first questions that should be asked in this situation, is the question of robustness: what happens if the assumptions on which the identification method is based, are not completely satisfied?
In order to illustrate possible consequences of such a situation, the following simulation example is presented.

Example 1.1

Consider a discrete–time dynamical system (S) with one input u and two outputs y_1 and y_2, defined by the following set of difference equations:

$$y_1(t+2)-2.7y_1(t+1)+2.12y_1(t)-0.16y_2(t) = -0.7u(t+1)+1.49u(t) \qquad (1.2a)$$

$$2.0y_1(t)-4.6y_1(t)+y_2(t+1) = -2.2u(t) \qquad (1.2b)$$

for all $t\in\mathbb{Z}$.
The corresponding transfer function $H_{yu}(z)$ has poles in $z=0.9$ and $z=0.9\pm0.1j$. The system is simulated with a unit variance white noise as input u(t), and the resulting output y(t) is disturbed with a noise term $\xi(t)$, according to:

$$\tilde{y}(t) = y(t) + \xi(t) \qquad (1.3)$$

with $\xi(t)$ an uncorrelated white noise signal with unit variance, that is not correlated with the input signal u(t). The signal to noise ratios on the outputs $\tilde{y}_1, \tilde{y}_2$ are about 50 dB. The measurement data $\{u(t),\tilde{y}(t)\}_{t=0,.,\infty}$ is considered as a basis

for an identification procedure.
For purpose of identification the two equations (1.2) are extended with equation error signals $e_1(t)$ and $e_2(t)$ and an estimated model is determined through the least squares identification method, by minimizing the expression

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} [e_1(t)^2 + e_2(t)^2] \tag{1.4}$$

over all models in the model set, parametrized by:

$$y_1(t+2)-a_1y_1(t+1)-a_2y_1(t)-a_3y_2(t) = b_1u(t+1)+b_2u(t)+e_1(t) \tag{1.5a}$$
$$-a_4y_1(t)-a_5y_1(t)+y_2(t+1) = b_3u(t)+e_2(t) \tag{1.5b}$$

for all $t\in\mathbb{Z}$, and all parameters $a_1,..,a_5,b_1,..,b_3\in\mathbb{R}$.
This least squares method implicitly assumes that the residual terms are stochastic and uncorrelated white noise sequences, while in this example the data sequence refers to the situation that the outputs y(t) and not the equations (1.2) have been disturbed by white noise. This noise contribution, leading to a signal to noise ration of 50 dB, shows that there is a very slight deviation from the assumptions implicitly incorporated in the method.

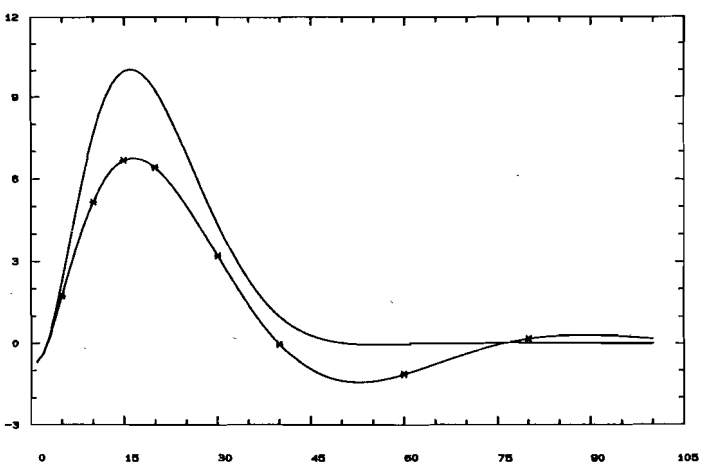

FIGURE 1.3 Markov parameters of the transfer function $H_{y_1u}(z)$ of the original process (S) (———), and of the estimated model (—×—).

In order to evaluate the estimated model, and to compare it with the original system, figure 1.3 shows the Markov parameters, i.e. the multivariable impulse response, of the transfer function between the input u and the first output signal y_1.
The poles of the transfer function $H_{yu}(z)$ of the estimated model are positioned at $z=0.922$ and $z=0.885\pm0.11j$. The results show that due to the very small noise terms on the simulated output signals, the asymptotic model deviates remarkably from the original process. □

Identification methods might be very sensitive with respect to the assumptions underlying the methods. From a methodological point of view this situation is not satisfactory. One can seriously dispute the validity of an identification method that is considered to generate a correct model, if a very small disturbance of the data (in example 1.1 a signal to noise ration of 50 dB) leads to essentially different models. The presence of these statistical assumptions in the fundamental formulation of the identification problem, has brought a number of authors to express their opinion on the state of the art quite strongly, as illustrated in the following quotations:

> "The subject is so underdeveloped at present that it is not possible to say very much about the identification of dynamical systems", (Kalman, 1982);

and

> "Notwithstanding the fact that identification theory and time series analysis have produced some very useful algorithms and important applications, it can be stated that there is a need to put a clear and rational foundation under the problem of obtaining models from time series. It is very much of an area where some of the first principles still need to be sorted out. In particular one should start by formalizing what is meant by an optimal (approximate) model" (Willems, 1986a).

There is a second reason for rejecting a statistical framework as a basic philosophy for solving an identification problem. In the course of years the conviction has been growing that processes to be modelled are in general far too complex to be modelled exactly by linear, time–invariant and finite dimensional models. On the other hand, the resulting models have to be relatively simple, in order to be applicable in a manageable way, e.g. in control system design. Consequently the modelling errors that we have to deal with, will not be caused mainly by random effects

like measurement noise, but will rather be due to the fact that our models are not complex enough.
Consequently, in this thesis the opinion of Willems (1986b) will be supported, stating that although there may be many situations in which a statistical framework indeed is a suitable one, it has many fundamental drawbacks as a general philosophy.

As an alternative approach the problem of system identification will be considered as a problem of deterministic approximation, in which the model(s) finally obtained, should be an optimal approximation of the process at hand, in a prespecified and well–defined sense.
During the identification procedure, the experimenter might contribute with reliable a priori knowledge on the process or with specific restrictions on the models to be obtained, and as such the introduction of statistical information on the process signals should also be considered: as possible additional information, but not as a basic philosophy for solving the identification problem.

When we position the identification problem in a nonstatistical context, considering it to be a problem of approximation, a completely different situation arises. In this situation explicit users' choices have to replace the implicit statistical assumptions discussed before. The consequences of these users' choices have to be investigated profoundly, in order to come up with criteria on the basis of which appropriate decisions can be made. As an example of a users' choice, note that the choice of different regressands in the linear regression models of example 1.1 (instead of the chose regressands $y_1(t+1)$ in eq. (1.5a) and $y_2(t+1)$ in eq. (1.5b)) definitely will lead to different models obtained from the identification method. In a statistical framework this is not recognized, since in the case of white equation error residuals the models obtained are invariant for a change of regressands.
Consequently, in order to arrive at well–defined optimal models, it has to be clarified which users' choices have to be made, and which users' choices influence the models to be obtained.

A general trend into the direction of considering approximate models instead of exact models, can also be found in the literature, however still mainly directed towards models that originate from statistical considerations. In the late seventies one started with considering properties of prediction error estimation schemes in situations that the system was not within the model set chosen, see e.g. Ljung (1978), Anderson, Moore and Hawkes (1978), Ljung and Van Overbeek (1978), and

Ljung and Caines (1979). Accepting the fact that models are approximative, it becomes an important question to analyse in which sense this approximation takes place in the different identification methods. In the frequency domain, an analysis on the properties of approximate prediction error models has been presented in Ljung (1985), Wahlberg and Ljung (1986) and Wahlberg (1987). In the time domain, results are obtained for approximate equation error models, by Mullis and Roberts (1976), Inouye (1983), Swaanenburg et.al. (1985) and Van den Hof and Janssen (1986,1987).
Since different identification methods show different approximation properties, a relevant question now also becomes:

> Are there any relations between identification methods and specific model applications, such as prediction, simulation, control etc.?

A first analysis of this question has led to some results on the comparison between equation error and output error identification methods, see e.g. Gevers and Bastin (1982), Damen *et al.* (1985), (1986) and Tomita *et al.* (1986).
Referring to this relation between intended model application and identification methods, one of the future challenges in the field of system identification will be to formulate identification methods that explicitly incorporate the intended model application (e.g. control system design), and that are directed towards yielding models that are best suited for this specific application.

1.2 PROBLEM STATEMENT

The core of the identification problem, as it will be considered in this thesis, can be visualized in figure 1.4.

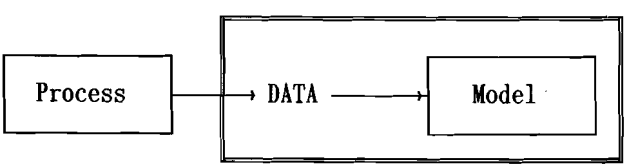

FIGURE 1.4 From process to model; the core of the identification problem.

In this thesis we will direct our attention to the second part of this figure: the construction of a model out of measurement data. Note that the first step, from process to data, is a step that we do not have complete knowledge of, since the data generating process is unknown. The second step, however, is a well–defined

step that allows a formal treatment without having a priori knowledge available. Restricting ourselves to this second step from data to model, the main parts of an identification procedure, as formulated in the previous section, now have been reduced to the following three choices:

1. the choice of a model set
2. the choice of a parametrization
3. the choice of an identification criterion

The central theme of this thesis can now be formulated as follows:

> Given a set of measurement data of the input and output variables of a multivariable dynamical system, construct a (set of) model(s) that in a well–defined sense optimally approximate(s) the available data sequence.

The expression "well–defined sense" should be interpreted in such a way that the models finally obtained from the identification procedure, should be based on the available data itself and on explicit users' choices, and not on implicit assumptions on the data and on the underlying process that are incorporated in the identification methods.
The basic choices 1–3 as mentioned above will be formalized by three notions: the model set $\mathcal{M}$, the parametrization $\tilde{M}$ and the identification criterion J.
As a derivative of the central theme, the following twofold problem statement can be given:

1) What are the requirements that have to be imposed on the different steps ($\mathcal{M}$,$\tilde{M}$,J) in an identification procedure, in order to obtain models that are a well–defined approximation of the available data sequence?

and

2) What are the consequences for these different steps ($\mathcal{M}$,$\tilde{M}$,J) in existing identification methods, when they are considered as methods for obtaining approximate models?

When elaborating on these questions, the following starting points will be taken into account.

- The system identification problem will be considered as a problem of deterministic approximation.
- The three basic choices: model set $\mathcal{M}$, parametrization $\tilde{M}$ and identification

criterion J, will be formalized in order to clearly characterize the different identification methods.
- The problem of parametrization will be incorporated in the problem of identification.
- For the existing identification methods, it has to be clarified which specific choices underlie these methods.
- It will be required to reformulate and to reconsider identification methods as procedures for deterministic approximate modelling.

In dealing with these aspects, the following restrictions will be made:
- Only models will be considered that are linear, time–invariant and finite dimensional, that are parametrized and that are represented in discrete time.
- No recursive identification schemes will be considered, since we are primarily interested in the fundamental aspects of identification, rather than algorithmic aspects.
- An a priori distinction will be made between input and output variables within the models.
- The question how to use a priori knowledge of the data generating process in the identification procedure will not explicitly be dealt with. However it has to be stressed that a priori knowledge and the intended model application are important aspects that should manifest itself in the different choices that have to be made ($\mathcal{M}$,$\tilde{\mathrm{M}}$,J). We will not primarily be concerned with the problem how to translate the intended model application into a suitable form for incorporating in the identification methods.

Summarizing this section, the work presented in this thesis will not give a complete answer to the identification problem (if ever possible). It will not be a manual with practical tips, nor a practical comparison of different identification methods. It will be an analysis that is intended to offer a better foundation under the system identification problem (in the opinion of the author), and that should offer an opportunity for a further development into a direction where specific goals of the obtained models can be incorporated in well–defined identification methods.

1.3 ELABORATION, JUSTIFICATION AND OVERVIEW OF CONTENTS

In order to discuss the construction of models in a fundamental way, there is a need of having a proper definition of the notion of model, and, even more general,

a need of having a proper definition of a dynamical system. Linear time–invariant and finite dimensional dynamical systems are commonly defined in terms of transfer functions, state space representations or difference equations, all having their own specific properties. However, all these definitions have one thing in common: they impose restrictions on the external signals of the dynamical system. In a dynamical system with inputs and outputs, not all pairs of input and output signals are admissible, but only those signals that are related through the laws of the dynamical system.
In the recent system theoretic work of Willems (1986a),(1988) this notion has been formalized and a dynamical system correspondingly is defined in terms of its behaviour, i.e. the set of admissible signal trajectories. This fundamental concept has been adopted in this thesis.
Despite the threshold that always has to be taken when entering a new formalism, the idea of describing dynamical systems on a level of signals has a number of important advantages.

* One unique definition of a dynamical system results, in which all current definitions, as mentioned above, are specific representations.
* A clear distinction is introduced between a dynamical system and its representation in terms of real coefficients, parameters, etc.
* The behaviour creates a joint aspect of different representations, giving the opportunity to clarify the relations and transformations between the different representations (like e.g. difference equations in the forward or backward shift operator).

Especially in view of the identification problem, the following remarks can be made with respect to this signal–based concept of a dynamical system:

* A description of a dynamical system in terms of admissible signals gives a good connection with the identification problem, in which signals are the starting-point.
* An important distinction is created between a set of dynamical systems (a set of models) and its representation (parametrization). This enables us to state clearly the relation and the distinction between the problems of parametrization and identification.
* There is a proper treatment of initial conditions which is beneficial for considering modelling problems of data sequences over a finite time interval.
* The abstract formulation of a dynamical system gives the opportunity to gener-

alize the current identification methods.

Based on this signal–based concept of a dynamical system, a framework will be constructed for the formulation of the identification problem. Within this framework the basic choices of model set ($\mathcal{M}$), parametrization ($\tilde{M}$) and identification criterion (J) will be defined, and their mutual relations and distinctions will be stated clearly. The purpose of the identification criterion is to select, given an available data sequence, that (those) model(s) out of the model set, that can be considered to be optimal for the available data sequence. Consequently the optimal models obtained are dependent on the specific model set and the identification criterion chosen. The parametrization takes care of a (unique) representation of the elements of the model set in terms of parameters; this is a problem of representation and consequently the parametrization itself should not influence the optimal models obtained.

Through the notion of a residual signal, as a natural extension of the signal–based concept of a dynamical system, a residual–based approach to the identification problem will be presented. This fictitious residual signal will be incorporated in the model descriptions in order to deal with the deviations between a measured data sequence and a data sequence that is admissible by a finite dimensional model. The way in which residuals enter the model descriptions, represent the locations within the model where the above mentioned deviations are discounted, and can be considered as a design variable for the experimenter.

The framework presented is very general and encompasses most existing identification techniques. Because of the clear conceptual separation of the identification and the parametrization problem, not only their mutual difference but also their relation are clarified. This evolves to the situation that the identifiability of a parametrization has to be reconsidered and has to be related to the identification criterion applied. A motivation for this relation can be found in the statement that a parametrization can only be identifiable if the corresponding models in the model set can be discriminated from each other by the identification criterion. The situation that results is depicted in figure 1.5, showing that a parametrization $\tilde{M}$ is strictly identifiable by the identification criterion J if the corresponding model set $\mathcal{M}$ is discriminable by J and the parametrization itself is bijective (see chapters 2 and 4 for the proper definitions).

(1) $\mathcal{M}$ is discriminable by J

(2) $\tilde{M}$ is bijective

$\Rightarrow \tilde{M}$ is strictly identifiable by J

FIGURE 1.5 Necessary conditions for a parametrization $\tilde{M}$ to be strictly identifiable by the identification criterion J.

For some identification methods (output error, prediction error methods), it will appear that condition (1) is relatively simple to satisfy and attention has to be devoted to condition (2), whereas in other methods (equation error methods) this will be the other way around.
With respect to this criterion–based approach to the identifiability problem, the philosophy presented in this thesis will be a further elaboration on earlier work as presented in Van den Hof (1987), (1988), (1989); some results will be closely related to corresponding results in Janssen (1988a).

In chapter two the basic concepts and definitions are introduced that will be used throughout this thesis. A basic methodology for dealing with an identification problem will be presented by using input–output–residual models. Commonly used notions, such as model set, identification criterion, parametrization and identifiability are reconsidered and clearly defined. A new notion of discriminability of models sets is introduced.
In chapter three the common identification methods are characterized within the presented framework, both with respect to the applied model sets, parametrizations, as well as with respect to the identification criteria. Model sets are classified in terms of properties of the corresponding residual signals; output error, prediction error and equation error type models are defined independent of any parametrization.
Chapter four shows an elaboration on the problem of constructing identifiable parametrizations for the standard least squares identification criterion. To this end the discriminability of model sets is analysed and sufficient conditions for discriminability are derived. It will be shown that different model sets (with different residual type models) have to be treated in an essentially different way when considering the identifiability question. Special attention is devoted to model sets of models with a fixed model order.
In chapter five some additional remarks are collected with respect to the considered identification methods, while conclusions are summarized in chapter six.

2. BASIC CONCEPTS

2.1 INTRODUCTION

In this chapter an overview will be given of the basic concepts to be used throughout this thesis. As motivated in the first chapter, a number of concepts will be adopted from Willems (1986a), especially those that refer to dynamical systems as collections of signal trajectories, being in accordance with the systems' equations. These basic definitions will be presented quite briefly in section 2.2; for a more extensive and philosophical treatment one is referred to Willems (1986a), (1988). Additionally the concept of input–output–residual model is introduced, to be used throughout this thesis as a central notion in dealing with identification problems. In section 2.3 a specific approach is formulated to attack the problem of modelling input–output data on the basis of residuals, positioning this problem in a context that appears to be very natural. The central aspects in this approach, the model set considered and the identification criterion, are defined in a general way, leaving the possibility for many different types of specification.
In section 2.4 the parametrization of model sets is considered, in relation with identifiability properties.

2.2 DYNAMICAL SYSTEMS AND INPUT–OUTPUT–RESIDUAL MODELS

First we will define the notion of a dynamical system.

Definition 2.2.1. Dynamical system S (Willems, 1986a,1988).
A *dynamical system* S is defined as a triple: S = $(T, W, \mathcal{B})$, with $T \subset \mathbb{R}$ the *time set*, W the *signal set*, i.e. the space in which the variables that are related to the system take on their values, and $\mathcal{B} \subset W^T$ the *behaviour* of the system, i.e. the space of all signal trajectories w: $T \rightarrow W$ that are compatible with the system. □

The behaviour $\mathcal{B}$ of a dynamical system S will generally be denoted by $\mathcal{B}(S)$.
In this thesis we will be dealing with discrete time system representations that are restricted to be linear, time–invariant and finite dimensional. The consequences of these restrictions are formulated in the following definition.

Definition 2.2.2. Set Σ of linear, time–invariant, finite dimensional systems on $\mathbb{I}$; (Willems, 1986a).

The set Σ of all *linear, time–invariant, finite dimensional systems on* $\mathbb{I}$ is defined by all dynamical systems $S=(T,W,\mathcal{B})$ that satisfy $T=\mathbb{I}$, W is a vector space, and $\mathcal{B}$ is a linear subspace of W^T that is closed in the topology of pointwise convergence, and that satisfies the shift–invariance property:

$w \in \mathcal{B}(S) \Leftrightarrow \sigma w \in \mathcal{B}(S) \Leftrightarrow \sigma^{-1}w \in \mathcal{B}(S)$ with the *shift operators* σ, σ^{-1} defined by: $(\sigma w)(t)=w(t+1)$, $t\in\mathbb{I}$ and $(\sigma^{-1}w)(t)=w(t-1)$, $t\in\mathbb{I}$. □

The choice for describing the systems on a time set $\mathbb{I}$, instead of e.g. $\mathbb{I}_+$, is motivated by the fact that it coincides with a symmetrical treatment of the shift operators σ, σ^{-1}. A choice $T=\mathbb{I}_+$ would lead to a shift–invariance requirement on $\mathcal{B}$, reflected only by the relation $w\in\mathcal{B} \Rightarrow \sigma w\in\mathcal{B}$, since σ^{-1} can not be defined properly on $\mathbb{I}_+$. This would cause an unbalanced treatment of the two shift operators. Under the additional assumption $\sigma\mathcal{B}=\mathcal{B}$, the two cases $\mathbb{I}$ and $\mathbb{I}_+$ coincide.

The signal set W will generally be chosen to be $\mathbb{R}^q$, with q the number of variables related to the system (inputs, outputs and possibly residuals).
Restrictions of signal variables and behaviours to time sets $\mathbb{I}_+$ or $\mathbb{I}_-$ will be denoted by w^+, $\mathcal{B}^+$and w^-, $\mathcal{B}^-$.

This concept of a linear, time–invariant finite dimensional system encompasses various definitions used in the literature such as the transfer function approach, representations with difference equations and the well known state space representation.
It can be shown (Willems, 1986a) that for every $S\in\Sigma$ the corresponding behaviour, denoted by $\mathcal{B}(S)$, can be represented by:

$$\mathcal{B}(S):=\{w\in W^{\mathbb{I}} \mid T(\sigma,\sigma^{-1})w=0,\ T\in\mathbb{R}^{g\times q}[z,z^{-1}]\}, \qquad (2.2.1)$$

with $\mathbb{R}^{g\times q}[z,z^{-1}]$ the ring of g×q polynomial matrices in the two indeterminates z and z^{-1}. The polynomial matrix $T(z,z^{-1})$ acts as an operator that maps $(\mathbb{R}^q)^{\mathbb{I}}$ into $(\mathbb{R}^g)^{\mathbb{I}}$. For applying (polynomial) matrix operations on w, as in equation (2.2.1), the i^{th} column of the matrix is supposed to operate on the i^{th} component of w, i.e. w is interpreted as a column vector. Any system $S\in\Sigma$ can be represented as the kernel of a polynomial matrix in the shift operators σ and σ^{-1}. The behaviour $\mathcal{B}(S)$ will be said to be induced by the polynomial matrix $T(z,z^{-1})$. This representation of $\mathcal{B}(S)$ will be used as a basic system representation throughout this thesis.

Example 2.2.3.
Consider a dynamical system S described by a polynomial representation:

$$y(t) + a_1y(t-1) + a_2y(t-2) = -b_0u(t) - b_1u(t-1) - b_2u(t-2) \qquad (2.2.2)$$

and $a_1,a_2,b_0,b_1,b_2 \in \mathbb{R}$. With $y(t)\in\mathbb{R}$, $u(t)\in\mathbb{R}$ it follows that $W=Y\times U$, $Y=U=\mathbb{R}$; $y,u \in \mathbb{R}^{\mathbb{Z}}$ and $w(t) = (y(t),u(t)) \in \mathbb{R}^2$, leading to $w \in (\mathbb{R}^2)^{\mathbb{Z}}$.
The behaviour $\mathcal{B}(S) \subset (\mathbb{R}^2)^{\mathbb{Z}}$ contains those trajectories $w=(y,u)$ that satisfy the equation (2.2.2) for all $t \in \mathbb{Z}$; this can be written as:
$\mathcal{B}(S):=\{w\in(\mathbb{R}^2)^{\mathbb{Z}} \mid R(\sigma,\sigma^{-1})w=0$, with $R(\sigma,\sigma^{-1})=[1+a_1\sigma^{-1}+a_2\sigma^{-2} \mid b_0+b_1\sigma^{-1}+b_2\sigma^{-2}]\}$. □

Note that the system S in this example 2.2.3 even has a number of specific properties that are not required in the previously given definitions. E.g. there is a causal relationship if we consider u to be an input and y to be an output; and given u(t), $t\in T$, on an interval $T = [t_1,\infty)$, then y(t) on T is uniquely specified by u(t) on T and a finite number of (initial) conditions. This latter effect is due to the fact that the system's variables are split up in inputs and outputs. We will first take a look at these specific properties of a dynamical system.

When considering the system's variables as inputs and outputs, (causes and effects), a special structure is laid upon the dynamical system, referring to the intuitive appeal that is connected to these notions. The character of an input signal is that it can be chosen freely, it is not bound by the system; the character of an output signal is that it is caused by the input, the system dynamics, and possibly a finite number of (initial) conditions. These notions can be formalized as follows.

Definition 2.2.4. Free variables (Willems, 1986a).
For any system $S\in\Sigma$ with signal set $W_1\times W_2$ and corresponding variables (w_1,w_2), the variables w_1 are called *free* if $\mathcal{B}_{w_1}(S) = W_1^{\mathbb{Z}}$, with

$$\mathcal{B}_{w_1}(S):=\{w_1\in W_1^{\mathbb{Z}} \mid \exists w_2\in W_2^{\mathbb{Z}},\ (w_1,w_2)\in\mathcal{B}(S)\}.$$ □

Free variables are not restricted by the dynamical system.

Definition 2.2.5. Processing (Willems, 1988).
For any system $S\in\Sigma$ with signal set $W_1\times W_2$ and corresponding variables (w_1,w_2), the expression w_2 *processes* w_1 is defined as:

$$\left.\begin{array}{l}(w_1,w_2'),\ (w_1,w_2'') \in \mathcal{B}(S)\\ w_2' = w_2''\end{array}\right\} \Rightarrow\ w_2' = w_2''$$

with $w_2'^{-} = w_2'|_{\mathbb{Z}_-}$. □

From this definition it follows that the expression "an output variable processes an input variable" means that once the input is specified, the output behaviour is finite dimensional, i.e. the output signal is specified by a finite number of initial conditions (see also example 2.2.9)

Definition 2.2.6. Input–output system (Willems, 1988).
A dynamical system $S=(T,W,\mathcal{B})\in\Sigma$ is called an *input–output system* $S\in\Sigma_{p,m}$ if $W=Y\times U$, with $Y=\mathbb{R}^p$, $U=\mathbb{R}^m$, while u is free and y processes u. □

For polynomial representations it follows that the behaviour of any input–output system can be written as:

$$\mathcal{B}(S) = \{w\in W^{\mathbb{Z}} \mid w=(y,u),\ P(\sigma,\sigma^{-1})y - Q(\sigma,\sigma^{-1})u = 0 \text{ with } P\in\mathbb{R}^{p\times p}[z,z^{-1}],\ Q\in\mathbb{R}^{p\times m}[z,z^{-1}],\ \det_{\mathbb{R}(z)}P\neq 0, \text{ with p, m the number of outputs, inputs of the system}\}. \quad (2.2.3)$$

$\det_{\mathbb{R}(z)}P$ is the determinant of P over the field $\mathbb{R}(z)$ of rational functions, which means that the determinant is considered as a rational function itself.
The requirement that y processes u, is reflected by the property that $P(z,z^{-1})$ is invertible as a rational matrix.
The rational function $H(z)=P(z,z^{-1})^{-1}Q(z,z^{-1})$ is the *transfer function* of the input-output system.
For two dynamical input–output systems S_1, S_2 induced by (P_1,Q_1) and (P_2,Q_2) as in equation (2.2.3), it can be shown that the two behaviours $\mathcal{B}(S_1),\mathcal{B}(S_2)$ are equal if and only if there exists a polynomial matrix U such that $[P_1(z,z^{-1})|-Q_1(z,z^{-1})] = U(z,z^{-1})[P_2(z,z^{-1})|-Q_2(z,z^{-1})]$, and $U\in\mathbb{R}^{pxp}[z,z^{-1}]$ unimodular over the ring $\mathbb{R}[z,z^{-1}]$, i.e. having an inverse that is again polynomial in the indeterminates z and z^{-1}.
In order to deal with the well known concept of causality we have to define:

Definition 2.2.7. Non–anticipation (Willems, 1988).
For any system $S\in\Sigma$ with signal set $W_1\times W_2$ and corresponding variables (w_1,w_2), it is defined that w_2 *does not anticipate* w_1 if

$$\{(w_1',w_2')\in\mathcal{B}(S),\ w_1''\in\mathcal{B}_{w_1'}(S) \text{ and } w_1'(t)=w_1''(t) \text{ for } t\leq 0\} \Rightarrow$$
$$\{\exists w_2'' \text{ such that } (w_1'',w_2'')\in\mathcal{B}(S) \text{ and } w_2''(t)=w_2'(t) \text{ for } t\leq 0\}. \quad \square$$

An input–output system will be called *causal*, if the output y does not anticipate the input u. In terms of the polynomial representation (2.2.3) an input–output system will be causal if and only if its transfer function H(z) is proper, which means that $\lim_{z\to\infty} H(z)$ exists.

We will need some other basic concepts related to general dynamical systems $S\in\Sigma$. In "classical" system theory the concept of controllability is related to the state space representation of a dynamical system. In this signal based framework it can be formulated as a property of the external behaviour of the system:

Definition 2.2.8. Controllable dynamical system. (Willems, 1988).
A dynamical system $S\in\Sigma$ is *controllable* if for all $w_1^-\in\mathcal{B}^-(S)$, $w_2^+\in\mathcal{B}^+(S)$ there exists a $t\in\mathbb{Z}_+$ and a $w:\mathbb{Z}\cap[0,t)\to W$ such that $w_1^- \wedge_0 w \wedge_t \sigma^{-t}w_2^+ \in \mathcal{B}(S)$. $\square$

The notation $\wedge$ denotes a concatenation of signals: $(w_1 \wedge_0 w_2)(t) := w_1(t)$ for $t<0$, and $w_2(t)$ for $t\geq 0$. In a controllable system any two admissible trajectories w_1, w_2 can be concatenated into a new admissible trajectory by insertion of a finite time trajectory w. It means that the behaviour at any time instant is not dependent on the behaviour at time instant $+\infty$ or $-\infty$. This notion of controllability is stronger than the well–known "pointwise state controllability to the origin", connected to state space systems (see e.g. Kailath, 1980). Definition 2.2.8 is stated in an input–output setting, and it actually requires both "pointwise state controllability to the origin" and "pointwise state controllability from the origin" (reachability); it is similar to the controllability of an input–output relation as presented in Blomberg and Ylinen (1983).

For any input–output system S, generated by polynomial matrices $P(z,z^{-1})$ and $Q(z,z^{-1})$ as in (2.2.3), the system S is controllable if and only if $(P(z,z^{-1}), Q(z,z^{-1}))$ are left coprime with respect to $\mathbb{R}[z,z^{-1}]$; this means that every left common factor in $(P(z,z^{-1}),Q(z,z^{-1}))$ is unimodular over the ring $\mathbb{R}[z,z^{-1}]$. It can be shown (Willems, 1988) that for two controllable input–output systems to be equal (having the same behaviour) it suffices that their transfer functions are equal.

Example 2.2.9.
Consider a dynamical system $S\in\Sigma_{1,1}$ with Y, $U = \mathbb{R}$ (single input, single output), with a behaviour $\mathcal{B}(S)$ defined by:

$$\mathcal{B}(S):=\{w\in(\mathbb{R}^2)^{\mathbb{Z}} \mid w=(y,u),\ (\sigma+a_1)y - (b_0\sigma+b_1)u = 0,\ a_1,b_0,b_1\in\mathbb{R}\setminus\{0\}\}.$$

The variable u is free ($\mathcal{B}_u(S)=\mathbb{R}^{\mathbb{Z}}$). If we take one specific $u\in U^{\mathbb{Z}}$ the output y then is restricted by the polynomial equation. By choosing a value for one time instant of y (e.g. y(0)), the variable y is completely determined by the equations:

$y(t+1)+a_1y(t)=b_0u(t+1)+b_1u(t)$, for all $t\in\mathbb{Z}$.

Consequently y processes u. Note that in this example it can be deduced by similar reasoning that y is free, since $\mathcal{B}_y(S)=\mathbb{R}^{\mathbb{Z}}$, and that u processes y.
For evaluation of controllability consider any signal $w^{1+}\in\mathcal{B}(S)|_{\mathbb{Z}\cap[1,\infty)}$. The signal w^{1+} is completely determined by a choice for u^{1+} and y(1). Similarly consider any signal $w^-\in\mathcal{B}(S)|_{\mathbb{Z}\cap(-\infty,-1]}$, which is completely determined by a choice for u^- and y(−1). Any value of w(0) = (y(0),u(0)) such that $w^- \underset{0}{\wedge} w(0) \underset{1}{\wedge} w^{1+} \in \mathcal{B}(S)$ has to fulfil: $y(0) - b_0u(0) = b_1u(-1) - a_1y(-1)$, and
$-a_1y(0) + b_1u(0) = -b_0u(1) + y(1)$.
There exists a solution for w(0) in this set of equations if and only if the determinant of the matrix $\begin{bmatrix}1 & -b_0\\ -a_1 & b_1\end{bmatrix}$ is unequal 0, which means that $b_1-a_1b_0 \neq 0$. This is exactly the condition that $(z+a_1)$ and (b_0z+b_1) should have no common factors (are left coprime). It can simply be verified that if there does not exist a proper w(0) for the interval [0,1), extension of the interval to $[0,t_1)$ with $t_1>1$ will not lead to any solutions. □

We will now proceed with our presentation of basic concepts related to dynamical systems, with the following definitions.

Definition 2.2.10. Autonomous dynamical system (Willems, 1986a).
A dynamical system $S\in\Sigma$ is called *autonomous* if there exists a linear map $f:\mathcal{B}^-(S)\to\mathcal{B}^+(S)$ such that for all $w\in W^{\mathbb{Z}}$:

$$\{w=w^- \underset{0}{\wedge} w^+ \in\mathcal{B}(S)\} \Leftrightarrow \{w^+=f(w^-)\}.$$ □

In all trajectories of the behaviour of an autonomous system, the future is implied by the past. Consequently the system does not have any free variables, and the

behaviour $\mathcal{B}(S)$ is a finite dimensional subspace of $W^{\mathbb{Z}}$.
In terms of a polynomial representation of $\mathcal{B}(S)$, as in equation (2.2.1), autonomy of S is equivalent with the existence of a square matrix $T(z,z^{-1})$ with $\det_{\mathbb{R}(z)} T(z,z^{-1})\neq 0$, representing the behaviour by $\mathcal{B}(S)=\{w\in W^{\mathbb{Z}} \mid T(\sigma,\sigma^{-1})w=0\}$.

Definition 2.2.11. Controllable part of a dynamical system; S_c.
The *controllable part* S_c of a dynamical system $S\in\Sigma$ with $S=(\mathbb{Z},W,\mathcal{B})$ is defined by $S_c:=(\mathbb{Z},W,\mathcal{B}_c(S))$ with
$\mathcal{B}_c(S):=\{w_1\in\mathcal{B}(S) \mid \exists\ w^-\in\mathcal{B}^-(S)$ and $t_1\in\mathbb{Z}_-$, $w^-(t)=0$ for $t\leq t_1$, and $(w^- \wedge_0 w_1)\in\mathcal{B}(S)\}$ □

The controllable part of a behaviour $\mathcal{B}$ consists of all trajectories that can be reached from zero. It can be shown that S_c again constitutes a dynamical system $S_c\in\Sigma$, acting as the largest controllable subsystem of $S\in\Sigma$. Evidently S is controllable if and only if $S_c=S$.
For a general input–output system $S\in\Sigma_{p,m}$, this controllable subsystem S_c is completely determined by the transfer function H(z) of S.

Definition 2.2.12. Observability of signals (Willems, 1988).
For any system $S\in\Sigma$ with signal set $W_1\times W_2$ and corresponding variables (w_1,w_2), it is defined that *w_2 is observable from w_1* if there exists a map $F: \mathcal{B}_{w_1}\to\mathcal{B}_{w_2}$ such that $\{(w_1,w_2)\in\mathcal{B}(S) \Leftrightarrow w_2=Fw_1\}$. □

The definition of observability shows that w_2 is completely specified once w_1 is known. For input–output systems induced by polynomial matrices $P(z,z^{-1})$ and $Q(z,z^{-1})$ as in eq. (2.2.3), the output variable will be observable from the input variable if and only if $P(z,z^{-1})$ is unimodular with respect to $\mathbb{R}[z,z^{-1}]$.

For input–output systems we will be interested in specifying a restricted part of the total behaviour of a system. When applying an input signal u to a system $S\in\Sigma_{p,m}$, we are interested in describing the corresponding output signal as a sum: $y = y_1 + y_2$, where y_1 is that part of the output that is directly caused by the input, and y_2 is an autonomous part of the output signal, not caused by the input but due to –what can be called– initial conditions of the system. In view of such a decomposition we define:

Definition 2.2.13. Initial zero part of a behaviour; $\mathcal{B}_0(S)$.
Let S be a dynamical system $S\in\Sigma$. The *initial zero part* of the behaviour $\mathcal{B}(S)$ is defined by: $\mathcal{B}_0(S) := \{w\in\mathcal{B}(S) \mid \exists t_1\in\mathbb{Z},\ w(t)=0 \text{ for } t<t_1\}$. □

In general $\mathcal{B}_0$ will not be the behaviour of a system within Σ, since it will generally not be closed. However it has to be noted that the closure of $\mathcal{B}_0$ is equal to $\mathcal{B}_c$, ($\overline{\mathcal{B}_0} = \mathcal{B}_c \subset \mathcal{B}$) and consequently $\mathcal{B}_0$ specifies $\mathcal{B}_c$ uniquely. The most important property of trajectories $w\in\mathcal{B}_0$, is reflected in the following proposition.

Proposition 2.2.14.
Let S be a dynamical system $S\in\Sigma$ with variables (y,u) such that y processes u.
Then $\{(y_1,u_1)\in\mathcal{B}_0,\ (y_2,u_2)\in\mathcal{B}_0 \text{ and } u_1=u_2\} \Rightarrow y_1=y_2$. □

For any "input signal" u there exists a unique "output signal" y such that $(y,u)\in\mathcal{B}_0$. Consequently it follows that within $\mathcal{B}_0$ the signal y is observable from the signal u.

Remark 2.2.15.
The unique signal y can be constructed by applying a Laurent series expansion around $z=\infty$ to the transfer function H(z) of the dynamical system:

$$H(z) = \sum_{k=t}^{\infty} M(k)\, z^{-k}, \qquad \text{for some } t\in\mathbb{Z},$$

and the signal y results from $y(k) = \sum_{i=t}^{\infty} M(i)\, u(k-i);\ k\in\mathbb{Z}$.

The matrices $\{M(k),\ k=t,..,\infty\}$ are called the *Markov parameters* of the transfer function H(z).
Note that there is a second way to construct a subset of $\mathcal{B}_c$ within which y is observable from u. In this alternative form the signals w are restricted to $w(t)=0$ for $t>t_1$ (end zero part), and the corresponding construction of y for a given u is via a Laurent expansion of the transfer function around $z=0$. Since we will generally be interested in causal processes, the choice for $\mathcal{B}_0$ as defined above is the most natural one. □

Until now signals and behaviours have been considered on the time set $\mathbb{Z}$. In many situations we will have to deal with signals that are only available on a restricted time interval. To this end we will use the time set $T_N=\mathbb{Z}\cap[0,N-1]$, with $N\in\mathbb{N}$, and restrictions of trajectories and behaviours to this set will be denoted by w^N, $\mathcal{B}^N$ etc. The same philosophy that has led to the introduction of the initial zero part of a

behaviour can now be applied to the behaviour on the interval T_N: decomposition of trajectories $(y^N,u^N)\in\mathcal{B}^N$ into a part in which y^N is uniquely implied by u^N, and a second part in which y^N is independent from u^N.

Proposition 2.2.16.
Consider a dynamical system $S\in\Sigma$ with variables (y,u) such that y processes u. For the given time set T_N there exists a direct sum decomposition of the behaviour $\mathcal{B}^N(S) = \mathcal{B}(S)|_{T_N}$, given by:

$$\mathcal{B}^N(S) = \mathcal{B}^N_{obs,y}(S) \oplus \mathcal{B}^N_{init,y}(S) \tag{2.2.4}$$

with $\mathcal{B}^N_{obs,y}(S) := \{w^N\in\mathcal{B}^N \mid \exists\, w_1,w_2\in W^{\mathbb{Z}},\ w_1=(y_1,u_1),\ w_2=(y_2,u_2),\ u_1=u_2=0,$ and

$$(w_1 \underset{0}{\wedge} w^N \underset{N}{\wedge} w_2) \in \mathcal{B}_0\}, \tag{2.2.5}$$

and $\mathcal{B}^N_{init,y}(S) := \{w^N\in\mathcal{B}^N \mid w^N=(y^N,u^N),\ u^N=0\}.$ (2.2.6) □

Proof. By inspection it follows that $\{w^N\in\mathcal{B}^N_{obs,y}(S) \wedge w^N\in\mathcal{B}^N_{init,y}(S)\} \Rightarrow w^N=0$. For a given $w^N\in\mathcal{B}^N(S)$ the corresponding element $w_1^N\in\mathcal{B}^N_{obs,y}(S)$ is unique because of the fact that y processes u. This proves the result. □

Example 2.2.17.
Consider the dynamical input–output system S as in example 2.2.9 with $W=Y\times U=\mathbb{R}^2$, and $\mathcal{B} := \{w\in(\mathbb{R}^2)^{\mathbb{Z}} \mid w=(y,u),\ (\sigma+a_1)y - (b_0\sigma+b_1)u = 0,\ a_1,b_0,b_1\in\mathbb{R}\setminus\{0\}\}$.
The direct sum decomposition of proposition 2.2.16 can now be specified according to:

$$\mathcal{B}^N_{obs,y}(S) := \{w^N\in(\mathbb{R}^2)^{T_N} \mid w^N=(y^N,u^N),\ y(0)=b_0u(0),\text{ and}$$
$$y(t)+a_1y(t-1)=b_0u(t)+b_1u(t-1);\ 1\le t\le N-1\};$$
$$\mathcal{B}^N_{init,y}(S) := \{w^N\in(\mathbb{R}^2)^{T_N} \mid w^N=(y^N,u^N),\ u^N=0,\ y(t)+a_1y(t-1)=0;\ 1\le t\le N-1\};$$

Note that for any $w^N\in\mathcal{B}^N$ there exists a unique decomposition $w^N=w_1^N + w_2^N$, with $w_1^N\in\mathcal{B}^N_{obs,y}(S)$ and $w_2^N\in\mathcal{B}^N_{init,y}(S)$. On the other hand, for any given $u^N\in U^{T_N}$ there exists a unique y^N such that $(y^N,u^N)\in\mathcal{B}^N_{obs,y}(S)$.
In Figure 2.1 a schematic representation of this decomposition is shown. □

Note that the decomposition described above is not restricted to systems with a proper transfer function. Also non–causal input–output systems can be handled

without any problem.

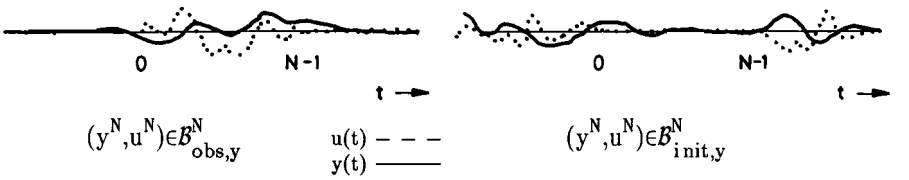

FIGURE 2.1. Schematic representation of $\mathcal{B}^N = \mathcal{B}^N_{obs,y} \oplus \mathcal{B}^N_{init,y}$ with $(y^N,u^N)\in\mathcal{B}^N$.

Another way of formulating the result of proposition 2.2.16, is to write it in a form that stresses the decomposition of the signal y for a given signal u:

$$\mathcal{Y}^N(u^N,S) = \mathcal{Y}^N_{obs}(u^N,S) \oplus \mathcal{Y}^N_{init}(S) \quad \text{for any } u^N\in U^{T_N}; \tag{2.2.7}$$

with

$$\mathcal{Y}^N(u^N,S) := \{y^N\in Y^{T_N} \mid w^N=(y^N,u^N) \in \mathcal{B}^N(S)\}; \tag{2.2.8}$$

$$\mathcal{Y}^N_{obs}(u^N,S) := \{y^N\in Y^{T_N} \mid w^N=(y^N,u^N) \in \mathcal{B}^N_{obs,y}(S)\}; \tag{2.2.9}$$

$$\mathcal{Y}^N_{init}(S) := \{y^N\in Y^{T_N} \mid w^N=(y^N,u^N),\ u^N=0,\ w^N \in \mathcal{B}^N_{init,y}(S)\}; \tag{2.2.10}$$

As a consequence of propositions 2.2.14 and 2.2.16 the trajectory space $\mathcal{Y}^N_{obs}(u^N,S)$ contains exactly one element. The trajectory space $\mathcal{Y}^N_{init}(S)$ contains those trajectories y^N that are generated by elements of the signal u outside the interval T_N, and the trajectories that are not generated by any signal u at all (autonomous behaviour of y).
Corresponding to remark 2.2.15 the signal $y^N\in\mathcal{Y}^N_{obs}(u^N,S)$ is determined by

$y(k) = \sum_{i=k-N+1}^{k} M(i)\, u(k-i)$, $k\in T_N$, with M(i) the Markov parameters of the corresponding transfer function.

After discussing the main concepts with respect to dynamical systems that will be used throughout this thesis, our attention will now be directed towards residual models.

Although the words "system" and "model" intuitively refer to different phenomena, their meaning in mathematical terms is, in fact, not essentially different. They both refer to mathematical relations between signals (variables). In our context we will use the word "system" in general to refer to (physical) processes to be modelled, or to formal dynamical systems as defined in the previous part of this section. The expression "model" will be used for the mathematical representation that is supposed to describe the essential characteristics of a (physical) process. In other words: a model will be a dynamical system to be used in modelling procedures. We will be using models that are linear, time–invariant and finite dimensional, that describe relations between three kinds of model variables: inputs (u), outputs (y) and residuals (e).

Definition 2.2.18. Input–output–residual (i/o/r)–model $M_{p,m,r}$.

An *input–output–residual model* or (i/o/r)–model $M_{p,m,r}$ is defined as a linear time–invariant finite dimensional dynamical system on $\mathbb{I}$, with $W = Y \times U \times E$, $Y = \mathbb{R}^p$, $U = \mathbb{R}^m$, and $E = \mathbb{R}^r$, where u and e are free and y processes (u,e). □

The signals $y \in Y^{\mathbb{I}}$, $u \in U^{\mathbb{I}}$, $e \in E^{\mathbb{I}}$ will be denoted as output, input and *residual*. As defined above we will always consider signals w that are defined as $w: \mathbb{I} \to W$, or $w \in W^{\mathbb{I}}$. Any such element w will be written as w = (y,u,e) or as w = (v,e) with $v \in V^{\mathbb{I}}$, $V = Y \times U$, and v=(y,u). The variables v will be the variables that can be measured, while residuals e in fact are artificially added to the model description to deal with modelling errors or external (stochastic) influences on the measurement data. The behaviour of an (i/o/r)–model $M_{p,m,r}$ will be denoted by $\mathcal{B}(M_{p,m,r})$. If there is no risk of misinterpreting, the indices p,m,r, will be omitted. The set of all (i/o/r)–models will be denoted by $\tilde{\Sigma}_{p,m,r}$ or $\tilde{\Sigma}$.

Definition 2.2.19. Set of (i/o/r)–models $\mathscr{M}_{p,m,r}$.

A set of (i/o/r)–models $\mathscr{M}_{p,m,r}$ is any collection of (i/o/r)–models $\{M^{(\alpha)}_{p,m,r} \in \tilde{\Sigma}_{p,m,r},\ \alpha \in I_\alpha\}$ with I_α an index set. □

If the subscripts are clear from the context, $\mathscr{M}_{p,m,r}$ will be written as $\mathscr{M}$.

Remark 2.2.20.

Let $\mathscr{M}_{p,m,r}$ be a set of (i/o/r)–models. It follows from definition 2.2.18 that two

models M_1, $M_2 \in \mathscr{M}_{p,m,r}$ will be equal ($M_1=M_2$) if and only if their behaviours are equal: $\mathscr{B}(M_1) = \mathscr{B}(M_2)$ □

Remark 2.2.21.
An (i/o/r)–model is a special form of a dynamical system with auxiliary variables, as defined in Willems (1986a). It can be interpreted in different ways:

1. In terms of linear input–output systems with disturbances, an (i/o/r)–model can be viewed as a dynamical system with input signals u, disturbance inputs e and output signals y. In view of this interpretation it is a natural condition to require that y processes (u,e).
2. In terms of dynamical models used for identification it is much more appropriate to consider an (i/o/r)–model as a model with as inputs the measured data consisting of u and y, and as output the residual e. A corresponding additional restriction on the system variables that supports this interpretation is the condition that e processes (y,u), while y and u are free.

The residual signals will play an important role in the modelling procedures that are going to be treated. They reflect the error terms that are required to incorporate in a measured data sequence, for creating a signal trajectory that is admissible for a model in the model set.
The models that are applied in this thesis will generally be required to satisfy both interpretations – and the corresponding restrictions – as mentioned above. This leads to the following special type of (i/o/r)–model.

Definition 2.2.22. Input–output–processing residual (i/o/pr)–model $M_{p,m}$.
An *input–output–processing residual model* or (i/o/pr)–model $M_{p,m}$ is defined as an (i/o/r)–model $M_{p,m} \in \tilde{\Sigma}_{p,m,p}$ for which holds that e processes (y,u) and y and u are free. □

Note that the fact that e processes (y,u) implies that the number of residuals has to be equal to the number of output signals (r=p). The set of all (i/o/pr)–models $M_{p,m}$ will be denoted by $\hat{\Sigma}_{p,m}$ or $\hat{\Sigma}$.

A further specification of (i/o/pr)–model sets follows in chapter 3.
An important concept for evaluation of admissible residual signals is reflected in the following definition.

Definition 2.2.23. Set of admissible residual signals for given v; $\mathcal{E}(v,M)$.
The *set of admissible residual signals* of an (i/o/r)–model M for a given measurement data sequence v is defined as:

$$\mathcal{E}(v,M):=\{e\in E^{\mathbb{Z}} \mid (v,e)\in\mathcal{B}(M)\}, \tag{2.2.11}$$

i.e. those residuals e that for a given v are part of the model behaviour. □

For (i/o/pr)–models the set $\mathcal{E}^N(v^N,M)$ can uniquely be decomposed for any $v^N\in V^{T_N}$ in accordance with proposition 2.2.16 and equations (2.2.8)–(2.2.10):

$$\mathcal{E}^N(v^N,M) = \mathcal{E}^N_{obs}(v^N,M) \oplus \mathcal{E}^N_{init}(M) \tag{2.2.12}$$

with the observable part

$$\mathcal{E}^N_{obs}(v^N,M) = \{e^N\in E^{T_N} \mid \exists w_1,w_2\in W^{\mathbb{Z}},\ w_1=(v_1,e_1),\ w_2=(v_2,e_2),\ v_1=v_2=0,\ w_1 \underset{0}{\wedge} (v^N,e^N) \underset{N}{\wedge} w_2 \in \mathcal{B}_0(M)\} \tag{2.2.13}$$

containing exactly one element, and the initial part:

$$\mathcal{E}^N_{init}(M) = \{e^N\in E^{T_N} \mid (0,e^N) \in \mathcal{B}^N(M)\} \tag{2.2.14}$$

independent of the measured signals v.

For the purpose of modelling on the basis of data, we will deal with (i/o/r)–models in this thesis. Nevertheless our main interest is still the relation between the measured external signals y and u of the process to be modelled. For this reason the i/o–part of an (i/o/r)–model will also be specified.

Definition 2.2.24. i/o–part of (i/o/r)–model; M^{io}.
Let M be an (i/o/r)–model, $M\in\tilde{\Sigma}$, with $M = (\mathbb{Z}, Y\times U\times E, \mathcal{B}(M))$. The i/o–part of M, denoted by M^{io} is defined by:

$$M^{io}:= (\mathbb{Z}, Y\times U, \mathcal{B}^{io}(M)) \tag{2.2.15}$$

with $$\mathcal{B}^{io}(M) := \{v\in V^{\mathbb{Z}} \mid (v,e)\in\mathcal{B}(M),\ e=0\}. \tag{2.2.16}$$ □

For evaluation of the i/o–part of a (i/o/r)–model, the residual component in the model is simply discarded.
As a final item in this section some equivalence relations between models will be presented for ease of reference in the rest of this thesis.

Definition 2.2.25. Model equivalences.
Consider two models M_1, $M_2 \in \tilde{\Sigma}$; the following equivalences are defined:

a. $M_1 \sim M_2$ $:= \mathcal{B}(M_1) = \mathcal{B}(M_2)$ behaviour–equivalence;

b. $M_1 \overset{t}{\sim} M_2$ $:= \mathcal{B}_c(M_1) = \mathcal{B}_c(M_2)$ controllable behaviour–(transfer–) equivalence;

c. $M_1 \underset{io}{\sim} M_2$ $:= \mathcal{B}^{io}(M_1) = \mathcal{B}^{io}(M_2)$ i/o behaviour–equivalence;

d. $M_1 \overset{t}{\underset{io}{\sim}} M_2$ $:= \mathcal{B}_c(M_1^{io}) = \mathcal{B}_c(M_2^{io})$ i/o controllable behaviour–(transfer–)equivalence; □

The equivalence relations defined above can be written in the following diagram, showing the interrelationships:

$$\begin{array}{ccc} \text{(a) } M_1 \sim M_2 & \Rightarrow & \text{(b) } M_1 \overset{t}{\sim} M_2 \\ \Downarrow & & \Downarrow \\ \text{(c) } M_1 \underset{io}{\sim} M_2 & \Rightarrow & \text{(d) } M_1 \overset{t}{\underset{io}{\sim}} M_2 \end{array}$$

FIGURE 2.2 Relationships between model equivalences.

By definition behaviour–equivalence is represented by model equality. Controllable behaviour equivalences (b) and (d) are called transfer equivalences because for input–output systems they refer to equality of the transfer functions involved. These different equivalence relations will be further specified in section 2.4 when considering polynomial matrix representations.

2.3 MODELLING OF DATA ON THE BASIS OF RESIDUALS

After having stated the basic definitions of dynamical systems and models we will now come to a part that is more directly related to the problem of system identification. As discussed in the introductory chapter 1, we will distinguish two central aspects in the problem of modelling input–output data.

1. The set of models $\mathcal{M} \subset \tilde{\Sigma}$ that is considered; i.e. the set of all models among which (a) best model(s) is (are) searched for, given the measured data, and
2. an identification criterion that selects "best" or "optimal" models from the set of models.

In section 2.2 the models have been presented that will be used; more about these models and their parametrizations will be said in section 2.4. In this section we will direct our main attention to the identification criterion.

First we will give a general definition of an identification criterion and discuss its implications in relation to the usual approaches presented in the literature. A few properties of interactions between model sets and identification criteria will be discussed. Subsequently our attention will be directed towards identification criteria that are based on a criterion function and, more specifically, based on a residual function.

Let there be given a data sequence: $v^N := v|_{T_N}$, with $T_N := \mathbb{Z} \cap [0,N-1]$, and any set of models $\mathcal{M}$. An identification criterion will be defined as follows.

Definition 2.3.1. Identification criterion J^N.
An *identification criterion* on a model set $\mathcal{M} \subset \tilde{\Sigma}$ over a time set $T_N = \mathbb{Z} \cap [0,N-1]$ is defined as an operator J^N: $V^{T_N} \times 2^{\mathcal{M}} \rightarrow 2^{\mathcal{M}}$ such that $J^N(v^N, \mathcal{M}^*) \subset \mathcal{M}^*$ for all $\mathcal{M}^* \subset \mathcal{M}$ and $v^N \in V^{T_N}$, where $2^{\mathcal{M}}$ is the set of all subsets of $\mathcal{M}$. □

Definition 2.3.2. Optimal model.
For a given data sequence $v^N \in V^{T_N}$, a model $\hat{M} \in \mathcal{M}$ is called an *optimal model* within $\mathcal{M}$ with respect to J^N if $\hat{M} \in J^N(v^N, \mathcal{M})$. □

Actually the identification criterion is defined as a selection rule. Given the data and a model set, the criterion J selects one or more optimal models from the model set. If J is defined on a model set $\mathcal{M}$, then the definition states that it is also defined on any subset of $\mathcal{M}$.
This definition is more general than usually presented in the literature. It allows a unifying approach to a number of well known system identification methods, and moreover it offers the possibility to incorporate extensions to them. As well known system identification methods we mention:

* Identification by criterion function minimization, e.g. prediction error methods, equation error methods, maximum likelihood methods etc., where the identification criterion is specified as an operation of minimizing a functional of residuals by varying parameter values; some of these methods will be treated in chapter 3. For a general overview see Ljung (1987).
* Identification by correlation techniques, e.g. instrumental variable methods, where the identification criterion acts as an operation of solving a set of equations containing data, instrumental variables and parameters; see Söderström and Stoica (1983) and Ljung (1987).

These two classes of identification methods, actually brought forward by a statistical approach to the system identification problem, aim at finding a unique model of the process data, i.e. a unique estimate of the process at hand. Definition 2.3.1 of an identification criterion encompasses both function minimization techniques as well as correlation techniques, and a general formulation for these two different types can be given:

* Identification by function minimization:

$$J^N(v^N,\mathcal{M}) = \arg\min_{M\in\mathcal{M}} f^N(v^N,M), \tag{2.3.1}$$

$$= \{\hat{M}\in\mathcal{M} \mid f^N(v^N,\hat{M}) \leq f^N(v^N,M) \text{ for all } M\in\mathcal{M}\}$$

with f^N a criterion function f^N: $V^{T^N} \times \mathcal{M} \rightarrow \mathbb{R}\cup\{\infty\}$ (2.3.2)

* Identification by correlation techniques:

$$\hat{M} \in J^N(v^N,\mathcal{M}) \Leftrightarrow g^N(v^N,\zeta^N,\hat{M}) = 0 \tag{2.3.3}$$

with $\zeta^N \in Z^{T^N}$ being (external) signals used for correlation (e.g. instrumental variables) with signal set Z, and g^N an operator

g^N: $V^{T^N} \times Z^{T^N} \times \mathcal{M} \rightarrow (\mathbb{R}\cup\{\infty\})^\nu$, and $\nu\in\mathbb{Z}_+\cap[1,\infty)$. (2.3.4)

Note that the correlation technique also can be formulated as a function minimization technique by considering the operator g^N as a collection of ν functions, and taking the criterion function to be minimized equal to $f^N = \sum_{i=1}^{\nu} (g_i^N)^2$. The models minimizing f^N then will only be accepted if the corresponding function value is 0. Our definition gives the possibility to explicitly formulate identification methods that allow nonunique identified models; an identification criterion could be chosen that explicitly generates a set of models as identification result e.g.:

$$\hat{M} \in J^N(v^N,\mathcal{M}) \Leftrightarrow f^N(v^N,\hat{M}) \leq a \tag{2.3.5}$$

with f^N as in eq.(2.3.2) and $a\in\mathbb{R}$ a prespecified fixed value.

One could think of f^N as being a measure of misfit for a model M given a data sequence v^N, and a as being a maximum tolerated value of this misfit, e.g. chosen based on the required accuracy of the model in view of its intended application. In this thesis, however, we will not elaborate on identification criteria like (2.3.5).

A second main characteristic of definition (2.3.1) is that it does not make use of parameters. This is in contrast with the usual approach where an identification criterion (identification method) is formulated as a parameter estimation method,

in terms of a mapping:

$$v^N \rightarrow \hat{\theta}_N \in D_{\mathcal{M}}$$

with v^N the available input–output data and $\hat{\theta}_N$ a parameter value in a parameter space $D_{\mathcal{M}}$ (see e.g. Ljung,1987). The abstract definition of a model (definition 2.2.18) offers the possibility to formulate an identification criterion regardless of any parametrization of models. Models are treated as models and not as sets of parameters. This – probably somewhat artificially looking – difference is an essential step in clearly relating and distinguishing the problems of modelling of data on the one hand and (uniquely) parametrizing a set of models on the other hand.

Since an identification criterion J^N operates on a model set $\mathcal{M}$, certain properties can be formulated with respect to the relation between these two notions. A number of criterion based model properties will be listed now, that are of interest for a proper analysis of identification methods.

Definition 2.3.3. Completely selectable model set.
A model set $\mathcal{M}$ is called *completely selectable* by an identification criterion J^N, defined on $\mathcal{M}$, if for all $M \in \mathcal{M}$ there exists a data sequence $v^N \in V^{T_N}$ such that $M \in J^N(v^N, \mathcal{M})$. □

A model set that is completely selectable by J^N does not contain models that can never become optimal for the identification criterion J^N. The corresponding identification criterion will be called completely selecting $\mathcal{M}$. As a consequence a model set $\mathcal{M}$ that is complete selectable by J^N satisfies:

$$\bigcup_{v^N \in V^{T_N}} J^N(v^N, \mathcal{M}) = \mathcal{M} \qquad (2.3.6)$$

It will be illustrated later on that the complete selectability of model sets will not pose any restrictions on the model sets and identification criteria that will be applied in this thesis.
As an analogy to the complete selectability property of models, we define the set of data sequences that are able to select optimal models within a model set.

Definition 2.3.4. Set of selecting data sequences.
Let $\mathcal{M}$ be a model set and J^N an identification criterion defined on $\mathcal{M}$.
Then the *set of selecting data sequences* for $\mathcal{M}$ and J^N is defined by

$$V^{T^N}_{J,\mathcal{M}} := \{v^N \in V^{T^N} \mid J^N(v^N,\mathcal{M}) \neq \emptyset\}. \qquad (2.3.7)\square$$

It is a desirable property property of J^N and $\mathcal{M}$, that $V^{T^N}_{J,\mathcal{M}}$ is a "large" set; in other words, the identification criterion should come up with a solution in terms of optimal models for many possible data sequences. It should e.g. not be true that $V^{T^N}_{J,\mathcal{M}}$ generically is empty for a data sequence v^+ in $(l_\infty)^{p+m}$.
A property of a model set and an identification criterion that is more restrictive than complete selectability, is the property of discriminability.

Definition 2.3.5. Discriminable model set.
A model set $\mathcal{M}$ is called *discriminable* by an identification criterion J^N, defined on $\mathcal{M}$, if for all $M \in \mathcal{M}$ there exists a data sequence $v^N \in V^{T^N}$ such that $J^N(v^N,\mathcal{M})=\{M\}$.□

If $\mathcal{M}$ is discriminable by J^N, then J^N is called *discriminating* on $\mathcal{M}$.
An identification criterion that is discriminating on a model set can distinguish between the different models in this set, in the sense that every model in $\mathcal{M}$ can be isolated by the identification criterion from the rest of the set. This concept of discriminability will appear to play a major role in the evaluation of identification methods and of the identifiability of parametrizations. A specific situation of lack of discriminability is formalized in the following definition.

Definition 2.3.6. Criterion based model equivalence.
Let $\mathcal{M}$ be a model set that is completely selectable by an identification criterion J^N. Two models $M_1, M_2 \in \mathcal{M}$ are called J^N–*equivalent on* $\mathcal{M}$, denoted $M_1 \overset{J^N}{\sim}_{\mathcal{M}} M_2$ if for all $v^N \in V^{T^N}$:

$$M_1 \in J^N(v^N,\mathcal{M}) \Leftrightarrow M_2 \in J^N(v^N,\mathcal{M}). \qquad \square$$

This definition states that if one of two J^N–equivalent models is optimal within $\mathcal{M}$, then automatically the other model is optimal too. The identification criterion J^N can not distinguish between J^N–equivalent models, irrespective of the data sequence v^N. Since the relation in definition 2.3.6 is reflexive, symmetric and transitive it is an equivalence relation on $\mathcal{M}$ (MacLane and Birkhoff, 1967).
By definition it follows that a model set that contains distinct models that are J^N–equivalent, can never be discriminable with respect to J^N. The converse, however, need not be true; absence of J^N–equivalent models in a model set does not necessarily lead to a discriminable model set. An example clarifying this phenome-

non is presented later on in example (2.4.15).

Proposition 2.3.7.
Let $\mathcal{M}$ be a model set that is discriminable by an identification criterion J^N. Then for two models M_1, $M_2 \in \mathcal{M}$:

$$M_1 \overset{J^N}{\underset{\mathcal{M}}{\sim}} M_2 \Leftrightarrow M_1 = M_2 \qquad \square$$

Proof. By definition. $\square$

Note that the notions of discriminating identification criteria and criterion based model equivalence as presented in definitions 2.3.5 and 2.3.6 are dependent on the model set on which the identification criterion is applied. If J^N is discriminating on $\mathcal{M}$ then it is not necessarily discriminating on a subset $\mathcal{M}_1 \subset \mathcal{M}$, and if two models M_1, $M_2 \in \mathcal{M}_1 \subset \mathcal{M}$, are J^N–equivalent on $\mathcal{M}$, then they are not necessarily J^N–equivalent on the subset $\mathcal{M}_1$. This is due to the fact that the selection procedure J^N can be essentially dependent on the (elements of the) model set on which the identification criterion is applied.

Definition 2.3.8. Regularly partitionable model set.
A model set $\mathcal{M}$ is called *regularly partitionable* for an identification criterion J^N, defined on $\mathcal{M}$, if for all $\mathcal{M}_1 \subset \mathcal{M}$ and $v^N \in V_N^T$:

$$\{J^N(v^N,\mathcal{M}) \cap \mathcal{M}_1 \neq \emptyset\} \text{ implies } J^N(v^N,\mathcal{M}_1) = J^N(v^N,\mathcal{M}) \cap \mathcal{M}_1 \qquad (2.3.8)\square$$

The interpretation of this definition is the following. If a model M is selected as optimal within a certain model set $\mathcal{M}$, then it is also selected as optimal within any subset $\mathcal{M}_1$ of $\mathcal{M}$ that contains M. Moreover all models that are selected as optimal within this $\mathcal{M}_1$, containing a part of $J^N(v^N,\mathcal{M})$, are also optimal within $\mathcal{M}$. If a model set $\mathcal{M}$ is regularly partitionable for J^N, then the optimal models within $J^N(v^N,\mathcal{M})$ can be determined by evaluating only subsets of $\mathcal{M}$. This is reflected in the following proposition.

Proposition 2.3.9.
Let $\mathcal{M}$ be a model set that is regularly partitionable for J^N, and let $\mathcal{M} = \bigcup_i \mathcal{M}_i$.

For any $v^N \in V_{J,\mathcal{M}}^{T_N}$ it follows that

$$J^N(v^N,\mathcal{M}) = J^N(v^N, \bigcup_i J^N(v^N,\mathcal{M}_i)) \qquad (2.3.9)\square$$

Proof. Let $\mathcal{M}$ be written as $\mathcal{M} = \mathcal{M}^{(1)} \cup \mathcal{M}^{(2)}$ with $\mathcal{M}^{(1)} = \bigcup_i J^N(v^N, \mathcal{M}_i)$. It follows from definition 2.3.8 that $J^N(v^N, \mathcal{M}) \subset \mathcal{M}^{(1)}$. Consequently $J^N(v^N, \mathcal{M}) \cap \mathcal{M}^{(1)} = J^N(v^N, \mathcal{M})$, leading to $J^N(v^N, \mathcal{M}) = J^N(v^N, \mathcal{M}^{(1)}) = J^N(v^N, \bigcup_i J^N(v^N, \mathcal{M}_i))$. □

Model sets that are regularly partitionable for J^N, have certain regularity properties that are not restrictive if we consider model sets and identification criteria that are common in dynamic system identification. Since model sets and identification criteria are defined in a very general setting, the property of being regularly partitionable is required to restrict to specific choices of these two phenomena. The regularity properties actually are motivated by the consideration that the selection of optimal models in a model set should be based on properties of the elements in the set, and not on properties of the set itself. If the selection of optimal models in a set is only dependent on the elements in the set, then the set of selected optimal models can be obtained as formulated in equation 2.3.9. If the question whether $M \in \mathcal{M}$ is optimal, is only dependent on M itself and not on $\mathcal{M}$, then equation 2.3.9 can be written as $J^N(v^N, \mathcal{M}) = \bigcup_i J^N(v^N, \mathcal{M}_i)$.

As formulated in the following proposition, regular partitionability of model sets leads to some specific properties for the notions of criterion based model equivalence and discriminability.

Proposition 2.3.10.

Let $\mathcal{M}$ be a model set that is completely selectable and regularly partitionable for an identification criterion J^N, defined on $\mathcal{M}$, and let $\mathcal{M}_1 \subset \mathcal{M}$.
Then for any two models $M_1, M_2 \in \mathcal{M}_1$:

$$M_1 \overset{J^N}{\sim}_{\mathcal{M}_1} M_2 \Rightarrow M_1 \overset{J^N}{\sim}_{\mathcal{M}} M_2. \tag{2.3.10}$$

If $V^{T^N}_{J,\mathcal{M}_1} \subset V^{T^N}_{J,\mathcal{M}}$ then also

$$M_1 \overset{J^N}{\sim}_{\mathcal{M}} M_2 \Rightarrow M_1 \overset{J^N}{\sim}_{\mathcal{M}_1} M_2 \tag{2.3.11}$$ □

Proof. Eq. (2.3.10) follows from eq. (2.3.8) by considering that $M_1 \overset{J^N}{\sim}_{\mathcal{M}} M_2$ is represented by the condition $\{M_1 \in J^N(v^N, \mathcal{M}) \Leftrightarrow M_2 \in J^N(v^N, \mathcal{M})\}$ for all $v^N \in V^{T^N}_{J,\mathcal{M}_1}$. If $M_1 \overset{J^N}{\sim}_{\mathcal{M}} M_2$, then only conclusions can be drawn for the properties of M_1, M_2 on $\mathcal{M}_1$

for data sequences $v^N \in V^{T_N}_{J,\mathcal{M}}$, and nothing can be said about M_1, M_2 for $v^N \notin V^{T_N}_{J,\mathcal{M}}$. With the assumption mentioned in the proposition, eq. (2.3.11) follows from eq. (2.3.8). □

Proposition 2.3.11.
Let $\mathcal{M}$ be a model set that is regularly partitionable for an identification criterion J^N, defined on $\mathcal{M}$, and let $\mathcal{M}_1 \subset \mathcal{M}$.
If $\mathcal{M}$ is discriminable by J^N, then $\mathcal{M}_1$ is also discriminable by J^N. □

Proof. By applying definitions 2.3.5 and 2.3.8. □

The model sets and identification criteria that are dealt with in this thesis, will be restricted to satisfy the properties of complete selectability and regular partitionability.
The concepts that have been introduced in this section will be illustrated in the following section (see e.g. examples 2.4.13, 2.4.15 and 2.4.19), after a presentation has been given of the parametrization problem. The implications of the presented propositions then also will be clarified.

Next we focus on the role of a criterion function f^N.
Many identification criteria evaluate the "performance" of a model for a given data sequence by a criterion function

$$f^N: V^{T_N} \times \mathcal{M} \to \mathbb{R}\cup\{\infty\} \tag{2.3.12}$$

and two general examples have already been given in (2.3.1)–(2.3.4). Such a criterion function f^N can act as a measure of fit, a measure of credibility, likelihood etc., dependent on the context in which it is formulated and dependent on the specific choice of f^N.

We will consider a criterion function f^N as in (2.3.12). With slight abuse of notation, (2.3.12) will be interpreted in such a way that f^N is allowed to be dependent on external variables, such as correlation signals or instrumental variables (as in (2.3.4)), filters that are applied to the data, weighting factors, etc. These external variables in f^N are considered to be implicitly incorporated in f^N, leaving the arguments v^N and M as the essential ones that are common for all kinds of choices of f^N.

In the next definition we will formalize the role of a criterion function.

Definition 2.3.12. Identification criterion based on a criterion function.
An identification criterion J^N, defined on a model set $\mathcal{M}$, is said to be *based on a criterion function* f^N: $V^{T^N} \times \mathcal{M} \rightarrow \mathbb{R} \cup \{\infty\}$, if for all $M_1, M_2 \in \mathcal{M}$:

$$\left.\begin{array}{l} M_1 \in J^N(v^N, \mathcal{M}) \\ f^N(v^N, M_1) = f^N(v^N, M_2) \end{array}\right\} \Rightarrow M_2 \in J^N(v^N, \mathcal{M})$$ □

This definition states that models within $\mathcal{M}$ with respect to their optimality are evaluated only on the basis of the corresponding criterion function values.

We can think of –straightforward– examples of identification criteria, based on criterion functions, as:

$$J^N(v^N, \mathcal{M}) = \arg \min_{M \in \mathcal{M}} f^N(v^N, M), \text{ or} \tag{2.3.13}$$

$$J^N(v^N, \mathcal{M}) = \arg \max_{M \in \mathcal{M}} f^N(v^N, M), \tag{2.3.14}$$

referring to e.g. least squares methods, prediction error methods (2.3.13), and maximum likelihood methods (2.3.14).

In choosing specific forms of criterion functions f^N, the residual signals e^N can act as an intermediate between data (v^N) and model (M) on one side, and the function value $f^N(v^N, M)$ on the other. In other words: criterion functions f^N can be chosen to measure characteristics of residuals. A residual that is compatible with the measured data and the model (i.e. $e^N \in \mathcal{E}^N(v^N, M)$) will be interpreted as an error signal, on the basis of which the value of the criterion function f^N is determined. Actually this is the reason for adding a residual signal to the model description. As a consequence, the notion of (i/o/r)–model, as defined in the previous section, incorporates the property of generating residuals that are admissible by a model, given the measurement data v^N; in these terms a model actually has the function of generating residuals. The choice for a specific type of residual signal that one would like to act as an intermediate for the criterion function f^N, can be made by considering specific model sets $\mathcal{M}$. In chapter three more will be said about the different types of residuals that can be chosen.
Next we formalize the notion of identification on the basis of residuals.

Definition 2.3.13. Identification criterion based on a residual function.
An identification criterion $J^N(v^N,M)$ is said to be *based on a residual function* ℓ^N if $J^N(v^N,M)$ is based on a criterion function $f^N(v^N,M)$ for which holds that for all v^N, M there exists a residual $e^N \in \mathcal{E}^N(v^N,M)$ such that $f^N(v^N,M) := \ell^N(e^N)$, with ℓ^N a *residual function* ℓ^N: $E^{T_N} \rightarrow \mathbb{R} \cup \{\infty\}$. □

Identification criteria that are based on residual functions take residual signals that are compatible with the data for a model M, as "measures of fit" between the data and the model; these residuals are contained in $\mathcal{E}^N(v^N,M)$.
In general these admissible residual signals will not be uniquely defined, and consequently an extra relation between f^N and ℓ^N is required to guarantee that f^N is a mapping (i.e. its function value is uniquely defined). A solution to this problem is to consider only a subset $\mathcal{F}^N(v^N,M) \subset \mathcal{E}^N(v^N,M)$ of the set of compatible residuals, such that $\mathcal{F}^N$ constitutes a mapping: $V^{T_N} \times \mathcal{M} \rightarrow E^{T_N}$.
Function f^N can then simply be written as:

$$f^N(v^N,M) = \ell^N(\mathcal{F}^N(v^N,M)). \tag{2.3.15}$$

One of the choices for $\mathcal{F}^N(v^N,M)$ could be:

$$\mathcal{F}^N(v^N,M) = \arg \min_{e^N \in \mathcal{E}^N(v^N,M)} \ell^N(e^N), \tag{2.3.16}$$

leading to

$$f^N(v^N,M) = \min_{e^N \in \mathcal{E}^N(v^N,M)} \ell^N(e^N). \tag{2.3.17}$$

However, instead of looking for the minimum also the maximum value of $\ell^N(e^N)$ over $\mathcal{E}^N(v^N,M)$ could be taken as measure of fit between data and model.

For (i/o/pr)–models the mapping $\mathcal{F}^N$ can be made more specific by using the direct sum property (see also eqs. (2.2.12)–(2.2.14)):

$$\mathcal{E}^N(v^N,M) = \mathcal{E}^N_{obs}(v^N,M) \oplus \mathcal{E}^N_{init}(M) \tag{2.3.18}$$

with $\mathcal{E}^N_{obs}(v^N,M)$ containing exactly one element, and $\mathcal{E}^N_{init}(M)$ representing the influence of initial conditions. Using this decomposition, eq. (2.3.16) can be rewritten as:

$$\mathcal{F}^N(v^N,M) = \arg \min_{e_i^N \in \mathcal{E}^N_{init}(M)} \ell(e_i^N + e_o^N), \quad \text{with } \{e_o^N\} = \mathcal{E}^N_{obs}(v^N,M) \tag{2.3.19}$$

leading to

$$f^N(v^N,M) = \min_{e_i^N \in \mathcal{E}^N_{init}(M)} \ell^N(e_i^N+e_o^N), \tag{2.3.20}$$

which is an alternative formulation of eq. (2.3.17) for (i/o/pr)–models.
This type of choice of a residual function stresses the fact that initial conditions are explicitly involved in the identification criterion. In a great number of situations these initial conditions are neglected or assumed to be zero, which is reflected by the choice:

$$\mathcal{F}^N(v^N,M) = \mathcal{E}^N_{obs}(v^N,M) \tag{2.3.21}$$

leading to:

$$f^N(v^N,M) = \ell^N(e_o^N). \tag{2.3.22}$$

This choice is brought forward by the consideration that only models with zero initial conditions are taken into account, thereby actually neglecting any influence of data that is outside the measurement interval; see e.g. Ljung (1987).

The specific choices that can be made for a residual function ℓ^N are wide–ranging. The most common choices for ℓ^N are quadratic functions:

$$\ell^N(e^N) = \frac{1}{N} \sum_{t=0}^{N-1} e^T(t)e(t) \tag{2.3.23}$$

for signals that have an infinite energy for $N\to\infty$ but a finite power, (e.g. stochastic signals with a constant variance) and

$$\ell^N(e^N) = \sum_{t=0}^{N-1} e^T(t)e(t) \tag{2.3.24}$$

for signals in the Hilbert space l_2, having a finite energy for $N\to\infty$.

Note that for finite values of N, both functions (2.3.23) and (2.3.24) can be given the interpretation of a norm on the space of residuals, by considering $\sqrt{\ell^N}$; for $N\to\infty$ only (2.3.24) can be interpreted as a norm on l_2, since (2.3.23) does not satisfy: $\ell^+(e^+)=0 \Rightarrow e^+=0$.
Similar to the situations for signals and behaviours, residual functions, criterion functions and identification criteria over the infinite time set $\mathbb{Z}_+$ will be denoted by ℓ^+, f^+ and J^+.
The quadratic functions are by far the most popular residual functions, in combination with specific choices of model sets leading to well known identification methods as least squares identification, prediction error methods and – under some extra conditions – maximum likelihood estimation.

The analysis presented in this section has mainly been directed towards criterion functions f^N that are mappings into $\mathbb{R}\cup\{\infty\}$. It has been mentioned before that identification methods based on correlation (e.g. IV–methods), can also be interpreted to be based on function minimization of f^N, although they actually incorporate criterion functions g^N that are mappings into $(\mathbb{R}\cup\{\infty\})^\nu$. Nevertheless the definitions given in this section can directly be extended to functions g^N.

2.4 PARAMETRIZATION AND IDENTIFIABILITY

2.4.1. Model sets and their parametrization.

One of the basic choices in an identification procedure is the choice for a set of models to be taken into account. In this subsection the aspects will be considered that play a role when choosing a set of models. Since we want to represent models by a (finite) number of parameters, the parametrization of these model sets has to be dealt with. Properties of parametrizations with respect to identifiability are discussed in the following subsection.

As mentioned above, the choice for a set of models $\mathcal{M}$ is one of the basic choices in an identification procedure; it is probably also the hardest one. No general rules can be given for deciding which model set to take in a specific situation. It will always be up to the experimenter to decide whether a model set that has been chosen leads to a satisfactory identification result or not. In this evaluation a combination has to be made of the knowledge (or prejudice) that is available on the process at hand, and an understanding of the identification procedure (Ljung, 1987). Consequently there will not be given any rules in this section that will tell how to choose model sets; some aspects will be passed in review that play a role in this process of choosing. Similar considerations can be found in Ljung (1987) and Janssen (1988a) with respect to parametrized model sets. In the scope of this thesis a distinction will be made between aspects that are related to the model set and those related to its parametrization.

Aspects related to the choice of a model set:

a. Model complexity.
   The model set should be large enough (complex enough) to be able to describe

the essential characteristics of the underlying process between input and output, or approximate these essential characteristics in a sufficiently accurate way. On the other hand a model set that is too large (too complex) will lead to models that are difficult to handle with respect to their intended use and, moreover, they will not be very reliable because they will incorporate too many (random) properties of the specific data sequence that are not representative for the underlying process. In Ljung (1987) this aspect is called the trade–off between flexibility and parsimony.
This aspect contains the problem of order and structure selection.

b. Construction of residual signal.
The construction of the residual signals is essential since generally identification criteria are based on these residuals. It determines the "locations" in the model where nonmeasurable fictitious disturbance terms are incorporated in the model behaviour. Consequently, it specifies how the distance is measured between a data sequence and a model.

c. A priori information.
The model set can (partly) be chosen based on a priori information on the underlying process. The information can be based on physical knowledge, but can also be gathered from previous experiments. One can think of models based on physical laws, models incorporating predefined time delays, static gain, etc. As opposed to this, model sets can be chosen to consist of so called black box models, not explicitly bearing any a priori information.

d. Intended use of the model.
The ultimate goal of the identification procedure will be to construct a model that is acceptable for the application that it has been built for. Whether the model will be used for the purpose of description, simulation, prediction, diagnostics, controller design etc., should have an influence on the choice of a model set, and particularly on the construction of the residual signal.

e. Properties of parametrizations.
The question whether a model set is easy to parametrize plays an important role in the determination of model sets. Actually the combination of an identification criterion J and a parametrized model set $\mathcal{M}_\theta$ determines the complexity of the algorithm that has to solve the identification problem.
More about this will be said when considering aspects of parametrizations.

Most often model sets for black box models have been chosen to contain models with a specified complexity, considering the McMillan degree of a model as a proper measure for this complexity. For a formal definition of this McMillan degree we refer to section 3.4. This measure for model complexity is independent of the parametrization of the models involved. Other choices for multivariable model complexity are often based on the parametrization of the models, e.g. in Ljung (1987) where degrees of polynomials in a polynomial description are considered, and in Backx (1987), where the degree of the minimal polynomial is chosen. In Willems (1987) a notion of model complexity is proposed that is also related to the McMillan degree, and that even constitutes a further specification of this, in order to classify models with equal McMillan degree. A further discussion on the complexity of models and model sets will follow in section 3.4.

A number of things has already been said about parametrizations. Now let us come to a formal introduction.

Definition 2.4.1. Parametrization $\tilde{M}$.
A *parametrization* $\tilde{M}$ for a model set $\mathcal{M}$, is a surjective mapping $\tilde{M}: \Theta \rightarrow \mathcal{M}$, with $\Theta \subset \mathbb{R}^d$ the parameter set, and $\mathcal{M}$ the parametrized set of models. □

In this thesis $\mathcal{M}$ is considered to be a set of (i/o/r)–models, so for $\theta \in \Theta$, $\tilde{M}(\theta)$ will be an (i/o/r)–model. A model set that is the image of a parametrization will be called a parametrized model set. In our context model sets are considered to have fixed values of the number of inputs, outputs and residuals. Consequently parametrizations will be considered whose images satisfy $Im(\tilde{M}) \subset \tilde{\Sigma}_{p,m,r}$. In view of algorithmic aspects, we will also restrict ourselves to parametrizations that are represented by differentiable mappings.

Model sets are being parametrized in order to make models manageable. The outcome of an identification procedure will be a collection of (real valued) parameters that will indicate the model(s) that should be considered as optimal.
Apart from numerical and algorithmic aspects, the choice of a parametrization should not influence the result of an identification procedure. In other words: $J^N(v^N, \mathcal{M})$ should be invariant for different choices of parametrizations for $\mathcal{M}$. The only, but very important, role of the parametrization will be to (uniquely) represent all models within the model set chosen.

We will now briefly summarize the properties and aspects that play a role in evaluating a parametrization.

a. Parametrization of intended model set.
The first and main requirement on a parametrization is that it has to parametrize the intended model set in such a way that all models that can be distinguished by the identification criterion J are represented in $Im(\tilde{M})$. This property will be formalized later on. It incorporates the question whether one is able to deal with specific restrictions, c.q. a priori information on the model set. A choice for a specific parametrization might be suggested by the kind of (physical) knowledge that is available from the process.

b. Algorithm complexity
The parametrization of a model set together with the identification criterion determine the kind of algorithms that are required for solving the identification problem. In this respect, the way in which the parameters enter a criterion function f^N are essential. The complexity of the algorithms can be evaluated by considering questions as e.g.: does the identification criterion have an explicit solution or does it have to be gathered by non–linear function minimization techniques (hill–climbing methods)?; do there exist local minima or not?; can the identification problem be solved accurately?

c. Identifiability.
The identifiability property of a parametrization is related to the question whether models can be uniquely distinguished by a set of parameters, and consequently whether as few as possible parameters are involved in the parametrization. This aspect will be more extensively discussed in the next subsection.

d. Ability for physical interpretation of parameters.
The fact whether parameters have a physical meaning could be an argument for considering specific parametrizations. However this seems only partly to be of importance; if the model set is chosen properly the parametrization only acts as an intermediate and the parameters used to represent the models during identification do not have to be the same parameters in which the experimenter is interested.

e. Conditioning of parametrization.
The conditioning of the parametrization is reflected by the influence that a

small deviation in a parameter has on the essential properties of the model. This influence can be taken with respect to the criterion function $f^N(v^N,\tilde{M}(\theta))$, used during identification, but also to a model validation criterion function, e.g. reflecting the ultimate use of the model. It is preferable that the effects of parameter variations for the different parameters are in a similar range. The conditioning of a parametrization is often confused with the identifiability of parameters, but should be clearly distinguished from the identifiability problem.

The choice of model set and the choice of a parametrization can strongly influence each other. In some situations a model set is chosen and a corresponding parametrization is constructed; this is in fact the most appealing philosophy.
In a number of other situations a parametrization is chosen and the model set that corresponds to this parametrization is accepted as a proper choice. Examples of these situations will be given in chapter 3.
Aspects of algorithm complexity have always played an important role in constructing model sets and corresponding parametrizations. The parametrization property of residuals being linear–in–the–parameters, together with a quadratic residual function induces an identification problem that is explicitly solvable. This is the basis for the popularity of the so called least squares identification methods. With respect to the conditioning of parametrizations, the use of simple shift operators in model descriptions with difference equations can lead to undesirable properties (see e.g. Åström and Wittenmark, 1984, p. 386). Motivated by these conditioning aspects, alternatives for these parametrizations have been proposed, e.g. in terms of ladder/lattice forms (Lee, 1980; Ljung and Söderström, 1983) or by parametrizations applying a so called delta operator (Goodwin, 1985).

Two specific types of parametrizations for dynamical systems will be introduced that will be used throughout this thesis for parametrizing (i/o/r)–model sets; the first type is based on a description in terms of polynomial matrices, the second type is based on the well known state space model.

Definition 2.4.2. Polynomial matrix parametrization.
A parametrization $\tilde{M}$ will be called a *polynomial matrix parametrization* of the model set $\mathcal{M} \subset \tilde{\Sigma}_{p,m,r}$ if $\tilde{M}$ satisfies: $\tilde{M} = \tilde{M}_p \circ \tilde{G}_p$ with mappings:

$\tilde{G}_p: \Theta \rightarrow \Theta_p$ with $\Theta \subset \mathbb{R}^d$, and $\Theta_p \subset \mathbb{R}^{g\times(p+m+r)}[z,z^{-1}]$, and

$\tilde{M}_p: \Theta_p \rightarrow \mathcal{M}$, with $\tilde{G}_p$ bijective, $\tilde{M}_p$ surjective, and $M = \tilde{M}_p(T)$ specified by:

$\mathcal{B}(M) := \{w \in (\mathbb{R}^{p+m+r})^{\mathbb{Z}} \mid T(\sigma,\sigma^{-1})w = 0\}$. □

Since $\tilde{G}_p$ is a bijection the polynomial matrix parametrization can be characterized by means of the mapping $\tilde{M}_p$. With slight abuse of the definitions there will also be spoken of the polynomial matrix parametrization $\tilde{M}_p$ (in stead of $\tilde{M}$). $M = \tilde{M}_p(T)$ will be called an (i/o/r)–model induced by polynomial matrix $T(z,z^{-1})$.

Proposition 2.4.3.
Let $M \in \tilde{\Sigma}_{p,m,r}$ be an (i/o/r)–model. Then there always exists a polynomial matrix $T \in \mathbb{R}^{p\times(p+m+r)}[z,z^{-1}]$ with full row rank ($\text{rank}_{\mathbb{R}(z)} T(z,z^{-1}) = p$), such that $M = \tilde{M}_p(T)$. □

Proof. See Willems (1988). □

A polynomial matrix parametrization that is restricted to full row rank polynomial matrices will be called a parametrization in full row rank polynomial matrix form. In the rest of this thesis we will generally deal with parametrizations in full row rank polynomial matrix form; so by writing $M = \tilde{M}_p(T)$ we will implicitly assume that T has full row rank.

Proposition 2.4.4.
Let $M_1, M_2 \in \tilde{\Sigma}_{p,m,r}$ be two (i/o/r)–models whose behaviours are induced by polynomial matrices $T_1, T_2 \in \mathbb{R}^{p\times(p+m+r)}[z,z^{-1}]$. Then $M_1 = M_2$ if and only if there exists a *unimodular matrix* $U(z,z^{-1})$ such that $T_1 = U\,T_2$. Unimodularity has to be considered over the ring $\mathbb{R}[z,z^{-1}]$, which means that $U(z,z^{-1})$ has a polynomial inverse in z and z^{-1}, i.e. $\det(U) = cz^d$ with $c \in \mathbb{R}\setminus\{0\}$, and $d \in \mathbb{Z}$. □

Proof. See Willems (1988). □

Polynomial parametrizations generate models whose behaviours are described by a set of equations: $T(\sigma,\sigma^{-1})w = 0$, or written in a decomposed form:

$$P(\sigma,\sigma^{-1})y = Q(\sigma,\sigma^{-1})u + R(\sigma,\sigma^{-1})e \qquad (2.4.1)$$

In commonly applied parametrizations use is made of only one of the shift operators σ or σ^{-1}. Of course these both types of representations can be easily transformed into one another under operations of behaviour–equivalence (premultiplica-

tion by a unimodular matrix with respect to $\mathbb{R}[z,z^{-1}]$). In this thesis the restrictions to the use of one shift operator σ or σ^{-1} will be considered as a special case of the general situation.
Model sets and corresponding polynomial matrix parametrizations with $R(z,z^{-1})$ equal to the identity matrix (so called equation error models or linear regression models) have become popular because of the linear relationship between the parameters of the model in the polynomial matrices P and Q, and the residual signal e. In chapter three more attention will be paid to these types of models.
We will generally deal with controllable (i/o/pr)–models, and consequently it is important to specify the property of controllability of models in terms of their representation in a polynomial matrix form.

Proposition 2.4.5.
Let $M\in\tilde{\Sigma}_{p,m,r}$ with $M=\tilde{M}_p(T)$ and $T(z,z^{-1})=[P|-Q|-R]\in\mathbb{R}^{p\times(p+m+r)}[z,z^{-1}]$.
Then the following expressions are equivalent.

(i) M is controllable;

(ii) P, Q, R are left coprime with respect to $\mathbb{R}[z,z^{-1}]$, i.e. all left common factors are unimodular with respect to $\mathbb{R}[z,z^{-1}]$;

(iii) $\operatorname{rank}_{\mathbb{C}} T(\lambda,\lambda^{-1}) = p$ for all $\lambda\in\mathbb{C}\setminus\{0\}$. □

Proof. See Willems (1988). □

Note that for an (i/o/pr)–model the notion of controllability is completely independent of the fact which signal (y or e) is considered as the output (processing) signal of the model.
The different model equivalences for (i/o/r)–models that were presented in section 2.2 can now also be specified in terms of polynomial matrix representations.

Proposition 2.4.6.
Consider two models M_1, $M_2 \in \tilde{\Sigma}_{p,m,r}$ such that $M_1=\tilde{M}_p(T_1)$, $M_2=\tilde{M}_p(T_2)$, with $T_1,T_2\in\mathbb{R}^{p\times(p+m+r)}[z,z^{-1}]$, $T_1 = [P_1|-Q_1|-R_1]$, and $T_2 = [P_2|-Q_2|-R_2]$;
then the following relations hold:

a. $M_1 \sim M_2 \Leftrightarrow T_1(z,z^{-1})=U(z,z^{-1})T_2(z,z^{-1})$ with $U(z,z^{-1})\in\mathbb{R}^{p\times p}[z,z^{-1}]$ and $U(z,z^{-1})$ unimodular over the ring $\mathbb{R}[z,z^{-1}]$;

b. $M_1 \overset{t}{\sim} M_2 \Leftrightarrow T_1(z,z^{-1})=U(z)T_2(z,z^{-1})$ with $U(z)$ a rational matrix: $U(z)\in\mathbb{R}^{p\times p}(z)$ and $\det_{\mathbb{R}(z)}U(z)\neq 0$;

c. $M_1 \overset{\gamma}{\sim}_o M_2 \Leftrightarrow$ $[P_1|-Q_1] = U(z,z^{-1})\,[P_2|-Q_2]$ with $U(z,z^{-1}) \in \mathbb{R}^{p\times p}[z,z^{-1}]$ and $U(z,z^{-1})$ unimodular over the ring $\mathbb{R}[z,z^{-1}]$;

d. $M_1 \overset{t}{\sim}_o M_2 \Leftrightarrow$ $[P_1|-Q_1] = U(z)\,[P_2|-Q_2]$ with $U(z)$ a rational matrix: $U(z) \in \mathbb{R}^{p\times p}(z)$ and $\det_{\mathbb{R}(z)} U(z) \neq 0$; □

Proof. The proof follows from results of Willems (1988). □

As an alternative parametrization a state space form will be considered.

Definition 2.4.7. Parametrization in state space form (A,B,C,D).
A parametrization $\tilde{M}$ will be called a *parametrization in state space form* (A,B,C,D) for the model set $\mathscr{M} \subset \tilde{\Sigma}_{p,m,r}$ if $\tilde{M}$ satisfies: $\tilde{M} = \tilde{M}_s \circ \tilde{G}_s$, with mappings:

$\tilde{G}_s : \Theta \to \Theta_s$ with $\Theta \subset \mathbb{R}^d$, and $\Theta_s \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times(m+r)} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times(m+r)}$, and

$\tilde{M}_s : \Theta_s \to \mathscr{M}$ with $\tilde{G}_s$ bijective, $\tilde{M}_s$ surjective, and $M = \tilde{M}_s(A,B,C,D)$ specified by

$$\mathscr{B}(M) := \{w \in (\mathbb{R}^{p+m+r})^{\mathbb{Z}} \mid \exists x \in (\mathbb{R}^n)^{\mathbb{Z}},\ w=(y,u,e),\quad \sigma x = Ax + B_u u + B_e e,\quad y = Cx + D_u u + D_e e \text{ with } B = [B_u|B_e] \text{ and } D = [D_u|D_e]\}.$$ □

In this parametrization, $x \in (\mathbb{R}^n)^{\mathbb{Z}}$ is called the state, and n the state space dimension. Similar to the situation for $\tilde{M}_p$ the parametrization in state space form can be characterized by the mapping $\tilde{M}_s$. The (i/o/r)–model $M = \tilde{M}_s(A,B,C,D)$ will be said to be induced by the quadruple of matrices (A,B,C,D), which will be called a *realization* of M. A realization of M which has a minimal state space dimension n is called a *minimal realization.* This situation seems to be very similar to the "classical" way of dealing with state space models. However there is a difference with respect to the minimality of the realizations. In "classical" terms a realization is minimal if it is observable and reachable, i.e. (C,A) is an observable pair and (A,B) is a reachable pair. In our context the state space parametrization acts as a representation of a model behaviour and a model behaviour can essentially contain noncontrollable parts, represented in a state space form by nonreachability of the pair (A,B). Consequently the minimality of a state space representation is now mainly due to observability of the pair (C,A) as reflected in the following proposition.

Proposition 2.4.8.
Let $M \in \tilde{\Sigma}_{p,m,r}$ be an (i/o/r)–model induced by a state space realization (A,B,C,D) with state space dimension n. (A,B,C,D) is a minimal realization if and only if:
a. (C,A) is an observable pair, i.e. $\text{rank}_{\mathbb{C}} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$, and
b. rank [A|B] = n. □

Proof. See Willems (1988). □

Condition b of proposition 2.4.8 can be given the interpretation that (A,B) should not have any unreachable poles in 0; equivalently $\text{rank}_{\mathbb{C}}\,[\lambda I - A \,|\, B] = n$ for $\lambda = 0$, or $A\mathbb{R}^n + Im(B) = \mathbb{R}^n$. Unreachable poles in 0 lead to nonminimality of the realization, because a corresponding state component can not contribute to the behaviour $\mathcal{B}(M)$. This can be visualized by considering that any value of this state component $x_i(t)$ for some $t=t_1$ would become 0 for $t=t_1+1$. Since this mode is not affected by any input signal and the considered time set is $\mathbb{Z}$, this $x_i(t_1)$ could never have become $\neq 0$.
Controllability of an (i/o/r)–model in (A,B,C,D) state space form follows straightforwardly.

Proposition 2.4.9.
Let $M \in \tilde{\Sigma}_{p,m,r}$ be an (i/o/r)–model induced by a state space realization (A,B,C,D) with state space dimension n. Then M is controllable if and only if

$$\text{rank}_{\mathbb{C}}\,[\lambda I - A|B] = n \quad \text{for all } \lambda \in \mathbb{C}\backslash\{0\}.$$ □

Note the exclusion of the point $\lambda=0$ in the condition of the proposition; this again is due to the fact that we consider the dynamical system M to be defined on the time axis $\mathbb{Z}$, in contrast with the situation for $\mathbb{Z}_+$ where the exclusion of $\lambda=0$ is not present. Note also that in the presented framework a nonsymmetrical treatment is created of the "classical" concepts of reachability/controllability and observability. Nonreachable parts in a state space realization are expressed in the model behaviour, whereas nonobservable parts or not.

We have chosen the state space model to be in a form that is generally applied for deterministic and stochastic dynamical models. State space model descriptions are very popular for several reasons. It is a compact description in which several properties of the model appear quite directly (e.g. poles, stability, controllability, pre-

dictions, etc.), and model simulation can be performed very easily. Moreover the theory related to state space models is quite extensive.

It has to be noted that a representation in state space form generates models that are more restrictive than the models that result from a polynomial matrix representation. Because of the specific construction of the parametrization, it is guaranteed that y processes (u,e), and moreover that y does not anticipate (u,e), whereas for polynomial matrix representations the polynomial matrix T has to be given extra restrictions to satisfy these requirements. In this respect state space models are less general than their polynomial counterparts. A generalization of state space representations to situations in which the restriction of nonanticipation is skipped, can be made by considering so called generalized state space models (Verghese, Lévy and Kailath, 1981). Representations in state space form that are as general as polynomial matrix representations are presented in Willems (1988). In the scope of this thesis we will restrict ourselves to the parametrizations as defined above.

Proposition 2.4.10.
Consider an (i/o/r)–model $M \in \tilde{\Sigma}_{p,m,r}$ such that $M=\tilde{M}_p(T)$. Then there exists a state space representation (A,B,C,D) such that $M = \tilde{M}_s(A,B,C,D)$ if and only if M satisfies that y does not anticipate (u,e). □

In order to describe (i/o/pr)–models, the representations considered have to satisfy additional conditions for the processing property of the residual signal. In a polynomial matrix representation (as in eq. 2.4.1), this is reflected by the restriction $\text{rank}_{\mathbb{R}(z)} R(z,z^{-1})=p$, while in an (A,B,C,D) state space form it requires $\text{rank}_{\mathbb{R}(z)}[D_e+C(zI-A)^{-1}B_e]=p$.

The two parametrizations as presented in this section, actually encompass a number of other parametrizations, like e.g. parametrizations in terms of Markov parameters. More specific choices for parametrizations will be presented in chapter 3, along with more explicit relations between the parametrizations $\tilde{M}_p$ and $\tilde{M}_s$. For this moment we will finish the presentation of parametrizations and we will focus on one of its most important properties: the identifiability.

2.4.2 Identifiability

Within the context of system identification the concept of identifiability has been a point of discussion ever since it was introduced, leading to a number of definitions that have been presented in the course of years. In Åström and Bohlin (1965) it was presented as a property of an identification situation, actually describing the ability to correctly estimate the parameters in a physical process. As such it was based on consistency properties of the corresponding identification method. Later on, Bellman and Åström (1970), Ljung (1976a) and Gustavsson, Ljung and Söderström (1977) discussed the so called structural identifiability, representing the situation that the identification criterion function has a unique global minimum; this definition still took the consistency property as a starting point: the unique global minimum had to represent the "true system". More or less parallel to this development there was a group of researchers analysing unique descriptions of model sets for models with a fixed McMillan degree (Glover and Willems, 1974; Rissanen, 1974; Denham, 1974; Guidorzi, 1975), leading to the use of canonical and pseudo–canonical (overlapping) parametrizations in system identification (Guidorzi, 1981; Guidorzi and Beghelli, 1982; van Overbeek and Ljung, 1982; Gevers and Wertz, 1984; Corrêa and Glover, 1984a, 1984b). In this development (structural) identifiability became a property of a parametrization not related anymore to an identification setup. Because of a growing interest in the approximate modelling aspects of system identification, accepting that the process at hand likely can not be modelled exactly, identifiability has become a uniqueness oriented property of a parametrization or parametrized model set.
Central in this property is a unique representation of a parametrized model set with respect to a model equivalence (Ljung, 1987; Gevers and Wertz, 1987b; Janssen, 1988a). As opposed to this (sometimes called structural–) identifiability which is not related to any experimental conditions or to any "true system", parameter–identifiability can be used for the property that (physical) parameters in an a priori structured model can uniquely be identified (Gevers and Wertz, 1987a). This parameter–identifiability indeed is dependent on the experimental conditions and on the "true system".
In this thesis identifiability will be considered as a property of a parametrization, in relation with an identification criterion.

Let us consider the situation of the following identification problem:
Given a data sequence v^N, an identification criterion J^N and a parametrization $\tilde{M}$ generating a parametrized model set $\mathcal{M}=Im(\tilde{M})$, we will focus on two models

$\tilde{M}(\theta_1)$, $\tilde{M}(\theta_2) \in J^N(v^N, \mathcal{M})$, with $\theta_1 \neq \theta_2$. Two situations can be distinguished:

1. $\tilde{M}(\theta_1)$ and $\tilde{M}(\theta_2)$ are both optimal models within $\mathcal{M}$ with respect to J^N for this specific data sequence v^N, although there apparently exist other data sequences for which one of the two models is optimal and the other is not;
2. $\tilde{M}(\theta_1)$ and $\tilde{M}(\theta_2)$ are optimal models within $\mathcal{M}$ with respect to J^N for any data sequence of any length N, for which one of the two is optimal.

Note that situation 1 is caused by a property of the particular data sequence; apparently this data sequence is not informative enough to distinguish between the two models $\tilde{M}(\theta_1)$ and $\tilde{M}(\theta_2)$ and between the parameter sets θ_1, θ_2.
On the contrary, situation 2 is caused by a property of the parametrized model set. The two parameter sets θ_1, θ_2 can never be distinguished, irrespective of the data. This situation is characterized by the expression that $\tilde{M}(\theta_1) \overset{J^N}{\underset{\sim}{\mathcal{M}}} \tilde{M}(\theta_2)$ for any value of N, which implies that $\tilde{M}(\theta_1) \overset{J^+}{\underset{\sim}{\mathcal{M}}} \tilde{M}(\theta_2)$.

It has to be noted that in this latter situation again two aspects can play a role :

a. $\tilde{M}(\theta_1) = \tilde{M}(\theta_2)$, showing that θ_1, θ_2 can never be distinguished by their corresponding models, and
b. $\tilde{M}(\theta_1) \neq \tilde{M}(\theta_2)$, showing that $\mathcal{M}$ is not discriminable with respect to J^+.

These two aspects of situation 2, described above, are taken into account when discussing the identifiability property of a parametrization. It deals with the question whether it is possible at all to find a unique parameter $\hat{\theta}$ as the solution of an identification problem, irrespective of the data. In line with this philosophy, identifiability is defined as a property of a parametrization and an identification criterion:

Definition 2.4.11. Strictly identifiable parametrization.
A parametrization $\tilde{M}$: $\Theta \rightarrow \mathcal{M}$ with $\Theta \subset \mathbb{R}^d$ is *strictly identifiable* by an identification criterion J^+ defined on $\mathcal{M}$ if:

* $\tilde{M}$ is a bijective mapping, and
* J^+ is discriminating on $\mathcal{M}$. □

The definition states that if a parametrization is identifiable by J^+ as defined above, all parameters in Θ can occur as the unique solution to an identification problem with identification criterion J^+. It can simply be shown that the conditions of definition 2.4.11 guarantee that:

$$\tilde{M}(\theta_1) \overset{J^+}{\underset{\sim}{\mathcal{M}}} \tilde{M}(\theta_2) \Leftrightarrow \tilde{M}(\theta_1) = \tilde{M}(\theta_2) \Leftrightarrow \theta_1 = \theta_2. \qquad (2.4.2)$$

This property of identifiability, which is in fact a unique parameter identifiability, is important for several reasons. In a situation of lack of this uniqueness, the model set apparently is overparametrized, which is not efficient, and no unique interpretation can be given to each of the estimated parameters. Moreover this non-uniqueness can easily lead to numerical problems in identification algorithms.

In a number of situations this property of strict identifiability appears to be a very severe restriction on parametrizations and therefore also a less strict version will be used:

Definition 2.4.12. Identifiable parametrization.
A parametrization $\tilde{M}$: $\Theta \rightarrow \mathcal{M}$ with $\Theta \subset \mathbb{R}^d$ is *identifiable* by an identification criterion J^+ defined on $\mathcal{M}$ if:

* $\tilde{M}(\theta_1) = \tilde{M}(\theta_2) \Rightarrow \theta_1 = \theta_2$ for almost all $\theta_1 \in \Theta$, and
* J^+ is discriminating on $\mathcal{M}$. □

The expression "for almost all $\theta_1 \in \Theta$" means that the property holds for all $\theta_1 \in \Theta_M \subset \Theta$, with Θ_M an open and dense subset of Θ. The corresponding property of $\tilde{M}$, reflected in the first condition of definition 2.4.12, will be denoted by the expression that $\tilde{M}$ is *almost bijective.*

For identifiability of $\tilde{M}$ it is required that $\mathcal{M} = Im(\tilde{M})$ is discriminable by J^+. Apparently this notion of discriminability is very fundamental, and in order to get a proper understanding of this concept the following example will be presented. At the same time we will take the opportunity to illustrate some of the other concepts that have been defined in section 2.3.

Example 2.4.13.
Consider a set $\mathcal{M}$ of (i/o/pr)–models with one input, one output and consequently one residual signal (SISO situation), defined by:

$$\mathcal{M} = \{M \in \hat{\Sigma}_{1,1} \mid M = \tilde{M}_p(T),\ T(z,z^{-1}) = [a_0 + a_1 z^{-1} \mid -b_0 - b_1 z^{-1} \mid -c],\ a_0, a_1, b_0, b_1, c \in \mathbb{R},\ ((a_0 + a_1 z^{-1}), (b_0 + b_1 z^{-1})) \text{ coprime}\} \qquad (2.4.3)$$

Since we are dealing with (i/o/pr)–models, the parameter c in $T(z,z^{-1})$ is not allowed to be equal to 0, and at least one of the parameters $\{a_0, a_1\}$ has to be unequal to 0.

As an identification criterion we will consider the well known least squares criterion:

$$J^+(v^+,\mathcal{M}) = \arg \min_{M\in\mathcal{M}} f^+(v^+,M) = \arg \min_{e^+\in\mathcal{E}^+(v^+,M)} \ell^+(e^+) \tag{2.4.4}$$

with

$$\ell^+(e^+) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e^2(t) \tag{2.4.5}$$

The *complete selectability* of $\mathcal{M}$ by the identification criterion J^+ follows straightforwardly from the observation that $J^+(v^+,\mathcal{M}) = \mathcal{M}$ for $v^+=0$. Consequently all models in $\mathcal{M}$ are selectable by J^+. Actually for this type of model sets and identification criteria the property of complete selectability does not yield any restrictions.

It is also straightforward that $\mathcal{M}$ is *regularly partitionable* for J^+. If a model $\hat{M}$ minimizes the residual function $f^+(v^+,M)$ over $M\in\mathcal{M}$, then it minimizes this same residual function over any subset of $\mathcal{M}$ that contains $\hat{M}$. Moreover all $M\in\mathcal{M}$ such that $f^+(v^+,M) = f^+(v^+,\hat{M})$ are optimal within $\mathcal{M}$. From a different approach one could also state that $\mathcal{M}$ is regularly partitionable for J^+ since for each $M\in\mathcal{M}$ the residual function $f^+(v^+,M)$ is independent of $\mathcal{M}$.

In order to evaluate the *discriminability* of $\mathcal{M}$ we distinguish between the following situations:

a. Data sequences v^+ for which holds that $f^+(v^+,M)>0$ for all $M\in\mathcal{M}$.
   For these data sequences it follows that $J^+(v^+,\mathcal{M})=\emptyset$ since $\ell^+(e^+)\to 0$ for $a_0,a_1,b_0,b_1\to 0$, or similarly $c\to\infty$, and the limit point $\ell^+(e^+)=0$ can not be reached within $\mathcal{M}$.

b. Data sequences v^+ for which holds that $f^+(v^+,M)=0$ for some $M\in\mathcal{M}$.
   Any such data sequence can be written as $v^+\in\mathcal{B}^+(M^{io})$ with $M\in\mathcal{M}$ and for such a data sequence it follows that
   $M_1\in J^+(v^+,\mathcal{M}) \Leftrightarrow \{\exists d\in\mathbb{R}\setminus\{0\}$ such that $(v,e)\in\mathcal{B}(M) \Leftrightarrow (v,e')\in\mathcal{B}(M_1)$ with $e=de'\}$.
   Consequently all models that lead to residual signals that are related by scaling factors, are always selected together in the identification criterion.

The model set $\mathcal{M}$ is not discriminable by J^+ since models having a scaled residual can never be distinguished from each other. It follows that for two models $M_1,M_2\in\mathcal{M}$: $M_1 \overset{J^+}{\underset{\mathcal{M}}{\sim}} M_2$ if and only if $\{\exists d\in\mathbb{R}\setminus\{0\}$ such that $(v,e)\in\mathcal{B}(M)\Leftrightarrow(v,e')\in\mathcal{B}(M_1)$ with $e=de'\}$. There is a freedom of scaling of the residual signals in the models that leads to nondiscriminability. In this example the set of selecting data sequences can be written as:

$$V^{+}_{J,\mathcal{M}} = \{ v^{+} \epsilon \mathcal{B}^{io,+}(M) \mid M \epsilon \mathcal{M} \} = \bigcup_{M \epsilon \mathcal{M}} \mathcal{B}^{+}(M^{io}) \qquad (2.4.6)\ \square$$

If a model set $\mathcal{M}$ is not discriminable by J^+, then there does not exist a (strictly) identifiable parametrization for $\mathcal{M}$. This is illustrated in the previous example showing that there exist different parameter sets that can not be distinguished by the identification criterion. In the example the lack of discriminability is due to the occurrence of distinct J^+–equivalent models in $\mathcal{M}$. In example 2.4.13 this lack of discriminability coincides with another undesirable property of the presented identification problem: for generic data sequences in $(\ell_\infty)^2$ no optimal models will be selected, i.e. the identification criterion $J^+(v^+,\mathcal{M})$ will be empty. Apparently the chosen $\mathcal{M}$ and J^+ do not lead to a useful identification problem.
The classical way to circumvent this problem is to restrict the considered model set by requiring that the polynomial matrix $T(z,z^{-1})$ in eq. (2.4.3) satisfies the additional constraint $a_0/c=1$. However it should be stresses that this is a specific choice out of many different alternatives and, moreover, that each different choice of restricting the model set will generally lead to different selected optimal models. In general terms it can be stated that if a model set $\mathcal{M}$ is not discriminable by J^+, then discriminability can be obtained by making a restriction to a discriminable subset $\mathcal{M}_1 \subset \mathcal{M}$.

The presence of distinct models in $\mathcal{M}$ that are J^+–equivalent leads to nondiscriminability. Consequently a first step in order to obtain discriminability is to remove these distinct J^+–equivalent models within $\mathcal{M}$. Following the definitions of sets of canonical forms, as present in the work of MacLane and Birkhoff (1967) and Guidorzi (1981), a straightforward way to remove such models is to consider a set of canonical forms $\mathcal{M}_1 \subset \mathcal{M}$ with respect to the equivalence relation $\overset{J^+}{\sim}_{\mathcal{M}}$. If this set of canonical forms is discriminable by J^+, the problem of constructing a strictly identifiable parametrization for $\mathcal{M}_1$ comes down to the construction of a bijective mapping that parametrizes $\mathcal{M}_1$. With respect to this approach the following remark has to be made.

Remark 4.2.14.

a. A model set that does not contain distinct J^+–equivalent models is not necessarily discriminable by J^+.
b. The identification procedure, in terms of the set of selected optimal models $J^+(v^+,\mathcal{M})$, will be dependent on the specific set of canonical forms with respect

to $\underset{\sim}{\overset{J^+}{\mathcal{M}}}$ that has been applied. □

In the remaining part of this section we will comment upon these remarks. First an example will be presented of a situation as meant in remark 2.4.14a.

Example 2.4.15.
Consider a set $\mathcal{M}$ of (i/o/pr)–models similar to the situation as in example 2.4.13, however now consisting of only two elements: $\mathcal{M} = \{M_1, M_2\}$ with

$$M_1 = \tilde{M}_p(T_1) \in \hat{\Sigma}_{1,1}, \text{ with } T_1(z,z^{-1}) = [a_0 + a_1 z^{-1} | -b_0 - b_1 z^{-1} | -1],$$

and $$M_2 = \tilde{M}_p(T_2) \in \hat{\Sigma}_{1,1}, \text{ with } T_2(z,z^{-1}) = [a_0 + a_1 z^{-1} | -b_0 - b_1 z^{-1} | -2],$$

with a_0, a_1, b_0, b_1 fixed real numbers, and $((a_0 + a_1 z),(b_0 + b_1 z))$ coprime. (2.4.7)
Consider the same least squares identification criterion as in example 2.4.13.
Note that for all data sequences v^+ such that $f^+(v^+, M) > 0$ for any $M \in \mathcal{M}$, it follows that $J^+(v^+, \mathcal{M}) = \{M_2\}$, whereas for all v^+ such that $f^+(v^+, M) = 0$ for any $M \in \mathcal{M}$, it holds that $J^+(v^+, \mathcal{M}) = \{M_1, M_2\} = \mathcal{M}$.

Consequently $\mathcal{M}$ is not discriminable by J^+, because there does not exist a data sequence v^+ such that $J^+(v^+, \mathcal{M}) = \{M_1\}$. Still $\mathcal{M}$ does not contain J^+–equivalent models. □

The – possibly rather artificial – example shows that the removal of distinct J^+–equivalent models out of the model set is not sufficient to arrive at discriminability. However we have the strong impression that in common situations of differentiable parametrizations with model sets having an infinite number of elements, the absence of these distinct J^+–equivalent models indeed is sufficient for discriminability. However this impression is not supported yet with a formal proof, and therefore in this thesis no results will be based on this assumption.
Nevertheless some attention will be paid to the reduction of a model set by considering a set of canonical forms with respect to $\underset{\sim}{\overset{J^+}{\mathcal{M}}}$, in order to investigate the influence of such a reduction on the set of selected optimal models.
In this investigation the following concept will appear to be illustrative.

Definition 2.4.16. Equivalent model sets.
Let $\sim$ be an equivalence relation defined on $\mathcal{M} \subset \tilde{\Sigma}$. This equivalence relation induces an equivalence relation on $2^{\mathcal{M}}$ by:
Two model sets $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{M}$ are equivalent, denoted $\mathcal{M}_1 \sim \mathcal{M}_2$, if

for all $M_1 \in \mathcal{M}_1$ there exists a $M_2 \in \mathcal{M}_2$, and

for all $M_2 \in \mathcal{M}_2$ there exists a $M_1 \in \mathcal{M}_1$, such that $M_1 \sim M_2$. □

Model sets that are equivalent with respect to a criterion based model equivalence, give similar results when applying them in an identification procedure. This is formalized in the following theorem.

Theorem 2.4.17.

Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two model sets such that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ is completely selectable and regularly partitionable for an identification criterion J^N, defined on $\mathcal{M}$.

If $\mathcal{M}_1 \overset{J^N}{\sim}_{\mathcal{M}} \mathcal{M}_2$ then

$$J^N(v^N, \mathcal{M}_1) \overset{J^N}{\sim}_{\mathcal{M}} J^N(v^N, \mathcal{M}_2) \qquad \text{for all } v^N \in V^{T^N}_{J,\mathcal{M}}. \tag{2.4.8}$$

Proof. For $v^N \in V^{T^N}_{J,\mathcal{M}}$ it follows that $J^N(v^N, \mathcal{M}) \neq \emptyset$. Suppose $J^N(v^N, \mathcal{M}) \cap \mathcal{M}_1 \neq \emptyset$; then because of the regular partitionability of $\mathcal{M}$: $J^N(v^N, \mathcal{M}_1) \subset J^N(v^N, \mathcal{M})$. For any $M_1 \in J^N(v^N, \mathcal{M}_1)$ there exists a $M_2 \in \mathcal{M}_2$ satisfying $M_2 \overset{J^N}{\sim}_{\mathcal{M}} M_1$, leading to $M_2 \in J^N(v^N, \mathcal{M})$ and consequently $M_2 \in J^N(v^N, \mathcal{M}_2)$. By reasons of symmetry this proves the result. □

The result of theorem 2.4.17 can be applied to a situation where a nondiscriminable model set $\mathcal{M}$ is reduced by excluding distinct J^N–equivalent models. Consequently one arrives at a subset $\mathcal{M}_1 \subset \mathcal{M}$ that satisfies $\mathcal{M}_1 \overset{J^N}{\sim}_{\mathcal{M}} \mathcal{M}$. For this situation the following result can be formulated.

Corollary 2.4.18.

Let $\mathcal{M}$ be a model set that is completely selectable and regularly partitionable for an identification criterion J^N, defined on $\mathcal{M}$.

Let $\mathcal{M}_1$ be a subset $\mathcal{M}_1 \subset \mathcal{M}$ that satisfies $\mathcal{M}_1 \overset{J^N}{\sim}_{\mathcal{M}} \mathcal{M}$; then

$$J^N(v^N, \mathcal{M}) = [J^N(v^N, \mathcal{M}_1)]_{\overset{J}{\sim}_{\mathcal{M}}}, \text{ for all } v^N \in V^{T^N}_{J,\mathcal{M}} \tag{2.4.9}$$

with $[J^N(v^N, \mathcal{M}_1)]_{\overset{J}{\sim}_{\mathcal{M}}} := \{M \in \mathcal{M} \mid \exists M_1 \in J^N(v^N, \mathcal{M}_1),\ M \overset{J^N}{\sim}_{\mathcal{M}} M_1\}$. (2.4.10) □

Moreover $\mathcal{M}_1$ can always be chosen in such a way that it is does not contain

J^N–equivalent models. □

Proof. The corollary follows rather directly from the proof of theorem 2.4.17. □

Note that the set of selected optimal models $J^N(v^N,\mathcal{M})$ is completely determined by the set $J^N(v^N,\mathcal{M}_1)$, however only for the data sequences $v^N \in V_{J,\mathcal{M}}^{T_N}$. Since $\mathcal{M}_1 \overset{J^N}{\sim}_{\mathcal{M}} \mathcal{M}$ it follows that $v^N \in V_{J,\mathcal{M}}^{T_N} \Rightarrow v^N \in V_{J,\mathcal{M}_1}^{T_N}$, and consequently $V_{J,\mathcal{M}}^{T_N} \subset V_{J,\mathcal{M}_1}^{T_N}$. As a result $J^N(v^N,\mathcal{M}_1)$ may generate more solutions than the ones that are determined by $J^N(v^N,\mathcal{M})$. In this sense a reduction of the model set may add additional selected optimal models. This is illustrated by again considering example 2.4.13.

Example 2.4.19.
Consider the situation of example 2.4.13 with

$$\mathcal{M} = \{M \in \hat{\Sigma}_{1,1} \mid M=\tilde{M}_p(T),\ T(z,z^{-1})=[a_0+a_1z^{-1}|-b_0-b_1z^{-1}|-c],\ a_0,a_1,b_0,b_1,c\in\mathbb{R},$$
$$((a_0+a_1z^{-1}),(b_0+b_1z^{-1}))\ \text{coprime}\} \qquad (2.4.11)$$

satisfying $V_{J,\mathcal{M}}^{+} = \bigcup_{M\in\mathcal{M}} \mathcal{B}^{+}(M^{io})$. (2.4.12)

Distinct J^+–equivalent models can be removed by considering $\mathcal{M}_1 \subset \mathcal{M}$ that satisfies the additional constraint that $a_0/c=1$ in $T(z,z^{-1})$ (2.4.11). For generic data sequences v^+ having a finite power, the identification criterion $J^+(v^+,\mathcal{M}_1)$ will be nonempty, and for the very specific data sequences $v^+ \in V_{J,\mathcal{M}}^{+}$ (eq. (2.4.12)), it holds that $J^+(v^+,\mathcal{M}_1) \overset{J^+}{\sim}_{\mathcal{M}} J^+(v^+,\mathcal{M})$. For generic data sequences apparently new selected optimal models are added by restricting the model set to $\mathcal{M}_1 \subset \mathcal{M}$. □

Referring to remark 2.4.14b it has to be noted that for two different model sets $\mathcal{M}_1$, $\mathcal{M}_2$ satisfying $\mathcal{M}_1 \overset{J^+}{\sim}_{\mathcal{M}} \mathcal{M}$ and $\mathcal{M}_2 \overset{J^+}{\sim}_{\mathcal{M}} \mathcal{M}$, (e.g. two different sets of canonical forms), the corresponding sets of optimal models $J^+(v^+,\mathcal{M}_1)$ and $J^+(v^+,\mathcal{M}_2)$ will generally be essentially different for data sequences that are not within $V_{J,\mathcal{M}}^{+}$. Consequently in this situation the choice for a specific set of canonical forms will clearly influence the set of selected optimal models.

The notions of discriminability of model sets by an identification criterion and of criterion based model equivalence are unusual in the literature on identification and parametrization. Criterion based model equivalence was introduced in Van den

Hof (1987), (1988) and a related model equivalence with respect to residual functions in Janssen (1988a). The two notions mentioned above constitute the main difference with the corresponding definitions of identifiability in the current literature (see e.g. Ljung, 1987, and Gevers and Wertz, 1987b). An equivalence relation based on an identification criterion, is explicitly incorporated in the concept of identifiability, whereas in the current literature this equivalence relation is fixed beforehand.
Further consequences of this situation will be discussed in chapter four.

2.5 DISCUSSION

In this chapter the basic concepts have been introduced that are required for analysing identification methods and corresponding parametrization problems.
The system theoretic framework of Willems (1986a), (1988) is adopted for describing dynamical systems in terms of linear spaces of admissible signal trajectories. The usual description of dynamical systems in terms of transfer functions, state space models, difference equations etc., all are encompassed in this approach, giving the opportunity to describe models and model sets independent of any parametrization. As a consequence, it becomes possible to present a clear relation between the problems of identification and parametrization.
For the notion of input–output–residual model as well as for the modelling of data on the basis of residuals, a completely non–stochastic framework has been chosen. Actually this choice represents a generalization of the existing stochastic models in which restrictions on the signals are imposed in terms of statistic/stochastic assumptions. If the introduction of such assumptions is desired, there is no single impediment in the presented approach to do so. However, in this thesis the approach is supported that the residuals do not a priori satisfy additional assumptions. They are interpreted as the modelling errors: that part of the measured data that remains unmodelled.
Consequently an (i/o/r)–model M can be looked upon from different points of view:

- in terms of a dynamical system it is represented by a model behaviour $\mathcal{B}(M)$, being a collection a signal trajectories that are compatible with the system's equations;
- in terms of identification on the basis of residuals it operates as a generator of compatible residual signals for a given measured data sequence v^N ($\mathcal{E}^N(v^N,M)$);

- in terms of model simulation it operates as a generator of output signals when input signals are supplied to the model, not considering any residual signals ($\mathcal{B}^{io}(M)$).

(i/o/r)–Models give us the opportunity to describe general identification problems. All currently used (stochastic) models can be incorporated, and general identification criteria can be applied to them, not even restricted to criteria with single model solutions (point estimates). Because of the signal based description of (i/o/r)–models, initial model conditions are implicitly incorporated.
Relational properties have been presented with respect to model sets and identification criteria, showing that the combination of model set and identification criterion leads to specific characteristics of an identification procedure, and that a separate treatment of these two notions is not appropriate. In this respect model sets and their parametrizations are considered in relation with the identification criterion, arriving at a criterion–based identifiability concept.

3. MODEL SETS AND IDENTIFICATION CRITERIA FOR RESIDUAL-BASED IDENTIFICATION

3.1 INTRODUCTION

In the previous chapter we have presented the basic concepts of a framework that is suitable for representing system identification methods in a fundamental and natural way. In this chapter 3 we are going to deal with the question how the different identification methods that are known from the literature, fit within this framework, and which specific choices underlie these methods.
Identification methods are based on specific choices for a model set, its parametrization, and an identification criterion. The combination of model set and identification criterion determines the result of the identification, i.e. the properties of the optimal models. The combination of parametrization and identification criterion determines the properties of the algorithm that is required for obtaining the set of selected optimal models. This chapter will be directed towards the model sets and the identification criteria.
The models that will be treated in this chapter are models that are often originally based on stochastic processes; however in our context there are no assumptions made about stochastic properties of signals and the models are purely seen as deterministic models. Although the framework allows a broader class of models, we will mainly restrict attention to the class of (i/o/pr)–models introduced in the previous chapter. The consequence of this restriction is that our models can be interpreted in two different ways:

a. as output–generating models, where the output signal y processes the inputs (u,e) and where the residual e consequently can be interpreted as a "disturbance" which acts on the dynamical system, and
b. as residual–generating models, where the residual signal e processes the measured data signals (y,u), and where e consequently can be interpreted as a modelling error.

This twofold interpretation is common in a great number of identification methods, see e.g. Ljung (1987).

In section 2.4.1 the different aspects have been listed that play a role in choosing a specific set of models to be applied in an identification problem. It has also been noted before, that actually there are two different approaches for arriving at a set of models and a corresponding parametrization:

- the first approach is to postulate a set of models, often restricted to models with a prescribed complexity, and subsequently to parametrize this set of models, and
- in the second approach the parametrization is taken as a starting point, e.g. because of favorable properties of this parametrization with respect to the identification algorithm; in this situation the resulting model set is accepted as a consequence of this parametrization.

In this chapter an overview will be given of model sets that can be used following either of these two approaches, and characteristic properties of these model sets will be discussed.

First a characterization of models will be presented in terms of the construction of the corresponding residual signals of the models (section 3.2). This characterization is relevant for both approaches mentioned above. In section 3.3, specific model sets will be discussed that are based on parametrizations. Section 3.4 subsequently contains a discussion on the complexity of models and model sets, leading to the introduction of a model order which is a generalization of both classical concepts of McMillan degree and minimal state space dimension. A related complexity measure introduced by Willems (1987) will be considered and will be compared with a specific minimal number of parameters. In this section also the advantage of polynomial matrix representations in the two shift operators σ and σ^{-1} will be exhibited by presenting a unifying theorem on the model order of model sets parametrized in either BDE–form (σ^{-1}) or in FDE–form (σ). Identification criteria are briefly considered in section 3.5.

In the discussion on models and model sets we will evidently use representations in terms of model behaviours. At the same time the consequence of several properties will be indicated in terms of the representations as defined in the previous chapter: polynomial matrix forms and (A,B,C,D) state space forms.

For purpose of reference to the existing literature, model properties will also be discussed in terms of the corresponding transfer functions. Because of the fact that a restriction will be made to (i/o/pr)–models, there are actually two transfer functions related to a given model $M \in \hat{\Sigma}$: one transfer function that considers y as the processing variable, and one transfer function considering e as the processing variable. For an (i/o/pr)–model $M \in \hat{\Sigma}_{p,m}$ with $M = \tilde{M}_p(T)$, $T=[P|-Q|-R]$, $T \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ and with $\mathcal{B}(M)$ consequently defined by

$$\mathcal{B}(M) := \{(y,u,e) \in \mathbb{R}^{(p+m+p)}\!\!\!\!\!\!\mathbb{I} \mid P(\sigma,\sigma^{-1})y - Q(\sigma,\sigma^{-1})u - R(\sigma,\sigma^{-1})e = 0\},$$

the following transfer functions will be considered:

$$H_y(z) = [H_{yu}(z) | H_{ye}(z)] := [\ P^{-1}Q \mid P^{-1}R\ ], \text{ and}$$
$$H_e(z) = [H_{ey}(z) | H_{eu}(z)] := [\ R^{-1}P \mid -R^{-1}Q\ ].$$

Note that both transfer functions represent only the controllable part of the (i/o/pr)–model M, as reflected in $\mathcal{B}_c(M)$ (see chapter 2). By considering transfer functions, consequently only properties of the controllable parts of (i/o/pr)–models can be considered. However this is not a severe restriction since we will restrict attention to controllable (i/o/pr)–models. This additional requirement of controllability is motivated by the fact that we want the residual signals to represent a modelling error that occurs when relating a measured data sequence to an (i/o/pr)–model. If the (i/o/pr)–model M is not controllable then a residual signal $e \in \mathcal{E}(v,M)$ contains a component that is not determined by any measured data v, nor by initial conditions. This part of the residual actually is caused by an autonomous part in the model behaviour which can not be considered to reflect modelling errors. Note that this controllability of an (i/o/pr)–model M does not imply that the i/o part of the model, M^{io}, is also controllable.

The restriction that will be made to controllable (i/o/pr)–models, has some consequences in terms of the representations to be considered. Following the results of chapter 2, these consequences, both with respect to the controllability of the models as well as with respect to the processing property of the residuals, are listed in table 3.1.

In this chapter the use of (i/o/pr)–models $M \in \hat{\Sigma}_{p,m}$ and general dynamical input–output systems $S \in \Sigma$ will be alternated. For some discussions the specific presence of a residual signal will be important. In other situations, as e.g. in the discussion on model complexity, it is appropriate to consider general input–output dynamical systems. In these situations (i/o/pr)–models can always be considered as special types of i/o dynamical systems $S \in \Sigma_{p,p+m}$, with y or e considered as the processing (output) variable.

	Restrictions on representations
Behaviour $\mathcal{B}(M)$ with $w=(y,u,e)\in\mathcal{B}(M)$	y processes (u,e); u,e free e processes (y,u); y,u free M controllable, (def.2.2.8)
Parametrization in full row rank polynomial matrix form: $T=[P\|-Q\|-R]\in\mathbb{R}^{p\times(p+m+p)}[z,z^{-1}]$	$\det_{\mathbb{R}(z)} P \neq 0$ $\det_{\mathbb{R}(z)} R \neq 0$ P,Q,R left coprime
Parametrization in minimal state space form with dimension n $x(k+1)=Ax(k)+[B_u\|B_e]\binom{u}{e}(k)$ $y(k)=Cx(k)+[D_u\|D_e]\binom{u}{e}(k)$	$\det_{\mathbb{R}(z)} [C(zI-A)^{-1}B_e+D_e] \neq 0$ rank $[\lambda I-A\ B_u\ B_e]$ = n for all $\lambda\in\mathbb{C}$ rank $\begin{bmatrix}\lambda I-A\\ C\end{bmatrix}$ = n for all $\lambda\in\mathbb{C}$
Transfer function $H_y(z) = [H_{yu}(z)\|H_{ye}(z)]$	$\det_{\mathbb{R}(z)} H_{ye}(z) \neq 0$
Transfer function $H_e(z) = [H_{ey}(z)\|H_{eu}(z)]$	$\det_{\mathbb{R}(z)} H_{ey}(z) \neq 0$

TABLE 3.1
A controllable (i/o/pr)–model $M\in\hat{\Sigma}_{p,m}$ in different representations.

3.2 CHARACTERIZATION OF MODELS BY RESIDUAL PROPERTIES

One of the important properties of an (i/o/pr)–model is the way in which the corresponding behaviour connects the residual signal to the input and output signals within the model. This residual signal is not physically measured and actually it is artificially added to the model behaviour in order to represent modelling

errors, i.c. influences on the data that are not represented by relations between inputs and outputs only. Therefore the connection between i/o signals and residual signals determines the "locations" in the model where disturbance terms between data and the i/o–part of an (i/o/pr)–model are represented.
In residual–based identification methods the identification criterion generates optimal models based on an evaluation of the residual signals only. They form the basis for the "measure of fit" between measured i/o–data and (i/o/r)–models. Consequently the location of these residual signals has a great influence on the properties of the optimal models finally obtained.

In this thesis three types of (i/o/pr)–models will be characterized with respect to the properties of their residual signals:

- Prediction error (PE) models;
- Output error (OE) or simulation error models; and
- Equation error (EE) models.

These expressions are generally known in the literature (see e.g. Ljung, 1987), though very often this model classification is strongly related towards specific model representations. In this thesis the different error–type models will be presented as general properties of controllable (i/o/pr)–models, irrespective of their representation.

Definition 3.2.1. k–step ahead prediction error (PE) model.
A controllable (i/o/pr)–model $M \in \hat{\Sigma}_{p,m}$ is called a *k–step ahead prediction error model*, if there exists a nonsingular matrix $L \in \mathbb{R}^{p \times p}$ such that for any $t_1 \in \mathbb{Z}$ and for any $w_1, w_2 \in \mathcal{B}(M)$ with $w_1=(y_1,u_1,e_1)$, $w_2=(y_2,u_2,e_2)$:

$$\left.\begin{array}{l} w_1(t) = w_2(t), \;\; t \leq t_1-k, \;\; k \in \mathbb{N} \\ u_1(t) = u_2(t), \;\; t_1-k+1 \leq t \leq t_1 \end{array}\right\} \Rightarrow \hat{y}_1(t_1) = \hat{y}_2(t_1), \quad \text{with } \hat{y}_i = y_i - Le_i, \; i=1,2$$

□

Signal $\hat{y}(t_1)$ can be interpreted as the output signal that is predicted by the model based on previous input data, and output and residual data up to time instant t_1–k. Since $e(t_1)$ is uniquely related to $y(t_1)$–$\hat{y}(t_1)$, $e(t_1)$ can be given the interpretation of a k–step ahead prediction error. Note that the expression "prediction" is used without any stochastic context.
The consequences for this model property in terms of the two basic representations are reflected in the following proposition.

Proposition 3.2.2.
Let M be a controllable (i/o/pr)–model, $M \in \hat{\Sigma}_{p,m}$. Then

a. If $M = \tilde{M}_P(T)$ with $T = [P|-Q|-R] \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, then M is a k–step ahead prediction error model if and only if:
   - $P^{-1}Q$ is a proper rational matrix, and
   - there exists a matrix $L \in \mathbb{R}^{p\times p}$, nonsingular, such that $(P^{-1}R-L)z^k$ is proper.

b. If $M = \tilde{M}_S(A,B,C,D)$ then M is a k–step ahead prediction error model if and only if
   - $D_e = L \in \mathbb{R}^{p\times p}$, nonsingular with $D=[D_u|D_e]$, and (3.2.1)
   - $CA^{j-1}B_e = 0$, $1 \le j \le k-1$. (3.2.2) □

Proof
(a). Following the definition above, the behaviour of a k–step ahead prediction error model can be described by a polynomial equation:

$$F(\sigma^{-1})\hat{y} = \sigma^{-k}G(\sigma^{-1})y + \sigma^{-k}K(\sigma^{-1})e + S(\sigma^{-1})u \quad (3.2.3)$$

with $F,G,K \in \mathbb{R}^{pxp}[z^{-1}]$, $S \in \mathbb{R}^{pxm}[z^{-1}]$, $\det_{\mathbb{R}(z)}F \neq 0$ and $F^{-1}S$, $F^{-1}G$ and $F^{-1}K$ proper rational matrices. Equivalent with (3.2.3) there can be written:

$$\{F(\sigma^{-1})-\sigma^{-k}G(\sigma^{-1})\}y - S(\sigma^{-1})u - \{\sigma^{-k}K(\sigma^{-1})+F(\sigma^{-1})L\}e = 0 \quad (3.2.4)$$

Because the model is controllable its behaviour is completely determined by its transfer function $P^{-1}[Q|R]$ that is restricted to be of the form:

$$P^{-1}[Q|R] = \{I - z^{-k}F^{-1}G\}^{-1}[F^{-1}S|L + z^{-k}F^{-1}K], \quad (3.2.5)$$

leading to the presented result.
(b). This result follows directly from (a) knowing that $P^{-1}R = C(zI-A)^{-1}B_e+D_e$ and the transfer function from input to output ($H_{yu}(z)$) by definition is proper for a model parametrized in state space form. □

Definition 3.2.3. Output error (OE) or simulation error model.
A controllable (i/o/pr)–model $M \in \hat{\Sigma}_{p,m}$ is called an *output error* or *simulation error model*, if there exists a nonsingular matrix $L \in \mathbb{R}^{p\times p}$ such that for any $t_1 \in \mathbb{Z}$ and for any $w_1,w_2 \in \mathcal{B}(M)$ with $w_1=(y_1,u_1,e_1)$, $w_2=(y_2,u_2,e_2)$:

$$\left.\begin{array}{l} u_1 = u_2 \\ \hat{y}_1(t) = \hat{y}_2(t),\ t \le t_1 \end{array}\right\} \Rightarrow \quad \hat{y}_1 = \hat{y}_2 \qquad \text{with } \hat{y}_i = y_i - Le_i,\ i=1,2$$

□

In this formulation, $\hat{y}$ can be given the interpretation of an output signal that is simulated by the model. The term "simulation" is used here in stead of "prediction", because of the fact that $\hat{y}$ is generated by the model and the input signal u

only, and not by y and e. The definition actually states that $\hat{y}$ processes u. Consequently e can be given the interpretation of an output or simulation error.

Proposition 3.2.4.
Let $M \in \hat{\Sigma}_{p,m}$ be a controllable (i/o/pr)–model. Then

a. If $M = \tilde{M}_p(T)$ with $T = [P|-Q|-R] \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, then M is an output error model if and only if:
   - $P^{-1}R = L \in \mathbb{R}^{pxp}$, nonsingular. (3.2.6)

b. If $M = \tilde{M}_s(A,B,C,D)$ then M is an output error model if and only if
   - $D_e = L \in \mathbb{R}^{pxp}$, nonsingular with $D=[D_u|D_e]$, and (3.2.7)
   - $CA^{j-1}B_e = 0$, $j \geq 1$. (3.2.8) □

Proof
(a). Using similar reasoning as in the proof of proposition 3.2.2 the model behaviour is characterized by:

$$F(\sigma^{-1})\hat{y} = G(\sigma^{-1})\hat{y} + S(\sigma^{-1})u \tag{3.2.9}$$

with $F,G \in \mathbb{R}^{pxp}[z^{-1}]$, $S \in \mathbb{R}^{pxm}[z^{-1}]$, $\det_{\mathbb{R}(z)} F \neq 0$ and $F^{-1}G$ a proper rational matrix.

Equivalent with (3.2.3) there can be written:

$$\{F(\sigma^{-1})-G(\sigma^{-1})\}y - S(\sigma^{-1})u - \{F(\sigma^{-1})-G(\sigma^{-1})\}Le = 0 \tag{3.2.10}$$

This shows that the transfer function $P^{-1}[Q|R]$ is restricted to be of the form:

$$P^{-1}[Q|R] = (I-F^{-1}G)^{-1}[F^{-1}S|(I - F^{-1}G)L], \tag{3.2.11}$$

leading to the result presented.
(b). This result follows directly from (a). □

Definition 3.2.5. Equation error (EE) model.
A controllable (i/o/pr)–model $M \in \hat{\Sigma}_{p,m}$ with variables w=(y,u,e) is called an *equation error model*, if e is observable from (y,u). □

An equation error model is characterized by the property that given measured data (y,u) a unique residual signal e results. Formulated in a way similar to the situations for PE and OE models, one can state that there exists a nonsingular matrix $L \in \mathbb{R}^{pxp}$, such that $\hat{y}=y-Le$ is observable from y and u.

Proposition 3.2.6.
Let $M \in \hat{\Sigma}_{p,m}$ be a controllable (i/o/pr)–model.

a. If $M = \tilde{M}_p(T)$ with $T = [P|-Q|-R] \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, then M is an equation error model if and only if $R(z,z^{-1})$ is unimodular with respect to $\mathbb{R}[z,z^{-1}]$.

b. If $M = \tilde{M}_s(A,B,C,D)$ with A having minimal dimension n, then M is an equation error model if and only if $\det_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B_e \\ -C & D_e \end{bmatrix} \neq 0$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. □

Proof (a). Since e is observable from y and u, there exist matrices P^*,Q^* such that the behaviour of the model is characterized by: $e = P^*(\sigma,\sigma^{-1})y - Q^*(\sigma,\sigma^{-1})u$. Knowing that under behavioural equivalence this set of equations can only be multiplied with a unimodular matrix, matrix R is restricted to be unimodular with respect to $\mathbb{R}[z,z^{-1}]$. The proof of part (b) is given in appendix A1. □

An overview of the three types of properties of controllable (i/o/pr)–models is presented in table 3.2, where also the consequences for the transfer functions $H_y(z)$ and $H_e(z)$ are listed.

It is possible that (i/o/pr)–models have several of the mentioned properties at the same time. Note e.g. that if an output error model has a proper transfer function $H_{yu}(z)$, it satisfies the requirements of being a k–step ahead prediction error model for *any* value of $k \in \mathbb{N}$. In that situation it actually behaves like a ∞–step ahead prediction error model. Similar to this example all combinations of PE, OE and EE properties are possible.

Each type of model presented, relates to specific qualities and therefore to a specific application of the model. Prediction error models are used in system identification if one is interested in using the identified model as a predictor of future behaviour over a prespecified prediction horizon (k). In most situations prediction error models are used with the prediction horizon k=1. Output error models should be used in system identification if one is interested in using the identified model for simulation purposes. Equation error models are actually based on mathematical considerations and are strongly related to linear regression models and the classical least squares problem. They are attractive from an algorithmic point of view. In terms of the intended use of the identified models, EE models could be considered for purposes of description of the original process, as discussed in Willems (1987) en Heij (1988).

The residual signals that we are dealing with are strongly related to the outputs y of the system to be modelled; they are also equal in dimension (r=p). This is directly caused by the fact that we consider (i/o/pr)–models.
Without going into detail we mention the possibility of using (i/o/r)–models with

	k–step ahead prediction error	Output error	Equation error
Behaviour $\mathcal{B}(M)$	$\exists L\in\mathbb{R}^{p\times p}$, nonsingular, such that for $w_1,w_2\in\mathcal{B}$ and any $t_1\in\mathbb{Z}$, with $\hat{y}_i=y_i-Le_i$, $i=1,2$: $\{u_1(t)=u_2(t),\ t\le t_1 \wedge w_1(t)=w_2(t),\ t\le t_1-k\} \Rightarrow \{\hat{y}_1(t_1)=\hat{y}_2(t_1)\}$	$\exists L\in\mathbb{R}^{p\times p}$, nonsingular, such that for $w_1,w_2\in\mathcal{B}$, with $\hat{y}_i=y_i-Le_i$, $i=1,2$: $\{u_1=u_2 \wedge \hat{y}_1(t)=\hat{y}_2(t), t\le t_1$ for any $t_1\in\mathbb{Z}\} \Rightarrow \hat{y}_1=\hat{y}_2$	for $w_1,w_2\in\mathcal{B}$ $\{u_1=u_2 \wedge y_1=y_2\} \Rightarrow e_1=e_2$
Parametrization in full row rank polynomial matrix form: $T(z,z^{-1}) = [P \mid -Q \mid -R]$	$P^{-1}Q$ is proper $(P^{-1}R-L)z^k$ is proper $L\in\mathbb{R}^{p\times p}$, nonsingular	$P(z,z^{-1})\cdot L = R(z,z^{-1})$ $L\in\mathbb{R}^{p\times p}$, nonsingular	$R(z,z^{-1})$ unimodular with respect to $\mathbb{R}[z,z^{-1}]$
Parametrization in minimal state space form $x(t+1)=Ax(t)+[B_u\mid B_e]\binom{u}{e}(t)$ $y(t)=Cx(t)+[D_u\mid D_e]\binom{u}{e}(t)$	$D_e = L\in\mathbb{R}^{p\times p}$, nonsingular $CA^{j-1}B_e = 0,\ 1\le j\le k-1$	$D_e = L\in\mathbb{R}^{p\times p}$, nonsingular $CA^{j-1}B_e = 0,\ j\ge 1$	$\det_{\mathbb{C}}\begin{bmatrix}\lambda I-A & B_e\\ -C & D_e\end{bmatrix}\neq 0$ for $0\neq\lambda\in\mathbb{C}$
Transfer function $H_y(z) = [H_{yu}(z)\mid H_{ye}(z)]$	$H_{yu}(z)$ is proper $(H_{ye}(z)-L)z^k$ is proper $L\in\mathbb{R}^{p\times p}$, nonsingular	$H_{ye}(z) = L$ $L\in\mathbb{R}^{p\times p}$, nonsingular	$H_{ye}(z)^{-1}\in\mathbb{R}^{p\times p}[z,z^{-1}]$ $H_{ye}(z)^{-1}H_{yu}(z)\in\mathbb{R}^{p\times m}[z,z^{-1}]$
Transfer function $H_e(z) = [H_{ey}(z)\mid H_{eu}(z)]$	$H_{eu}(z)$ is proper $(H_{ey}(z)-L^{-1})z^k$ is proper $L^{-1}\in\mathbb{R}^{p\times p}$	$H_{ey}(z) = L^{-1}$ $L\in\mathbb{R}^{p\times p}$	$H_{ey}\in\mathbb{R}^{p\times p}[z,z^{-1}]$ $H_{eu}\in\mathbb{R}^{p\times m}[z,z^{-1}]$

TABLE 3.2 Specifications of PE, OE and EE type of (i/o/pr)–models.

residuals that are related to the inputs (r=m). A combined approach is followed when using errors–in–variables models, relating the residuals to both inputs and outputs (r=p+m). There is a growing interest in these latter type of models, as witnessed by some recent references: Kalman (1982), Anderson and Deistler (1987), De Moor (1988) and Beghelli, Guidorzi and Soverini (1988).

3.3 MODEL SETS BASED ON PARAMETRIZATIONS

3.3.1 Introduction

In this subsection a short overview will be given of model sets that are used in system identification and that are based on specific parametrizations. This means that the parametrizations are taken as a starting point. Model sets are constructed by considering specific restrictions on the set of polynomial matrices Θ_p in a polynomial matrix parametrization, or by considering specific restrictions on the set of realizations Θ_s in an (A,B,C,D) state space parametrization.

First we have to go through some notational definitions on polynomial matrices, that will be used in the remainder of this chapter.

Definition 3.3.1

Consider a polynomial matrix $T \in \mathbb{R}^{pxq}[z,z^{-1}]$. Denote with:

T_{i*} := the i^{th} row of $T(z,z^{-1})$, $i \in \underline{p}$; with $\underline{p} := \mathbb{Z} \cap [1,p]$;

T_{*j} := the j^{th} column of $T(z,z^{-1})$, $j \in \underline{q}$;

$\delta_{ij}^{(u)}(T)$, $\delta_{ij}^{(l)}(T)$:= the maximum, minimum, power of z in $T_{ij}(z,z^{-1})$, $i \in \underline{p}; j \in \underline{q}$;

$\delta^{(u)}(T)$, $\delta^{(\ell)}(T)$:= the maximum, minimum, power of z in $T(z,z^{-1})$;

$\nu_i^{(u)}(T)$, $\nu_i^{(\ell)}(T)$:= the maximum, minimum, power of z in $T_{i*}(z,z^{-1})$, $i \in \underline{p}$;

$\mu_j^{(u)}(T)$, $\mu_j^{(\ell)}(T)$:= the maximum, minimum, power of z in $T_{*j}(z,z^{-1})$, $j \in \underline{q}$;

$\delta_{ij}(T) := \delta_{ij}^{(u)}(T) - \delta_{ij}^{(\ell)}(T)$; $\delta(T) := \delta^{(u)}(T) - \delta^{(\ell)}(T)$;

$\nu_i(T) := \nu_i^{(u)}(T) - \nu_i^{(\ell)}(T)$;

$\mu_j(T) := \mu_j^{(u)}(T) - \mu_j^{(\ell)}(T)$; □

Definition 3.3.2.

Let $T(z,z^{-1})\in\mathbb{R}^{p\times q}[z,z^{-1}]$ with row degrees $\nu_i^{(u)}$ and $\nu_i^{(\ell)}$ and let T be written as:

$$T(z,z^{-1}) = \mathrm{Diag}(z^{\nu_1^{(u)}},..,z^{\nu_p^{(u)}})\cdot\Gamma_{hr} + T_{r1}(z,z^{-1}),$$

and $$T(z,z^{-1}) = T_{r2}(z,z^{-1}) + \mathrm{Diag}(z^{\nu_1^{(\ell)}},..,z^{\nu_p^{(\ell)}})\cdot\Gamma_{lr},$$

with $\Gamma_{hr},\Gamma_{lr}\in\mathbb{R}^{p\times q}$ and $T_{r1},T_{r2}\in\mathbb{R}^{p\times q}[z,z^{-1}]$ satisfying $\nu_i^{(u)}(T_{r1}) < \nu_i^{(u)}$ and $\nu_i^{(\ell)}(T_{r2}) > \nu_i^{(\ell)}$ for $i\in\underline{p}$, then

Γ_{hr}:= the leading row coefficient matrix of $T(z,z^{-1})$, denoted by $\Gamma_{hr}(T)$; and

Γ_{lr}:= the trailing row coefficient matrix of $T(z,z^{-1})$, denoted by $\Gamma_{lr}(T)$. □

A corresponding definition with respect to the columns of $T(z,z^{-1})$ is also presented.

Definition 3.3.3.

Let $T(z,z^{-1})\in\mathbb{R}^{p\times q}[z,z^{-1}]$ with column degrees $\mu_j^{(u)}$ and $\mu_j^{(\ell)}$ and let T be written as:

$$T(z,z^{-1}) = \Gamma_{hc}\cdot\mathrm{Diag}(z^{\mu_1^{(u)}},..,z^{\mu_q^{(u)}}) + T_{c1}(z,z^{-1}),$$

and $$T(z,z^{-1}) = T_{c2}(z,z^{-1}) + \Gamma_{lc}\cdot\mathrm{Diag}(z^{\mu_1^{(\ell)}},..,z^{\mu_q^{(\ell)}}),$$

with $\Gamma_{hc},\Gamma_{lc}\in\mathbb{R}^{p\times q}$ and $T_{c1},T_{c2}\in\mathbb{R}^{p\times q}[z,z^{-1}]$ satisfying $\mu_j^{(u)}(T_{c1}) < \mu_j^{(u)}$ and $\mu_j^{(\ell)}(T_{c2}) > \mu_j^{(\ell)}$ for $j\in\underline{q}$, then

Γ_{hc}:= the leading column coefficient matrix of $T(z,z^{-1})$, denoted by $\Gamma_{hc}(T)$; and

Γ_{lc}:= the trailing column coefficient matrix of $T(z,z^{-1})$, denoted by $\Gamma_{lc}(T)$. □

We can now generalize the well known properties of row properness and column properness of polynomial matrices (Wolovich, 1974) to the situation of polynomial matrices in two shift operators.

Definition 3.3.4. Row proper polynomial matrix.

A polynomial matrix $T\in\mathbb{R}^{p\times q}[z,z^{-1}]$ is called *row proper with respect to* $\mathbb{R}[z]$ ($\mathbb{R}[z^{-1}]$) if rank $\Gamma_{hr}(T)$=p (rank $\Gamma_{lr}(T)$=p).

T is called *bilaterally row proper* if it is row proper with respect to $\mathbb{R}[z,z^{-1}]$, i.e. rank $\Gamma_{hr}(T)$ = rank $\Gamma_{lr}(T)$ = p. □

The notion of bilaterally row properness of a polynomial matrix was first presented in Willems (1986a).

Definition 3.3.5. Column proper polynomial matrix.
A polynomial matrix $T \in \mathbb{R}^{p \times q}[z,z^{-1}]$ is called *column proper with respect to* $\mathbb{R}[z]$ ($\mathbb{R}[z^{-1}]$) if rank $\Gamma_{hc}(T)=q$ (rank $\Gamma_{lc}(T)=q$).
T is called *bilaterally column proper* if it is column proper with respect to $\mathbb{R}[z,z^{-1}]$, i.e. rank $\Gamma_{hc}(T)$ = rank $\Gamma_{lc}(T)$ = q. □

Parametrizations in polynomial matrix form are generally not used in a representation with both shift operators σ and σ^{-1}. It is common to apply either one of the two shift operators and to represent the applied shift operator in the name of the model representation concerned. In accordance with the nomenclature as presented in Janssen (1988a), the following definitions are given.

Definition 3.3.6. Parametrization in backward difference equation (BDE) form.
A polynomial matrix parametrization $\tilde{M}_p: \Theta_p \rightarrow \mathcal{M}$ with $\mathcal{M} \subset \tilde{\Sigma}_{p,m,r}$ is said to be in *backward difference equation (BDE) form*, if $\Theta_p \subset \mathbb{R}^{p \times (p+m+r)}[z^{-1}]$. □

Definition 3.3.7. Parametrization in forward difference equation (FDE) form.
A polynomial matrix parametrization $\tilde{M}_p: \Theta_p \rightarrow \mathcal{M}$ with $\mathcal{M} \subset \tilde{\Sigma}_{p,m,r}$ is said to be in *forward difference equation (FDE) form*, if $\Theta_p \subset \mathbb{R}^{p \times (p+m+r)}[z]$. □

In order to make a clear distinction between the two sets of polynomial matrices, we will denote T_b as a general element $T_b \in \mathbb{R}^{p \times (p+m+r)}[z^{-1}]$, and T_f as a general element $T_f \in \mathbb{R}^{p \times (p+m+r)}[z]$.
Note that for any model $M \in \tilde{\Sigma}$ a representation in either BDE–form or FDE–form always exists. Both representations can be transformed into one another by pre-multiplication of the rows in the polynomial matrices by (positive or negative) powers of z. This shift operation will not change the system behaviour, since it corresponds with a unimodular premultiplication.
However the two representations do make a difference when we consider model sets that are defined by specific restrictions on the polynomial matrices T_b or T_f. In general, model sets based on BDE or FDE parametrizations are constructed by restricting the degrees of the different polynomial entries in the matrices T_b or T_f. However similar restrictions on the structures of these polynomial matrices will

generally lead to different model sets.

3.3.2 Model sets based on BDE–parametrizations

Sets of (i/o/pr)–models that are based on BDE–parametrizations are represented in terms of polynomial matrices $T(z^{-1})=[P(z^{-1})\,|-Q(z^{-1})\,|-R(z^{-1})]\in\mathbb{R}^{p\times(p+m+p)}[z^{-1}]$. Corresponding parametrizations are very often restricted to the situation that $P(z^{-1})$ and $R(z^{-1})$ are monic polynomials, see Ljung (1987), i.e. :

$$P(z^{-1}) = I + P_{-1}z^{-1} + \ldots + P_{-n_P}z^{-n_P}$$
$$\text{and} \quad R(z^{-1}) = I + R_{-1}z^{-1} + \ldots + R_{-n_r}z^{-n_r} \tag{3.3.1}$$

with $-n_P = \delta^{(\ell)}(P)$, $-n_r = \delta^{(\ell)}(R)$.
This specification automatically meets both requirements of (i/o/pr)–models, i.e. $\det_{\mathbb{R}(z)}P(z^{-1})\neq 0$ and $\det_{\mathbb{R}(z)}R(z^{-1})\neq 0$, and additionally it guarantees that y does not anticipate u, i.e. there is a causal relationship between the input u and the output y. It can easily be verified that the specification (3.3.1) brings the models into a 1–step ahead prediction error form. However by adding some extra restrictions, the model set can easily be formulated in an output error form (e.g. by the restriction $R(z^{-1})=P(z^{-1})$) or in an equation error form (e.g. by $R(z^{-1})=I$, corresponding to $n_r=0$).

Model sets based on BDE–parametrizations are obtained by considering all polynomial matrices in the specific class as specified above, that are restricted in complexity by prescribing the maximum degrees of the polynomial matrices involved:

$$\begin{aligned}
&\delta^{(\ell)}_{ij}(P) \geq -n_p^{(ij)}\,, \quad i,j\in\underline{p};\\
&\delta^{(\ell)}_{ij}(Q) \geq -n_q^{(ij)}\,, \quad i\in\underline{p};j\in\underline{m};\\
&\delta^{(\ell)}_{ij}(R) \geq -n_r^{(ij)}\,, \quad i,j\in\underline{p};
\end{aligned} \tag{3.3.2}$$

with $n_p^{(ij)}$, $n_q^{(ij)}$, $n_r^{(ij)}$ a priori specified integers ≥ 0.
In the so called *full polynomial forms* there is no difference between the polynomial degrees of the different entries in the matrices, leading to the restrictions: $\delta^{(\ell)}(P) \geq -n_p$, $\delta^{(\ell)}(Q) \geq -n_q$ and $\delta^{(\ell)}(R) \geq -n_r$ (see e.g. Hannan, 1976).
In the so called *prescribed maximum lag forms* more structural specifications are given for the model set, by prescribing the maximum column degrees of the polynomial matrices: $\mu_j^{(\ell)}(P) \geq -n_p^{(j)}$, $\mu_j^{(\ell)}(Q) \geq -n_q^{(j)}$, $\mu_j^{(\ell)}(R) \geq -n_r^{(j)}$, i and j running over the proper integer interval (see e.g. Deistler, 1983).

Scalar forms are restricted to the special case that $P(z^{-1}) = p(z^{-1})I_p$, with $p(z^{-1})\in\mathbb{R}[z^{-1}]$ and I_p the p×p identity matrix (see Hannan, 1976, Gevers and Wertz, 1987b).

Without any further restrictions on the polynomials P, Q and R, the model sets as indicated above are also denoted as ARMAX model set, while specific names are used in the literature for model sets in which P, Q and R satisfy some additional restrictions, e.g.:

* ARX models: restriction $R(z^{-1})=I$ leading to a set of EE models;
* Finite impulse response (FIR) or Markov parameter models: restriction $H_{yu}(z)\in\mathbb{R}^{p\times m}[z,z^{-1}]$, e.g. by fixing $P(z^{-1})=I$.

In Ljung (1987) a basic model set for SISO systems is applied which can be defined in a multivariable form by:

$$H_{yu}(z)=A(z^{-1})^{-1}F(z^{-1})^{-1}B(z^{-1});\ H_{ye}(z)=A(z^{-1})^{-1}D(z^{-1})^{-1}C(z^{-1}); \qquad (3.3.3)$$

with A,B,C,D,F polynomial matrices of proper sizes, with prescribed polynomial degrees, and with all polynomials monic except B.

This so called *ABFCD model set* can be represented in BDE–form by:

$$P(z^{-1}) = \tilde{F}(z^{-1})D(z^{-1})A(z^{-1}); \qquad (3.3.4)$$

$$Q(z^{-1}) = \tilde{D}(z^{-1})B(z^{-1}); \qquad (3.3.5)$$

$$R(z^{-1}) = \tilde{F}(z^{-1})C(z^{-1}); \qquad (3.3.6)$$

where $\tilde{F}$ and $\tilde{D}$ are determined by $\tilde{F}(z^{-1})D(z^{-1})=\tilde{D}(z^{-1})F(z^{-1})$.

This special form has the advantage that it is simple to separately parametrize the contributions of the dynamics between u and y on the one hand, and e and y on the other hand. E.g. by fixing $A(z^{-1})=I$, the transfer functions $H_{yu}(z)$ and $H_{ye}(z)$ are independently parametrized. This separation of system dynamics is quite common in many situations where the residual e is considered as a disturbance signal acting on the measured output y as an additive output noise.

Specific choices of subsets of this ABFCD–form can be made by fixing polynomial matrices to I. For an overview of choices see Ljung (1987).

As an intrinsic advantage of model sets based on BDE–parametrizations it can be mentioned that these model sets have very simple representations in terms of the different residual forms PE, OE and EE, while keeping up with the restriction of a causal relationship between input u and output y:

1–step PE:	Standard form; no restrictions on $P(z^{-1})$, $Q(z^{-1})$, $R(z^{-1})$;
OE:	Restriction $R(z^{-1})=P(z^{-1})$;
EE:	Restriction $R(z^{-1})$ is unimodular with respect to $\mathbb{R}[z,z^{-1}]$, e.g. $R(z^{-1})=I$.

The corresponding restrictions on the BDE–parametrization are easily dealt with in an identification algorithm.
Model sets based on BDE–parametrizations are also treated in Gevers (1986), Gevers and Wertz (1987b), Bokor and Keviczky (1987) and Janssen (1988a).

3.3.3 Model sets based on FDE–parametrizations

Sets of (i/o/pr)–models that are based on FDE–parametrizations are represented in terms of polynomial matrices $T_f(z)=[P(z)|-Q(z)|-R(z)] \in \mathbb{R}^{p\times q}[z]$. In many respects model sets based on these parametrizations are dual forms of the model sets based on BDE forms, however there are some important differences. In a dual situation of the BDE–forms, as presented in the previous subsection, it would be natural to restrict the polynomial matrices P(z) and R(z) to be monic (compare eq. 3.3.1), leading to:

$$P(z) = I + P_1 z + P_2 z^2 + \ldots + P_{n_p} z^{n_p}$$

and $$R(z) = I + R_1 z + R_2 z^2 + \ldots + R_{n_r} z^{n_r} \qquad (3.3.7)$$

with $n_p=\delta^{(u)}(P)$ and $n_r=\delta^{(u)}(R)$,
thereby automatically satisfying the requirements $\det_{\mathbb{R}(z)}P(z)\neq 0$, $\det_{\mathbb{R}(z)}R(z)\neq 0$.
There are two direct consequences of this choice, that differ essentially from the situation of BDE–parametrized model sets:

1. Restricting P(z) to be monic does not guarantee a causal relationship between input u and output y; in other words: the transfer function $H_{yu}(z)=P(z)^{-1}Q(z)$ is not necessarily proper. In order to arrive at a proper transfer function, additional restrictions have to be imposed on P(z) and Q(z).
2. A similar consequence follows for the transfer function $H_{ey}(z)$. Since restriction (3.3.7) does not automatically yield a proper transfer function $H_{ey}(z)$, additional restrictions are required in order to arrive at a 1–step ahead prediction error form. Extrapolating from definition 3.2.1, models satisfying restrictions (3.3.7) can be considered to represent a 1–step *backwards* prediction error form. These backwards prediction error models could be defined based on the forward prediction error forms, allowing negative values of the prediction horizon. In terms of the restrictions on polynomial matrix parametrizations (3.2.2), a k–step backward prediction error model would require $(P^{-1}R–L)z^{-k}$ to be a proper rational matrix with $L\in\mathbb{R}^{p\times p}$, nonsingular.

These two consequences for model sets satisfying (3.3.7), have motivated a treatment of FDE parametrizations that is different from the corresponding BDE pa-

rametrizations. An additional argument for this separate treatment, and indeed a very important one, will be given in the next section 3.4, when discussing the complexity of models in terms of the McMillan degree.

In order to be able to arrive at model sets based on FDE–forms that intrinsically satisfy conditions of properness of the transfer functions $H_{yu}(z)$ and $H_{ye}(z)$, parametrizations will be considered that satisfy restrictions on the leading row coefficient matrices, by requiring that matrices P and R are row proper.
As in the situation of BDE parametrizations, the FDE forms are generally specified by prescribing the maximum degrees of the polynomial entries in the polynomial matrices:

$$\begin{aligned} &\delta^{(u)}_{ij}(P) \le n_p^{(ij)}, \quad i,j\in\underline{p}; \\ &\delta^{(u)}_{ij}(Q) \le n_q^{(ij)}, \quad i\in\underline{p};j\in\underline{m}; \\ &\delta^{(u)}_{ij}(R) \le n_r^{(ij)}, \quad i,j\in\underline{p}; \end{aligned} \tag{3.3.8}$$

Model sets that are defined in this way and that are restricted by requiring that matrices P and R are row proper with respect to $\mathbb{R}[z]$ can simply be brought into prediction error forms, as formulated in the following proposition.

Proposition 3.3.8.
Let $\mathcal{M}$ be a model set based on an FDE parametrization, defined by a set of integers $\{n_p^{(i)},n_q^{(i)},n_r^{(i)}\}$, $i\in\underline{p}$, according to

$$\begin{aligned} \mathcal{M} := \{M\in\hat{\Sigma}_{p,m} \mid \mathcal{B}(M)=\tilde{M}_p(T), \text{ with } T=[P|-Q|-R]\in\mathbb{R}^{pxq}[z]; \\ \nu_i^{(u)}(P)=n_p^{(i)},\ \nu_i^{(u)}(R)=n_r^{(i)},\ \nu_i^{(u)}(Q)\le n_q^{(i)},\ i\in\underline{p}, \\ \text{and P and R row proper with respect to } \mathbb{R}[z]\} \end{aligned} \tag{3.3.9}$$

then $\mathcal{M}$ is a set of k–step ahead prediction error models if and only if:

(i) $n_q^{(i)} \le n_p^{(i)}$, $i\in\underline{p}$; and (3.3.10)

(ii) $n_r^{(i)} = n_p^{(i)}$, $i\in\underline{p}$; and (3.3.11)

(iii) $[P^{-1}R - \{\Gamma_{hr}(P)\}^{-1}\Gamma_{hr}(R)]z^k$ is a proper rational matrix. (3.3.12)

For k=1 condition (iii) is automatically satisfied. □

Proof. See appendix A2. □

The result of proposition 3.3.8 shows that for additional restrictions to row proper matrices P and R, 1–step ahead prediction error models can be obtained by simple restrictions on the row degrees of the polynomial entries in matrix T(z), as reflect-

ed in (3.3.10) and (3.3.11). Formulations for FDE parametrizations in terms of output error or equation error type of residuals are even less involved; these situations actually are similar to the situation of BDE–forms, by requiring $R(z) = P(z)$ for output error models, and $R(z) = I_p$ for equation error models.

3.3.4. Model sets based on state space representations

In this thesis we have restricted ourselves to the use of the specific type of state space representations as presented in chapter 2:

$$\begin{aligned} x(k+1) &= A\,x(k) + [B_u|B_e]\;col(u(k),e(k)) \\ y(k) &= C\,x(k) + [D_u|D_e]\;col(u(k),e(k)) \end{aligned} \tag{3.3.13}$$

with the restrictions of minimality as reflected in proposition 2.4.8.
The requirement of the state space representation to induce an (i/o/pr)–model is generally realized by restricting (3.3.13) to $D_e=I_p$. This restriction creates a specific 1–step ahead prediction error model; under the additional assumption that the residual e is the result of a zero mean Gaussian stochastic process, this form is known as an innovations representation (Gevers and Kailath, 1973, Ljung, 1987).
As denoted in section 3.2, k–step ahead prediction error and output error models require the simple restriction that D_e is nonsingular. Equation error type of models are rather artificial in a state space representation; they are actually based on polynomial parametrizations. Model sets of models parametrized in a state space form are generally restricted by an upper bound of the state space dimension n. In this representation the state space dimension is the straightforward measure of complexity of models. A further discussion on this complexity follows in section 3.4.
At this moment we want to point at two specific aspects of state space representations.
The first one is the possibility to separately parametrize the contributions of the dynamics between u and y on the one hand, and e and y on the other hand, as for the ABFCD model set in equations (3.3.3)–(3.3.6). This can be done by choosing the following model structure:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(k) + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u(k) \\ e(k) \end{bmatrix} \\ y(k) &= [C_1\;C_2]\,x(k) + [D_u\;D_e] \begin{bmatrix} u(k) \\ e(k) \end{bmatrix} \end{aligned} \tag{3.3.14}$$

In (3.3.14), (A_1,B_1,C_1,D_u) represents the dynamics between the system variables u and y, whereas (A_2,B_2,C_2,D_e) represents the dynamics between e and y.

As a second aspect we would like to point at a specific set of models that is not directly based on the state space dimension of its representations, but on the degree of the minimal polynomial of A. The specific model set is parametrized by the state space form:

$$x(k+1) = \begin{bmatrix} 0 & I_p & & 0 \\ \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & 0 \quad I_p \\ -a_r I_p & -a_{r-1} I_p & \cdot \cdot & -a_1 I_p \end{bmatrix} x(k) + \begin{bmatrix} B_{u1} & B_{e1} \\ B_{u2} & B_{e2} \\ \cdot & \cdot \\ \cdot & \cdot \\ B_{ur} & B_{er} \end{bmatrix} \begin{bmatrix} u(k) \\ e(k) \end{bmatrix}$$

$$y(k) = [\, I_p \; 0 \; \cdot \; \cdot \; 0 \,]\; x(k) + [D_u \; D_e] \begin{bmatrix} u(k) \\ e(k) \end{bmatrix} \tag{3.3.15}$$

The transfer function connected to this representation:

$$H_y(z) = C(zI-A)^{-1}B + D \tag{3.3.16}$$

can be expanded in a formal power series around $z=\infty$, leading to:

$$H_y(z) = \sum_{k=0}^{\infty} M_k z^{-k} \tag{3.3.17}$$

with $\{M_k\}_{k=0,..,\infty}$ the Markov parameters of the transfer function. It follows from the representation (3.3.15) that these Markov parameters can be written as:

$$M_0 = D \tag{3.3.18a}$$

$$M_i = [B_{u_i} | B_{e_i}], \quad i \in \underline{r} \tag{3.3.18b}$$

$$M_{r+j} = \sum_{i=1}^{r} -a_i M_{r+j-i}, \qquad j \geq 1 \tag{3.3.18c}$$

where $\{a_i\}_{i=1,r}$ are the coefficients of the minimal polynomial of A.

The model set is determined by an a priori chosen value of the integer r, the degree of the minimal polynomial of A.

This specific parametrization for a set of models has been introduced by Gerth (1972), and is in an output error form ($B_{e_i}=0$, $i \in \underline{r}$) further developed and applied in Backx (1987). It has favorable properties in terms of the complexity of corresponding identification algorithms, which is also discussed in chapter 5.

3.4. MODEL COMPLEXITY

3.4.1. Introduction

The complexity of models is an important concept in the identification problem. This notion of complexity has to be understood as a measure for the 'size' of models. The more complex a model is, the more severe restrictions are imposed on the admissible signals, the more coefficients it requires to represent the model in any representation, the more the model 'explains' from a measured data sequence, and consequently the better it is able to represent this measured data sequence.
For two dynamical systems $S_1, S_2 \in \Sigma$, having equal signal set W, it can be stated that if $\mathcal{B}(S_1) \subset \mathcal{B}(S_2)$, then S_1 is more complex than S_2. In this sense the dynamical system that imposes no restrictions on the admissible signals ($\mathcal{B} = W^{\mathbb{Z}}$) is the least complex system (actually it explains nothing).
In view of the intended use of the model, the complexity of a model can also be viewed as a measure of model 'cost'. The more complex a model is, the more expensive it will be with respect to its intended application. In this respect it is favorable to construct 'cheap' models, having small complexities. Moreover the purpose of modelling data sequences, is to represent a data sequence by a compact model that is less complex than the data sequence itself. It has to be stressed that a procedure of modelling should incorporate a considerable amount of data reduction, in order to obtain models that do not depend heavily on random effects within the data.
Apparently there is a trade–off between the two aspects mentioned above: the more complex the model, the better it represents the data, but the higher the model cost. In an identification problem, a proper choice has to be made regarding this trade–off.
The way in which we measure the model 'cost', or in other words the most suitable definition of model complexity, will be dependent on the application of the model that the experimenter has in mind. Dependent on this intended use, different aspects of the models will have to be considered for evaluation of the model cost. Consequently no general statements can be made on the choice for a model complexity, without taking account of the application of the model. The ambiguity that can arise when dealing with model complexities will be illustrated in example 3.4.1.

The discussion on model complexity will be pursued on the basis of a general in-

put–output dynamical system $M \in \Sigma_{p,m}$ with m inputs and p outputs. The presence of a specific residual signal is not really relevant in this discussion. Moreover, any (i/o/pr)–model can be considered to be an input–output dynamical system with input (u,e) and output y. Although the (i/o/pr)–models in this chapter are restricted to be controllable, we will not make this assumption for the systems $S \in \Sigma_{p,m}$ to be considered in this section. This is motivated by the fact that we want to be able to discuss the complexity of i/o parts of (i/o/pr)–models, while these i/o–parts are not necessarily controllable.

Example 3.4.1.

Consider two dynamical systems $S_1, S_2 \in \Sigma_{1,1}$, induced by polynomial matrices T_1, T_2, according to:

$S_1 = \tilde{M}_p(T_1)$, $T_1(z,z^{-1}) = [1 | z^{-5}]$;

$S_2 = \tilde{M}_p(T_2)$, $T_2(z,z^{-1}) = [1 + a_1 z^{-1} + \cdot + a_5 z^{-5} | -b_1 z^{-1} - \cdot - b_5 z^{-5}]$, $a_i, b_i \in \mathbb{R} \setminus \{0\}$, $i \in \underline{5}$.

Consider also the following possible choices for the complexity of a dynamical system $S \in \Sigma$:

1. $c^{(1)}(S)$: the dimension of a minimal (A,B,C,D) state space representation of S.
2. $c^{(2)}(S)$: the McMillan degree of the transfer function of the system.
3. $c^{(3)}(S)$: the minimal number of nonzero coefficients in a polynomial matrix representation of the system.

Complexity measures $c^{(1)}$ and $c^{(2)}$ will be determined by the maximum power of z^{-1} in $T(z,z^{-1})$, leading to $c^{(1)}(S_1) = c^{(2)}(S_1) = 5$, and $c^{(1)}(S_2) = c^{(2)}(S_2) = 5$. Consequently, in terms of $c^{(1)}$ and $c^{(2)}$, S_1 and S_2 are equally complex.

$c^{(3)}$ only measures the number of nonzero coefficients in T, leading to $c^{(3)}(S_1) = 2$ and $c^{(3)}(S_2) = 11$. In terms of $c^{(3)}$, S_2 is more complex than S_1.

A *general* statement on the suitability of these complexity measures can not be given without taking account of the purpose (and consequently the 'cost') of the dynamical system. □

When an identification procedure generates a set of optimal models $\mathcal{M}^*$ within the original model set $\mathcal{M}$, it is favorable to consider the least complex models in $\mathcal{M}^*$ as the best (and cheapest) representation of the data. Therefore it is appropriate for identification purposes to consider model sets of models with equal complexity; if models should be compared with respect to their usefulness for representing the data sequence, it is fair to search between models with the same level of complexi-

ty. Note that in this respect, there exists a difference between the complexity measures as chosen in example 3.4.1. For $c^{(1)}$ and $c^{(2)}$ it is relatively simple to parametrize model sets of all models with a prespecified model complexity; considering $c^{(3)}$, it is hardly possible to construct a model set of all models with an a priori chosen fixed number of coefficients, without restricting the positions of these coefficients in any parametrization. Especially for black box models it is therefore more common to consider complexity measures as $c^{(1)}$ and $c^{(2)}$, and to construct a sequence of models sets $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdot \cdot$ where $\mathcal{M}_i$ contains all models with complexity $\leq i$.

A different notion of complexity that is relevant in an identification problem is the notion of complexity of a *model set*. This notion should be understood as a measure for the 'size' of a model set. Contrary to the notion of model complexity, referring to the costs of application of the model, the complexity of a model set refers to the costs that are related to the identification procedure itself. The larger the model set, the more 'expensive' it is to apply the identification criterion. Note that a model set that is parametrized with 2 parameters, is much easier to deal with than a model set that is parametrized with 36 parameters, although in view of the model complexity, the models in the first model set might be more complex.
Of course there can be a direct link between the two different complexities. In situations that a model complexity has been defined and a model set contains all models with a given complexity, the two notions actually coincide. The more complex the model set, the more complex the models that are contained. However in situations in which a different philosophy is followed the two notions can essentially differ. In order to illustrate this we consider the following example.

Example 3.4.2.
Consider the following two model sets:

$\mathcal{M}_1(n) := \{M \in \Sigma_{1,1} \mid M = \tilde{M}_p(T_1)$ with
$T_1(z,z^{-1}) = [1 + a_1 z^{-1} + \cdot + a_n z^{-n} \mid -b_1 z^{-1} - \cdot - b_n z^{-n}]$, $a_i, b_i \in \mathbb{R}$, $i \in \underline{n}$, and the two polynomials in T_1 being left coprime$\}$;

$\mathcal{M}_2(n) := \{M \in \Sigma_{1,1} \mid M = \tilde{M}_p(T_2)$ with
$T_2(z,z^{-1}) = [1 + a_1 z^{-1} + \cdot + a_n z^{-n} \mid -b_n z^{-n}]$, and $a_i, b_n \in \mathbb{R}$, $i \in \underline{n}\}$

for some $n \in \mathbb{N}$.
In terms of the complexity measures $c^{(1)}, c^{(2)}$ as presented in example 3.4.1, all

models in $\mathcal{M}_1(n)$ and $\mathcal{M}_2(n)$ have complexity n. Moreover note that all models within $\mathcal{M}_2(n)$ are also contained in $\mathcal{M}_1(n)$. In other words, for all $M_2 \in \mathcal{M}_2(n)$ there exists a $M_1 \in \mathcal{M}_1(n)$ such that $\mathcal{B}(M_1) \subset \mathcal{B}(M_2)$. Therefore it is appealing to consider $\mathcal{M}_1(n)$ to be more complex than $\mathcal{M}_2(n)$. As a measure for this complexity we can consider the minimal number of free parameters that is required for parametrizing the model set; $\mathcal{M}_1(n)$ is parametrized by 2n+1 parameters, and model set $\mathcal{M}_2(n)$ by n+1 parameters. □

The minimal number of parameters that is required for parametrizing a model set, can act as a measure of complexity of model sets. However this notion can only be defined properly for specific model sets, as will be formalized later on in this section 3.4.
We would like to point at the difference between complexity of models and complexity of model sets. Counting the number of independent and free parameters in a parametrization refers to the property of a model set, since numbers of parameters are connected with parametrizations, and parametrizations parametrize model sets and not models.

In section 3.3 a number of model sets has been presented already, within which one could distinguish an implicit notion of model complexity, such as e.g. certain polynomial degrees in polynomial matrix representations, or the degree of the minimal polynomial in a state space representation. In this section we want to focus on more general notions of model complexity that are not directly brought forward by one specific parametrization. In section 3.4.2 we will first elaborate on the concepts of McMillan degree and minimal state space dimension, leading to a generalized model order as a measure of complexity for general dynamical systems. A further specification of this model order is discussed by considering structure indices, required for the construction of model sets that can be parametrized by uniquely identifiable and differentiable parametrizations. Finally we will pay attention to the minimal number of parameters for model sets with prespecified sets of structure indices, and a link will be established with the notion of model complexity as introduced in Willems (1987).

3.4.2. Model order and structure indices

We will start this section by considering the McMillan degree of a rational matrix which is interpreted as the transfer function of a dynamical input–output system.

Definition 3.4.3. McMillan degree δ_M.

The McMillan degree of a rational matrix $H(z)\in\mathbb{R}^{pxm}(z)$ is defined as

$\delta_M(H)$:= the sum of the polar degrees at all poles of H(z). □

For finite poles this sum of the polar degrees can be determined by the sum of the degrees of the denominator polynomials in the Smith–McMillan form of H(z). The polar degree at infinity has to be added to arrive at the McMillan degree (see e.g. Kailath, 1980). There exists a close connection between the McMillan degree of a transfer function and the minimal state space dimension that is required for representing the corresponding input–output system. This is formulated in the following proposition.

Proposition 3.4.4.

Let M be an input–output dynamical system $M\in\Sigma_{p,m}$ that is controllable and causal, and let $H(z)\in\mathbb{R}^{pxm}(z)$ be the proper transfer function of M. Then

$\delta_M(H) = \delta(h_c)$ with $h_c\in\mathbb{R}[z,z^{-1}]$ the least common denominator of all minors of H;

= the minimal state space dimension of any realization (A,B,C,D) of M.□

The proposition is a classical result from the theory of dynamical systems, see e.g. Rosenbrock (1970) and Kailath (1980).

The McMillan degree is considered to be a measure of model complexity that is related to the transfer function of an i/o dynamical system. Therefore it takes account only of the controllable part of a dynamical system and it neglects any noncontrollability. The state space dimension of a minimal (A,B,C,D) state space representation is only defined for causal systems, related to proper transfer functions H(z). For this reason it is appropriate to consider a closely related measure of complexity that takes account of both noncontrollable as well as noncausal parts in a system. To this end we will use the following notion of model order, formulated in terms of the model behaviour.

Definition 3.4.5. Model order n(M).

Let $M=(\mathbb{T}, W, \mathcal{B})$ be a dynamical system $M\in\Sigma_{p,m}$. Then define the order of M as:

$$n(M) := \sum_{N=0}^{\infty} (\dim(\mathcal{B}^N) - \dim(\mathcal{B}^{N-1}) - m) \quad \text{with } \dim(\mathcal{B}^{-1}):=0. \qquad (3.4.1)$$

The expression (3.4.1) is proposed by Willems (1986a). The connection of n(M) with the McMillan degree and the minimal state space dimension is presented in theorem 3.4.7. For purpose of notation we first have to define the concepts of minor degrees in a polynomial matrix.

Definition 3.4.6. Minor degrees.
Let T be a polynomial matrix $T\in\mathbb{R}^{pxq}[z,z^{-1}]$, then denote with
$\pi_i^{(u)}(T)$:= the highest power of z over all i×i–minors of $T(z,z^{-1})$;
$\pi_i^{(\ell)}(T)$:= the lowest power of z over all i×i–minors of $T(z,z^{-1})$; □

Theorem 3.4.7.
Let M be an input–output system $M\in\Sigma_{p,m}$, with $M=\tilde{M}_p(T)$, T=[P|–Q] and $T\in\mathbb{R}^{px(p+m)}[z,z^{-1}]$. Then

a. $n(M) = \pi_p^{(u)}(T) - \pi_p^{(\ell)}(T)$; (3.4.2)

b. there exists a permutation matrix $X\in\mathbb{R}^{qxq}$ such that
   - $M^*=\tilde{M}_p(TX)$ is a causal i/o system with m inputs and p outputs, and
   - $M^*=\tilde{M}_s(A,B,C,D)$, with (A,B,C,D) a minimal realization with state space dimension $n=n(M)=n(M^*)$;

c. $n(M) \geq \delta_M(P^{-1}Q)$ with equality if and only if $\operatorname{rank}_{\mathbb{C}} T(\lambda,\lambda^{-1})=p$ for all $0\neq\lambda\in\mathbb{C}$, or equivalently M is controllable.

d. $n(M) \leq \sum_{i=1}^{p} \nu_i(T)$ with equality if and only if T is bilaterally row proper. □

Proof. Parts (b) and (d) are proved in Willems (1986a); the proofs of parts (a) and (c) are given in appendix A3.

Note that the model order n(M) is invariant for any shift in polynomial matrix T, i.e. for premultiplication of T with z or z^{-1}.
It follows from this theorem that n(M) is a generalization of both the McMillan degree of a transfer function and the minimal state space dimension of a state space representation. If M is causal then the permutation matrix X can be chosen the identity matrix and n(M)=n. If M is non–causal, the signal variables w can be reordered such that the reordered system M^* is causal and again allows a state space representation with minimal dimension n(M). The model order n(M) is invariant for any nonsingular constant column operation on T. Note that $\mathcal{B}(M^*)$ is

constructed according to $w\in\mathcal{B}(M^*) \Leftrightarrow Xw\in\mathcal{B}(M)$.
For some specific situations the expression (3.4.2) can be further simplified.

Proposition 3.4.8.
Let M and T be as in theorem 3.4.7. Then
$\pi_p^{(u)}(T) = \delta^{(u)}(\det(P))$ if and only if $P^{-1}Q$ has no poles in $z=\infty$;
$\pi_p^{(\ell)}(T) = \delta^{(\ell)}(\det(P))$ if and only if $P^{-1}Q$ has no poles in $z=0$. □

Proof. See appendix A4.

Note that the absence of poles in $z=\infty$ refers to the situation of a proper transfer function, and consequently to a causal system. The results of theorem 3.4.7 and proposition 3.4.8 contain a generalization of the results on the McMillan degree of MFD and ARMA models as reported in Janssen (1988a,1988b), where separate results were derived for models represented in one of the shift operators σ or σ^{-1}. A generalized model representation in both shift operators leads to an overall theory that covers both situations of (FDE)– and (BDE)–parametrizations, where σ and σ^{-1} are treated separately. The connections with the results of Janssen (1988a,1988b) are shown in the following corollary.

Corollary 3.4.9.
Let M be an input/output system $M\in\Sigma_{p,m}$, with

$$M=\tilde{M}_p(T_f), \quad T_f=T_f(z)=[P_f|-Q_f]\in\mathbb{R}^{p\times(p+m)}[z], \quad \text{and}$$
$$M=\tilde{M}_p(T_b), \quad T_b=T_b(z^{-1})=[P_b|-Q_b]\in\mathbb{R}^{p\times(p+m)}[z^{-1}].$$

a. If $\operatorname{rank}_{\mathbb{C}} T_f(\lambda)=p$ for $\lambda=0$ then $\pi_p^{(\ell)}(T_f) = 0$;
b. If $\operatorname{rank}_{\mathbb{C}} T_b(\lambda)=p$ for $\lambda=0$ then $\pi_p^{(u)}(T_b) = 0$;
c. If (P_f,Q_f) is left coprime with respect to $\mathbb{R}[z]$, and $P_f^{-1}Q_f$ has no poles in $z=\infty$ then $\delta_M(P_f^{-1}Q_f) = n(M) = \delta^{(u)}(\det(P_f))$;
d. If (P_b,Q_b) is left coprime with respect to $\mathbb{R}[z^{-1}]$ and $P_b^{-1}Q_b$ has no poles in $z=0$ then $\delta_M(P_b^{-1}Q_b) = n(M) = -\delta^{(\ell)}(\det(P_b))$. □

Proof. See appendix A5.

There is a certain symmetry in the expressions for BDE– and FDE–representations of models. However there are still some important differences. In order to be able

to extract the model order from the (autoregressive) parts P_b,P_f only, different restrictions on the transfer functions have to be satisfied. In the case of the FDE representation this restriction is the quite natural condition of causality of the system (the system should not contain anticipations). In the case of BDE representations the condition of H(z) having no poles in z=0 (the system should not contain any delays) is much more restrictive. This is exactly the reason why model sets of causal models with specified model order n(M) are much easier to parametrize in FDE–form than in BDE–form.

In view of the properties of corresponding parametrizations, specification of this model order n(M) is not sufficient as a basis for the construction of suitable model sets. It is a well known result that for multi output systems (p>1), it is not possible to construct one single identifiable and differentiable parametrization for all causal dynamical systems with prespecified value of n(M), see Hazewinkel and Kalman (1976). However it is possible to parametrize this model set by a finite number of differentiable and identifiable parametrizations. Two philosophies that can be followed in the construction of such parametrizations are reflected in the parametrizations based on canonical (nonoverlapping) forms (see e.g. Guidorzi, 1981) and the parametrizations based on pseudo–canonical (overlapping) forms (see e.g. Guidorzi and Beghelli, 1982; van Overbeek and Ljung, 1982; Gevers and Wertz, 1984; Corrêa and Glover, 1984a, 1984b). In both situations the model set $\mathcal{M}$ of all models with a fixed order n, is considered as the union of a finite number of model sets $\mathcal{M} = \bigcup_i \mathcal{M}_i$, where each model set $\mathcal{M}_i$ is characterized by a set of structure indices, and where for each model set $\mathcal{M}_i$ an identifiable parametrization can be constructed that is differentiable. The terms "canonical" (nonoverlapping) and "pseudo–canonical" (overlapping) straightforwardly refer to the situation that the model sets $\mathcal{M}_i$ are disjoint ($\mathcal{M}_i \cap \mathcal{M}_j$ is empty for $i \neq j$) or overlapping ($\mathcal{M}_i \cap \mathcal{M}_j$ is nonempty for $i \neq j$).
In order to arrive at these model sets $\mathcal{M}_i$, an additional decomposition of the model order n is applied, by splitting up n(M) as the sum of a number of structure indices. We will now take a closer look at these structure indices.

Definition 3.4.10. Observability indices $(\gamma_i)_{i=1,..,p}$.
Let (A,B,C,D) be a state space representation of a dynamical system $M \in \Sigma_{p,m}$, with $C = [c_1^T\ c_2^T\ \cdot\ \cdot\ c_p^T]^T$. Consider the observability matrix
$\mathcal{O} = [C^T\ A^TC^T\ \cdot\ \cdot\ (A^T)^{n-1}C^T]^T$ and evaluate the linearly independent rows in $\mathcal{O}$

from top to bottom. Order these independent rows as

$$\{c_1, c_1A, .., c_1A^{\gamma_1-1}, .., c_p, .., c_pA^{\gamma_p-1}\}$$

then the integers $(\gamma_i)_{i=1,..,p}$ are defined to be the *observability indices* of the realization (A,B,C,D). □

The notation $(\gamma_i)_{i=1,..,p}$ refers to a mapping $\mathbb{Z} \cap [1,p] \rightarrow \mathbb{Z}_+$, which means that the ordering of the observability indices is of importance, (e.g. $(2,1) \neq (1,2)$). When writing $\{\gamma_i\}_{i=1,..,p}$ we will refer to a set with p elements, where consequently any ordering of the elements is not relevant.

In the case of a minimal realization (A,B,C,D) it follows directly that

$$\sum_{i=1}^{p} \gamma_i = n(M). \tag{3.4.3}$$

Corresponding structure indices can also be defined on the basis of transfer functions. The definition of these – so called – left dynamical indices will be adopted from Forney (1975), Kailath (1980) and Janssen (1988a), and will not be further elaborated here. For more details the reader is referred to these references.

Definition 3.4.11. Left dynamical indices of H(z), $\{\kappa_i\}_{i=1,..,p}$.
The set of *left dynamical indices* $\{\kappa_i\}_{i=1,..,p}$ of a rational matrix $H(z) \in \mathbb{R}^{pxm}(z)$ with $\text{rank}_{\mathbb{R}(z)} H = p$, is defined as the set of row degrees of any minimal polynomial basis in $\mathbb{R}[z]$, for the rational vector space generated by the rows of the matrix $[I_p | H(z)]$. Minimality of this basis is considered with respect to the sum of the row degrees. □

The name "left dynamical indices" for the structure indices in definition 3.4.11 has been proposed by Janssen (1988a), where it is also proved that these left dynamical indices of H(z) sum up to its McMillan degree:

$$\sum_{i=1}^{p} \kappa_i = \delta_M(H). \tag{3.4.4}$$

Again, as in the case of the model order n(M), both sets of structure indices can be related to each other in terms of a polynomial matrix representation, in order to deal with both noncontrollable and noncausal dynamical systems.

Proposition 3.4.12.
Let M be an input–output system $M\in\Sigma_{p,m}$, with $M=\tilde{M}_p(T)$, $T=[P|-Q]$, $T\in\mathbb{R}^{p\times(p+m)}[z,z^{-1}]$, and T bilaterally row proper.

a. The set of row degrees $\{\nu_i(T)\}_{i=1,..p}$ is invariant under premultiplication of T with a polynomial matrix that is unimodular with respect to $\mathbb{R}[z,z^{-1}]$;
b. The set of row degrees $\{\nu_i(T)\}_{i=1,..p}$ is equal to the set of left dynamical indices $\{\kappa_i\}_{i=1,..p}$ of $P^{-1}Q$, if and only if (P,Q) is left coprime with respect to $\mathbb{R}[z,z^{-1}]$;
c. The set of row degrees $\{\nu_i(T)\}_{i=1,..p}$ is equal to the set of observability indices $\{\gamma_i\}_{i=1,..p}$ of any minimal state space realization (A,B,C,D) that satisfies $\tilde{M}_s(A,B,C,D) = \tilde{M}_p(TX)$ and $X\in\mathbb{R}^{q\times q}$ a permutation matrix. □

Proof. See appendix A6.

Part a of the proposition actually shows that the set of row degrees of T is an invariant of the system M.
Note that equality as meant in parts b and c of the proposition, refers to the sets of indices, i.e. equality of the indices up to ordering.
As shown in proposition 3.4.12 the row degrees of any full row rank and bilaterally row proper polynomial matrix act as a further structural specification of dynamical systems with a model order $n(M)=\sum_{i=1}^{p}\nu_i(T)$. Consequently we can formalize this set of row degrees as a specific set for a given input–output dynamical system.

Definition 3.4.13. Left structure indices of M; $\{\rho_i\}_{i=1,..p}$.
Let M be an input–output system $M\in\Sigma_{p,m}$, and let $T\in\mathbb{R}^{p\times(p+m)}[z,z^{-1}]$.
The set of *left structure indices* of M, denoted by $\{\rho_i\}_{i=1,..p}$, is defined as the set of row degrees $\{\nu_i(T)\}_{i=1,..p}$ of any bilaterally row proper matrix T that satisfies $M=\tilde{M}_p(T)$. □

As mentioned before, structure indices can generate a decomposition of the model set of all models with a prespecified model order. It will be discussed in chapter 4 that the structure indices that we have dealt with so far, are related to the construction of such a decomposition in disjoint parts (sets of canonical forms).

These structure indices are related to row degrees in a corresponding polynomial matrix representation. Under some additional conditions, the column degrees of such polynomial matrices can also be given an interpretation of structure indices, referring to a decomposition of the above mentioned model set in overlapping parts. This is formulated in the following proposition.

Proposition 3.4.14.
Let M be an input–output system $M\in\Sigma_{p,m}$, with $M=\tilde{M}_p(T)$, $T=[P|-Q]$ and $T\in\mathbb{R}^{px(p+m)}[z]$.
If M is a causal system, then $n(M) \leq \sum_{i=1}^{p} \mu_i^{(u)}(P)$ with equality if P is column proper with respect to $\mathbb{R}[z]$, and $\text{rank}_{\mathbb{C}}[P(\lambda)|-Q(\lambda)]=p$ for $\lambda=0$. □

Proof. This proposition follows from the proofs of theorem 3.4.7a, proposition 3.4.8, and corollary 3.4.9a. □

Under the conditions mentioned in the proposition, the column degrees of the matrix P also sum up to the order n(M), just as in the case of the left structure indices of M, being determined by the row degrees of T. Model sets that are based on specific sets of these column degrees will also be considered in the sequel of this thesis.

3.4.3. Number of parameters and complexity

When we are dealing with model sets that contain all models with a prespecified model order, it is very simple to order the different model sets according to their complexity. The higher the order of the models in the set, the higher the model complexity and consequently the higher the complexity of the model set.
However the question can be asked how to order model sets of models with prespecified structure indices summing up to a fixed model order n(M), as illustrated in the following example.

Example 3.4.15.
Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two sets of input–output dynamical systems, defined by:
$\mathcal{M}_1 := \{M\in\Sigma_{2,2} |$ M has a minimal realization with a set of observability indices

$(\gamma_1,\gamma_2) = (3,1)\}$;

$\mathcal{M}_2$:= $\{M\in\Sigma_{2,2}|$ M has a minimal realization with a set of observability indices

$(\gamma_1,\gamma_2) = (2,2)\}$;

Both model sets only contain models of the same model order n(M)=4. However the two model sets are clearly different. Which model set is more complex? □

We will formulate an answer to this question by evaluating the minimal number of parameters that is required for a parametrization of the model sets. However in order to be able to deal with a minimal number of parameters, a model set has to satisfy additional constraints. In general terms, the notion of a minimal number of parameters is not a properly defined measure. However the model sets that we will be dealing with will appear to satisfy these additional constraints.

Definition 3.4.16. Minimal number of parameters; $\tau(\mathcal{M})$.
Let $\mathcal{M}$ be a set of input–output dynamical systems, $\mathcal{M}\subset\Sigma$, such that $\mathcal{M}$ is homeomorphic[1] with an open and dense subset of $\mathbb{R}^\tau$, with $\tau=\tau(\mathcal{M})\in\mathbb{N}$. Then
$\tau(\mathcal{M})$:= the *minimal number of parameters* that is required for parametrization of $\mathcal{M}$. □

If $\mathcal{M}$ satisfies the conditions as formulated in the definition, then there exist $\tau(\mathcal{M})$ parameters that are independent from each other, and that generically can be given arbitrary values. Note that all model sets that are based on (almost) bijective parametrizations in which the parameters can be given arbitrary real values, independently from each other, automatically meet the requirements as formulated in definition 3.4.16.

We will now consider the minimal number of parameters that is required for the parametrization of a model set that is specified by the observability indices $(\gamma_i)_{i=1,..,p}$. An implicit expression for this number of parameters is known in the literature. In this thesis an explicit expression will be formulated.

Theorem 3.4.17.
Consider the set of input–output dynamical systems defined by:
$\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}$:= $\{M\in\Sigma_{p,m}|$ $M=\tilde{M}_s(A,B,C,D)$, with (A,B,C,D) a minimal realization

[1]Homeomorphic means that there exists a homeomorphism, i.e. a continuous bijective mapping whose inverse is continuous.

with observability indices $(\gamma_i)_{i=1,..,p}\}$. (3.4.5)

Then $\tau(\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}) = 2\sum_{k=1}^{p} k\gamma'_k + (m-1)n + pm + \frac{1}{2}(p^2 - \sum_{i=1}^{d} t_i^2) - s$ (3.4.6)

with: $(\gamma'_k)_{k=1,..,p}$ the ordered set of observability indices; $\{\gamma'_i\}_{i=1,..,p} = \{\gamma_i\}_{i=1,..,p}$, with $\gamma'_1 \geq \gamma'_2 \geq \cdot\cdot \geq \gamma'_p$.

$n \in \mathbb{Z}_+$ is the model order of the models in $\mathcal{M}$, $n = \sum_{i=1}^{p} \gamma_i$;

$d \in \underline{p}$ is the number of distinct values in $\{\gamma_i\}_{i=1,..,p}$;

t_i, $i \in \underline{d}$ is the number of appearances of each of these distinct values in $\{\gamma_i\}_{i=1,..,p}$;

$s \in \mathbb{Z}_+$ is the minimal number of permutations of adjacent observability indices γ_k, γ_{k+1} in $(\gamma_i)_{i=1,..,p}$ for $k \in \underline{p-1}$, that is required for creating the ordered set of indices $(\gamma'_i)_{i=1,..,p}$. □

Proof. See appendix A7.

Theorem 3.4.17 presents a new and (more or less) explicit expression for the minimal number of parameters that is required for parametrizing a model set based on observability indices. Given a set of observability indices, the number of parameters is maximal if the set is ordered according to $\gamma_1 \geq \gamma_2 \geq \cdot\cdot \geq \gamma_p$, and minimal for a set that is ordered in reverse direction. In the situation $\gamma_1 = \gamma_2 = \cdot\cdot = \gamma_p$, it can be verified that expression (3.4.6) yields the well known result:

$$\tau(\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}) = n(p+m) + pm \qquad (3.4.7)$$

which equals the maximum number of coefficients that may be required for representing a dynamical system with order n.
In order to illustrate expression (3.4.6) the following example will be presented.

Example 3.4.18.
Consider the model set $\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}$ with m=p=3, γ_1=3, γ_2=1, γ_3=6.
Now considering equation (3.4.6), there are two terms that are dependent on m: mn+mp. In this example n=10 and mn+mp = 39. This number of parameters refers to the parameters required in the matrices B (=mn) and D (=mp) in a canonical observability form parametrization of the considered model set.

The remaining number of parameters:

$$2 \sum_{k=1}^{p} k\gamma_k' - n + \frac{1}{2}(p^2 - \sum_{i=1}^{d} t_i^2) - s \qquad (3.4.8)$$

refers to the parameters required in the matrix A of the parametrization mentioned above.

In this example $\gamma_1'=6$, $\gamma_2'=3$, $\gamma_3'=1$; $t_1=t_2=t_3=1$, and $s=2$ since it takes two permutations within the sequence of observability indices to arrive at an ordered set: $(3,1,6) \rightarrow (3,6,1) \rightarrow (6,3,1)$.

Substituting these numbers in equation (3.4.8) gives $2\times15-10+\frac{1}{2}(9-3)-2 = 21$.

Together with the previously obtained 39 parameters, this leads to a total of 60. □

Comparing the number of parameters that is required for parametrizing model sets $\mathcal{M}_1$, $\mathcal{M}_2 \subset \Sigma_{2,2}$ with observability indices (3,1) and (2,2), as formulated in example 3.4.15, it follows that $\tau(\mathcal{M}_1) = 19$, and $\tau(\mathcal{M}_2) = 20$. Apparently, in view of the minimal number of parameters, model sets are more complex if the observability indices are closer together.

A different approach to deal with the problem of ordering model sets with models having the same model order, is presented in the work of Willems (1987) and Heij (1988), where use is made of a notion of complexity of a dynamical system, that is able to distinguish between dynamical systems having the same model order. We will briefly present this notion.

Definition 3.4.19. Complexity of a dynamical system $M\in\Sigma$.

Let $M=(\mathbb{I}, W, \mathcal{B})$ be a dynamical system $M\in\Sigma$ with $W=\mathbb{R}^q$. Then the complexity of M is defined as a mapping $c:\Sigma\rightarrow(\mathbb{R}^+)^{\mathbb{I}_+}$, with $c(M):=(c_t(\mathcal{B}(M), t\in\mathbb{I}_+)$ and

$$c_t(\mathcal{B}(M)) := \frac{1}{t+1} \dim \mathcal{B}^t. \qquad (3.4.9)$$ □

Note that $\mathcal{B}^t := \mathcal{B}|_{\mathbb{I}\cap[0,t]}$.

Complexity of a dynamical system $M\in\Sigma$ has been defined as a sequence of real numbers. It can be shown that for sufficiently large values of $t\in\mathbb{I}_+$,

$c_t(\mathcal{B}(M)) = m + \frac{n}{t+1}$, with m the number of inputs, and n=n(M), the model order.

This measure of complexity will be illustrated later on in more detail; first we will add a rule of partial ordering of c(M).

Definition 3.4.20. Partial ordering of c(M); (Willems, 1987).
Let M_1, $M_2 \in \Sigma$ be two dynamical systems. Define a partial ordering of the corresponding complexities $c(M_1)$, $c(M_2)$ by:

$\{c(M_1) \geq c(M_2)\}$ if and only if $\{c_t(\mathcal{B}(M_1)) \geq c_t(\mathcal{B}(M_2))$, for all $t \in \mathbb{Z}_+\}$. □

Since this ordering is partial, there exist situations that neither $c(M_1) \geq c(M_2)$, nor $c(M_1) \leq c(M_2)$ holds true. In Willems (1987) two suggestions are presented for making this partial ordering a total one; however in this thesis we will restrict attention to the ordering as defined in definition 3.4.20.
In view of this partial ordering three levels of decision can be distinguished in the ordering of complexities of two models.
For two dynamical systems $M_1 \in \Sigma_{p_1,m_1}$, $M_2 \in \Sigma_{p_2,m_2}$, it follows that (Willems, 1987):

1. If $c(M_1) \geq c(M_2)$ then $m_1 \geq m_2$.
2. If $c(M_1) \geq c(M_2)$ and $m_1 = m_2$ then $n(M_1) \geq n(M_2)$.
3. If $m_1 = m_2$ and $n(M_1) = n(M_2)$ then the ordering of complexity is completely determined by the sets of left structure indices of M_1 and M_2.

In this thesis we will always consider models with a prespecified number of inputs and outputs, and consequently situations 2 and 3 show that the complexity c(M) acts as a further specification of the model order n(M). Further analysis of situation 3, as described above, shows the following result.

Proposition 3.4.21.
Let $M_1, M_2 \in \Sigma_{p,m}$ with left structure indices $\{\rho_i^{(1)}\}_{i=1,.p}$ and $\{\rho_i^{(2)}\}_{i=1,.p}$.
Consider the partial ordering of model complexities $c(M_1)$, $c(M_2)$ as defined in definition 3.4.20, then

$$\{c(M_1) \geq c(M_2)\} \text{ if and only if } \{\sum_{i=1}^{p} \min(\rho_i^{(1)}, t) \geq \sum_{i=1}^{p} \min(\rho_i^{(2)}, t), \text{ for all } t \in \mathbb{Z}_+\}. \quad (3.4.10)$$ □

Proof see Willems (1987).

Note that the complexity c(M) is evaluated on the basis of the set of left structure indices of M. Knowing the left structure indices of the models, there is a simple rule for evaluating the ordering of the complexities. A consequence of this is, that it does not make any difference if these structure indices are reordered. This is in

contrast with the expression for the number of parameters of model sets with prespecified observability indices, where a reordering of indices indeed makes a difference. Now let us return to the situation of example 3.4.15.

Example 3.4.22.
Consider the two model sets of example 3.4.15 and consider two models $M_1 \in \mathscr{M}_1$ with observability indices $\gamma_1^{(1)}=3$, $\gamma_2^{(1)}=1$, and $M_2 \in \mathscr{M}_2$ with observability indices $\gamma_1^{(2)}=2$, $\gamma_2^{(2)}=2$. For the considered models $\{\gamma_1^{(i)},\gamma_2^{(i)}\} = \{\rho_1^{(i)},\rho_2^{(i)}\}$, i=1,2, (see proposition 3.4.12c), we can evaluate the complexities of the models on the basis of these observability indices. Evaluation of the expression on the right hand side of (3.4.10) gives the following result:

t	0	1	2	3	4	5	6
M_1	0	2	3	4	4	4	4
M_2	0	2	4	4	4	4	4

TABLE 3.3 Expression $\sum_{i=1}^{p} \min(\rho_i,t)$ for t=0,.,6, for two models M_1, M_2 with $\rho(M_1)=\{3,1\}$, $\rho(M_2)=\{2,2\}$.

This leads to the conclusion that $c(M_1) \leq c(M_2)$. □

The example shows a result that is similar to the situation of considering the minimal number of parameters of the model sets. In this current situation models are more complex when the left structure indices are closer together. In other words: the more 'generic' a system, the more complex it is. It has been stressed in Willems (1987) and Heij (1988) that this measure of complexity is not based on any number of parameters, but only on dimensions of trajectory spaces.
However the impression exists that there is a close relationship between the model complexity c(M) of a dynamical system with left structure indices $\{\rho_i(M)\}_{i=1,.,p}$ and the minimal number of parameters $\tau(\mathscr{M}_{p,m,\gamma_1,.,\gamma_p})$, with $\{\rho_i\}_{i=1,.,p}=\{\gamma_i\}_{i=1,.,p}$,

and $\gamma_1 \geq \gamma_2 \geq \cdot \geq \gamma_p$. The ordering of these observability indices can be made feasible by considering that in the expression for the model complexity, the ordering of the left structure indices is not relevant. This means that all possible orderings of the corresponding observability indices have to be taken into account. Since the ordering of observability indices in decreasing order generates the maximum number of required parameters, this worst–case situation has to be taken as a starting point. In the following example a possible relation between model complexity and minimal number of parameters is illustrated.

Example 3.4.23.

Consider the class of models with 3 outputs, 3 inputs and model order n(M)=9, and with all models M satisfying: $\rho_i(M) \geq 1$ for $i \in \underline{p}$, and $\sum_{i=1}^{p} \rho_i(M) = 9$.

In the following table the expression $\sum_{i=1}^{p} \min(\rho_i, t)$ is evaluated for the different possible sets of structure indices, in order to evaluate the model complexity.
A comparison is made with the minimal number of parameters required, as reflected by $\tau(\mathcal{M}_{p,m,\gamma_1,.,\gamma_p})$ with $\{\rho_i\}_{i=1,.,p} = \{\gamma_i\}_{i=1,.,p}$, and $\gamma_1 \geq \gamma_2 \geq \cdot \geq \gamma_p$.
It follows from table 3.4 that $c(M_1) \leq c(M_2) \leq c(M_3) \leq c(M_4)$, and $c(M_5) \leq c(M_6) \leq c(M_7)$, while $c(M_4) \not\leq c(M_5)$ and $c(M_5) \not\leq c(M_4)$. If we evaluate the complexity by the minimal number of parameters, we come to the same conclusion, taking notice of the fact that for M_4 and M_5 the corresponding number of parameters is equal. □

However the relation between c(M) and $\tau(\mathcal{M})$ as suggested above is not as close as might be expected from the previous example. This is shown in the following counter example.

Example 3.4.24.

Consider two models M_1, $M_2 \in \Sigma_{5,5}$, having 5 inputs and 5 outputs, and model order $n(M_1)=n(M_2)=20$, and with left dynamical indices $\rho(M_1)=\{7,5,4,3,1\}$ and $\rho(M_2)=\{8,3,3,3,3\}$. For both models the expression $\sum_{i=1}^{5} \min(\rho_i, t)$ will be evaluated, in order to determine which of the two models is more complex than the other. Table 3.5 shows that neither $c(M_1) \leq c(M_2)$ nor $c(M_2) \leq c(M_1)$. If we evaluate the number of parameters that is required to parametrize model sets with observability

t	1	2	3	4	5	6	7	8	τ
$\rho(M_1)=\{7,1,1\}$	3	4	5	6	7	8	9	9	53
$\rho(M_2)=\{6,2,1\}$	3	5	6	7	8	9	9	9	56
$\rho(M_3)=\{5,3,1\}$	3	5	7	8	9	9	9	9	58
$\rho(M_4)=\{5,2,2\}$	3	6	7	8	9	9	9	9	59
$\rho(M_5)=\{4,4,1\}$	3	5	7	9	9	9	9	9	59
$\rho(M_6)=\{4,3,2\}$	3	6	8	9	9	9	9	9	62
$\rho(M_7)=\{3,3,3\}$	3	6	9	9	9	9	9	9	63

TABLE 3.4. Expression $\sum_{i=1}^{p} \min(\rho_i,t)$ for t=1,.,8, for seven models M_1, ., M_7 with $n(M_j)=9$ for all j=1,.,7; minimal number of parameters $\tau(\mathcal{M}_{p,m,\gamma_1,.,\gamma_p})$ with $\{\gamma_i\}_{i=1,.,p}=\{\rho_i\}_{i=1,.,p}$ and $\gamma_1 \geq \cdot \geq \gamma_p$.

t	1	2	3	4	5	6	7	8	9	τ
M_1	5	9	13	16	18	19	20	20	20	207
M_2	5	10	15	16	17	18	19	20	20	209

TABLE 3.5. Expression $\sum_{i=1}^{5} \min(\rho_i,t)$ for t=1,.,9, for two models M_1,M_2 with $\rho(M_1)=(7,5,4,3,1)$, $\rho(M_2)=(8,3,3,3,3)$; minimal number of parameters $\tau(\mathcal{M}_{p,m,\gamma_1,.,\gamma_p})$ with $(\gamma_i)_{i=1,.,5} = (\rho_i)_{i=1,.,5}$.

indices (7,5,4,3,1) ($\mathcal{M}_1$) or (8,3,3,3,3) ($\mathcal{M}_2$), it follows that $\tau(\mathcal{M}_1)=207$ and $\tau(\mathcal{M}_2)=209$. Apparently there is a distinction in the number of parameters τ, whereas the complexity c(M) does not distinguish between the two models. □

Example 3.4.24 shows that evaluation of the complexity of a set of models with prespecified model order n(M), by using the complexity measure c(M) or alternatively by using the minimal number of parameters $\tau(\mathcal{M}_{p,m,\gamma_1,..,\gamma_p})$, with $\mathcal{M}$ as defined in (3.4.5), can lead to different results.
With respect to this item, the following conjecture will be stated.

Conjecture 3.4.25.
Consider two dynamical systems M_1, $M_2 \in \Sigma_{p,m}$. Let M_1, M_2 have left structure indices $\{\rho_i^{(1)}\}_{i=1,..,p}$, $\{\rho_i^{(2)}\}_{i=1,..,p}$, such that $\sum_{i=1}^{p} \rho_i^{(1)} = \sum_{i=1}^{p} \rho_i^{(2)} = n$, and denote the ordered left structure indices by:

$$\{\eta_i^{(1)}\}_{i=1,..,p} = \{\rho_i^{(1)}\}_{i=1,..,p}, \text{ with } \eta_1^{(1)} \geq \eta_2^{(1)} \geq \cdot\cdot \geq \eta_p^{(1)}, \text{ and}$$
$$\{\eta_i^{(2)}\}_{i=1,..,p} = \{\rho_i^{(2)}\}_{i=1,..,p}, \text{ with } \eta_1^{(2)} \geq \eta_2^{(2)} \geq \cdot\cdot \geq \eta_p^{(2)}.$$

a. If $c(M_1) \leq c(M_2)$ then $\tau(\mathcal{M}_{p,m,\eta_1^{(1)},..,\eta_p^{(1)}}) \leq \tau(\mathcal{M}_{p,m,\eta_1^{(2)},..,\eta_p^{(2)}})$;

b. If $\tau(\mathcal{M}_{p,m,\eta_1^{(1)},..,\eta_p^{(1)}}) \leq \tau(\mathcal{M}_{p,m,\eta_1^{(2)},..,\eta_p^{(2)}})$ then
   - $c(M_1) \leq c(M_2)$ or
   - $c(M_1) \not\leq c(M_2)$ and $c(M_2) \not\leq c(M_1)$; □

Apparently a decision for one model M_2 to be more complex than M_1, in terms of c(M), where M_1 and M_2 have the same model order, is always confirmed by the corresponding number of parameters $\tau(\mathcal{M})$ as defined before. On the other hand if c(M) can not decide for an ordering of complexities between M_1 and M_2, $\tau(\mathcal{M})$ possibly can. As such this measure $\tau(\mathcal{M})$ can extend the partial ordering of complexities of models and model sets to a total ordering.

3.5 IDENTIFICATION CRITERIA

In section 2.3 we have presented a very general definition of an identification criterion, considering the criterion as a selection rule that determines optimal models

out of a model set. This general definition allows a unifying approach to a great number of system identification methods. However in this thesis we will mainly restrict attention to the use of the most straightforward choice of an identification criterion, based on a (least squares) residual function. The restriction to this situation should not be interpreted as an uncritical support of these methods, but merely as an investigation into methods that are the most popular.
The following identification criterion will be considered:

$$J^N(v^N,\mathcal{M}) = \arg\min_{M\in\mathcal{M}} \min_{e^N\in\mathcal{E}^N(v^N,M)} \ell^N_{LS}(e^N) \tag{3.5.1}$$

with ℓ^N_{LS} a residual function $\ell^N_{LS}: E^{T^N} \to \mathbb{R}\cup\{\infty\}$, being specified by

a. $$\ell^N_{LS2} = \sum_{t=0}^{N-1} e^T(t)e(t), \tag{3.5.2}$$

for signals in the Hilbert space l_2, having a finite energy for $N\to\infty$, or

b. $$\ell^N_{LS\infty} = \frac{1}{N}\sum_{t=0}^{N-1} e^T(t)e(t), \tag{3.5.3}$$

for signals that have an infinite energy for $N\to\infty$, but that have a finite power, like stochastic signals with a constant variance.

For finite values of N, both expressions (3.5.2), (3.5.3) will lead to the same identification result $J^N(v^N,\mathcal{M})$.

Since we are dealing with (i/o/pr)-models, the set of admissible residual signals $\mathcal{E}^N(v^N,M)$ for a given model M, as in eq. (2.2.12), can be uniquely decomposed by:

$$\mathcal{E}^N(v^N,M) = \mathcal{E}^N_{obs}(v^N,M) \oplus \mathcal{E}^N_{init}(v^N,M) \tag{3.5.4}$$

where $\mathcal{E}^N_{obs}(v^N,M)$ contains exactly one element, representing that part of $\mathcal{E}^N(v^N,M)$ that is compatible with v^N in M under zero initial conditions, and $\mathcal{E}^N_{init}(v^N,M)$, representing that part of $\mathcal{E}^N(v^N,M)$ that is caused by initial conditions, i.e. elements of v(t) for t outside the interval $T_N=\mathbb{Z}\cap[0,N-1]$.
Restricting the minimization in (3.5.1) to the set $\mathcal{E}^N_{obs}(v^N,M)$, comes down to the identification of models under the restriction that initial conditions are fixed to zero, as formulated in section 2.3.
For notational convenience the following expressions will be used:

$$J^{+}_{LS\infty}(v^{+},\mathcal{M}) := \arg\min_{M\in\mathcal{M}} \min_{e^{+}\in\mathcal{E}^{+}(v^{+},M)} \ell^{+}_{LS\infty}(e^{+}) \tag{3.5.5}$$

$$\text{with} \qquad \ell^{+}_{LS\infty}(e^{+}) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e^{T}(t)e(t). \tag{3.5.6}$$

The minimization of $\ell^{+}_{LS\infty}(e^{+})$ as formulated in equation (3.5.6) should be interpreted in such a way that all residuals $e^{+}\in\mathcal{E}^{+}(v^{+},M)$ are considered such that the limit in (3.5.6) indeed exists. A similar expression can be used for J^{+}_{LS2}.

As mentioned already in chapter 2, it is quite straightforward that these identification criteria, applied to (i/o/r) model sets, lead to a situation in which the model sets are completely selectable and regularly partitionable by the identification criterion, as discussed in section 2.3.

In the literature some identification criteria are known that apparently are extensions of the least squares criteria as presented above. We can think of criteria that use weighted residuals e' in the residual function $\ell^{N}_{LS}(e')$ with $e'=\Omega e$ and $\Omega\in\mathbb{R}^{p\times p}$, Ω nonsingular, or even a filtered residual e'' being the output of a chosen i/o dynamical system with e as input (see e.g. Ljung, 1987). It has to be noted that all such identification methods can be represented in the form (3.5.1) by a proper adjustment of the model set $\mathcal{M}$ that is taken into account.

Least squares based identification methods actually can be interpreted to have a statistical background. Under the additional (but very often unrealistic) assumption that a modelling error e is a realization of a zero mean Gaussian white noise process with variance $\sigma^2 I$, it comes down to a maximum likelihood estimator. However also without any statistical assumptions, a sum of squared residuals can act as a measure of approximation, and consequently as an indication of the quality of the model with respect to the representation of a data sequence.
In contrast with this, identification by instrumental variable methods actually is brought forward by statistical assumptions on the data. Consequently it is much harder to give it an interpretation in terms of approximation without these statistical assumptions. For purpose of reference, we will formulate a basic instrumental variable identification criterion in the context of this thesis.

A basic instrumental variable identification criterion J_{IV} is specified by:

$$\hat{M}\in J_{IV}^{N}(v^N,\mathcal{M}) \Leftrightarrow \{e^N\in\mathcal{E}^N(v^N,\hat{M});\ \frac{1}{N}\sum_{t=0}^{N-1} z_i(t)e_i(t)=0,\ \text{with } z_i\in(\mathbb{R}^{n_i})^{\mathbb{Z}},\ i\in\underline{p},\ n_i\in\mathbb{Z}_+\}. \tag{3.5.7}$$

The signals z_i are called the instrumental variables; they are external signals that have to satisfy some additional restrictions in order to guarantee that there indeed exists a solution $\hat{M}\in J^N(v^N,\mathcal{M})$. These restrictions are both with respect to the properties of the signals z_i and with respect to their number n_i.
In basic instrumental variable methods, the instrumental variables z_i are chosen to be delayed or filtered input signals u. An extensive treatment of IV–methods can be found in Söderström and Stoica (1983).

Without any further comment we will present some other identification criteria that can simply be formulated in our approach, but that will not be elaborated upon in this thesis:

a. Identification methods based on maximum tolerated residual function.
Let $\mathcal{M}$ be a set of (i/o/r)–models; let an identification criterion be defined by:
$J^N(v^N,\mathcal{M}) = \{\hat{M}\in\mathcal{M}|\ \exists e^N\in\mathcal{E}^N(v^N,\hat{M}),\ \ell^N(e^N)\le a,$ with a a prespecified maximum tolerated level of the residual function $\ell^N\}$. (3.5.8)

b. Total least squares identification (see e.g. Golub and Van Loan, 1980; Van Huffel, 1987).
Let $\mathcal{M}$ be a set of (i/o/r)–models with $E=\mathbb{R}^{p+m}$; then
$$J_{TLS}^{N}(v^N,\mathcal{M}) = \arg\min_{M\in\mathcal{M}} \ell_{LS}^{N}(e^N) \text{ with } e^N\in\mathcal{E}^N(v^N,M) \text{ and } (v^N-e^N)\in\mathcal{B}^{io,N}(M). \tag{3.5.9}$$

c. Frisch scheme (see e.g. Kalman, 1982; de Moor, 1988; Beghelli, Guidorzi and Soverini, 1988).
Let $\mathcal{M}$ be a set of (i/o/r)–models with $E=\mathbb{R}^{p+m}$; then
$J_F^N(v^N,\mathcal{M}) = \{\hat{M}\in\mathcal{M}|\ \exists e^N\in\mathcal{E}^N(v^N,\hat{M})$ with $\frac{1}{N}\sum_{t=0}^{N-1} e(t)e^T(t)=\text{diag}(s_1^2,\ .\ ,s_{p+m}^2)$, $s_i\in\mathbb{R}$, $i\in\underline{p+m}$, such that $\frac{1}{N}\sum_{t=0}^{N-1} e(t)\hat{v}^T(t)=0$ for $\hat{v}^N=v^N-e^N$, and $\hat{v}^N\in\mathcal{B}^N(M^{io})\}$. (3.5.10)

Note that in the criteria b and c the number of residuals equals the total number of inputs and outputs, which necessitates a wider model class than the class of

(i/o/pr)–models as generally used in this thesis.
For an alternative and slightly different approach to the construction of identification criteria for descriptive and predictive modelling, the reader is referred to Heij (1988).

3.6 DISCUSSION

In this chapter an overview has been given of model sets and identification criteria that can be used for residual–based identification. General characteristics of models have been presented in terms of their residual signals, leading to a classification in k–step ahead prediction error (PE) models, output error (OE) models and equation error (EE) models. Each of these types of models reflects a specific way of connecting a residual signal to the system's inputs and outputs u and y in the model behaviour. Correspondingly, each type of model represents a specific way of dealing with modelling errors. A choice for one type of model determines the "location" in the model where deviations between model description and measured data are discounted. These residual properties of models have been defined on the level of model behaviour, which makes it possible to evaluate them in different representations of models, as in polynomial matrix, state space or transfer function representations.
Several choices of model sets that are commonly proposed in the literature, are positioned in the presented framework. The complexity of models has been considered, which has led to the introduction of a model order that is a generalization of both the classical concepts of McMillan degree and minimal state space dimension. This model order is defined both for models that are not controllable and models that are not causal. This approach is extended to corresponding structure indices. By exploiting the advantages of polynomial matrix representations in the two shift operators σ and σ^{-1}, it has been possible to combine the separate theories on the McMillan degree and the structure indices of models represented in an FDE–form (in σ) or in a BDE–form (in σ^{-1}), into a generalized theory.
The notion of model complexity, as introduced by Willems (1987), has been considered, and a comparison has been made with the minimal number of parameters that is required for parametrizing model sets with prescribed observability indices. Using either one of these two concepts as a measure of model complexity, it is possible to order model sets of models with a fixed model order but with different sets of structure indices.
Identification criteria have been slightly passed in review, and it is shown that our

general notion of an identification criterion encompasses a lot of existing identification techniques, from function minimization and instrumental variable techniques, to identification of errors–in–variables models in the Frisch scheme.

4. ON DISCRIMINABILITY AND IDENTIFIABLE PARAMETRIZATIONS

4.1 INTRODUCTION

Before we come to a discussion on identifiable parametrizations for the model sets as described in section 3.2, attention has to be paid to the problem of discriminability of model sets. As defined in section 2.3 a model set is called discriminable by an identification criterion J^N if the identification criterion J^N can distinguish between the different models in the model set.
For model sets that are not discriminable by the identification criterion, it is not possible to construct uniquely identifiable parametrizations. In view of this, discriminability is a prerequisite for identifiability.
It has been illustrated in example 2.4.13 that (i/o/pr)–model sets without any additional restriction are generally not discriminable by a standard least squares identification criterion. This is due to the fact that there is a freedom of scaling in the residual signal. Moreover in the example mentioned, this lack of discriminability of $\mathcal{M}$ coincides with the fact that $V_{J,\mathcal{M}}^{T,N} = \bigcup_{M \in \mathcal{M}} \mathcal{B}^N(M^{io})$, i.e. $J^N(v^N,\mathcal{M})$ only leads to selected optimal models for data sequences that have been generated by the input–output part of one of the models in the set itself (data sequences that have been generated with residual zero)! Consequently $J^N(v^N,\mathcal{M})$ will generally be an empty set. It has to be stressed that this situation is very undesirable, since the identification criterion only offers the possibility of exact modelling, and not of approximate modelling.
Therefore discriminability of model sets is not only required for arriving at identifiable parametrizations, but it also important in order to guarantee that one is dealing with a nontrivial identification problem. Both items of discriminability and identifiability will be considered in this chapter. We will restrict attention to the asymptotic least squares identification criterion, and the model sets that will be considered are the different types of model sets as discussed in chapter three.
The organization of this chapter is schematically represented in figure 4.1.
The representation within this figure is in accordance with the definition of an identifiable parametrization, as formulated in definition 2.4.12.
When discussing the three different topics (a), (b) and (c), as denoted in the figure, the different types of model sets, (PE), (OE) and (EE), will be separately paid attention to.

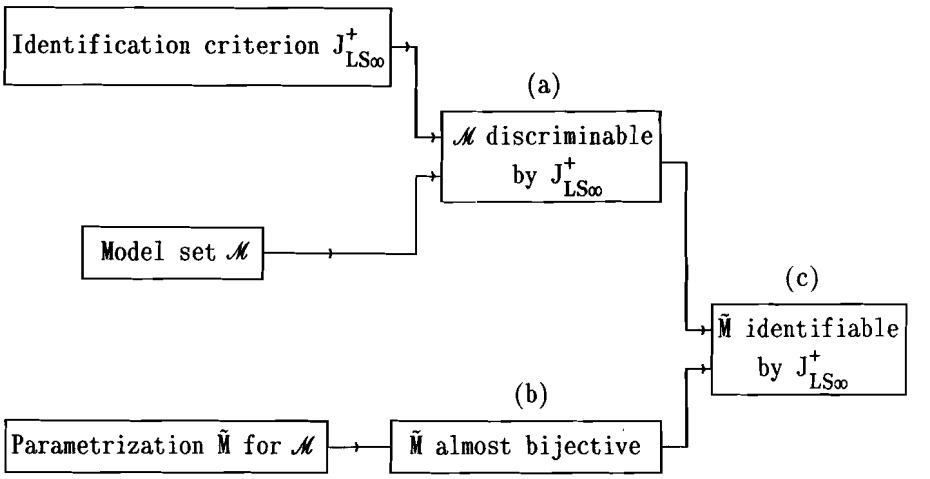

FIGURE 4.1 Schematic representation of the requirements for a parametrization $\tilde{M}$ of $\mathcal{M}$ to be identifiable by $J^+_{LS\infty}$.

In section 4.2 it will be discussed how to construct model sets that are discriminable by $J^+_{LS\infty}$ (part (a) in fig. 4.1); this property is not dependent on any parametrization of the model set, but only refers to the situation that the identification criterion can distinguish between the different models in the set. The second part of this chapter, section 4.3, deals with the aspects that are related to the construction of parametrizations (parts (b) and (c) in fig. 4.1). In the separate treatment of PE and OE model sets (section 4.3.2) and EE model sets (section 4.3.3), first the construction of (almost) bijective parametrizations will be discussed (part (b) of fig. 4.1). This property is not dependent on the identification criterion, but refers to the situation that (almost) all models in the model set are represented by a unique set of parameters. In the same sections 4.3.2 and 4.3.3, the results presented so far, will be combined in order to deal with the identifiability of parametrizations.

Special attention will be paid to model sets of models with a prespecified model order n(M).

Note that the final result of an identification procedure, i.e. the set of optimal models finally obtained, is only dependent on $\mathcal{M}$ and $J^+_{LS\infty}$, and is not influenced by the parametrization. Consequently, in view of the topics that will be treated in this chapter, only the way of constructing discriminable model sets will influence

the set of selected optimal models.
The choice for considering the identification criterion over an infinite data length is motivated by the following reasoning. When dealing with model sets and parametrizations that are discriminable, respectively identifiable by an identification criterion, it is undesirable that these properties have to reconsidered every time the length of the data interval has changed. Therefore, in an identification situation we will consider these notions to be independent of the specific length of the data interval, by evaluating the situation for $N \to \infty$.

4.2 DISCRIMINABILITY OF MODEL SETS FOR LS IDENTIFICATION

4.2.1. Introduction

In this section 4.2, the discriminability of model sets will be discussed in view of the least squares identification criterion $J^+_{LS\infty}$. In section 2.4 a number of examples has already been presented, showing undesirable consequences in identification procedures, if model sets are not discriminable. In order to get a better understanding of what the discriminability problem is actually about, we will again recall example 2.4.15.

Example 4.2.1.
Consider a set $\mathcal{M}$ of (i/o/pr)–models consisting of two elements: $\mathcal{M} = \{M_1, M_2\}$
with $M_1 = \tilde{M}_p(T_1) \in \hat{\Sigma}_{1,1}$, with $T_1(z,z^{-1}) = [a_0 + a_1 z^{-1} | -b_0 - b_1 z^{-1} | -c_1]$,
and $M_2 = \tilde{M}_p(T_2) \in \hat{\Sigma}_{1,1}$, with $T_2(z,z^{-1}) = [a_0 + a_1 z^{-1} | -b_0 - b_1 z^{-1} | -c_2]$,
with $a_0, a_1, b_0, b_1, c_1, c_2$ fixed real numbers, $((a_0 + a_1 z), (b_0 + b_1 z))$ coprime, and $|c_2| > |c_1| > 0$.
Considering the identification criterion $J^+_{LS\infty}$, and taking account of the fact that $|c_2| > |c_1| > 0$, it follows straightforwardly that M_2 will always lead to a value of the residual function $\ell^+_{LS\infty}$ that is smaller than the corresponding residual function generated by M_1. Consequently $M_1 \in J^+_{LS\infty}(v^+, \mathcal{M}) \Rightarrow M_2 \in J^+_{LS\infty}(v^+, \mathcal{M})$ and $\mathcal{M}$ is not discriminable by $J^+_{LS\infty}$. For data sequences $v^+ \in \bigcup_{M \in \mathcal{M}} \mathcal{B}^+(M^{io})$ it will hold that $J^+_{LS\infty}(v^+, \mathcal{M}) = \{M_1, M_2\}$, while for all other data sequences in $V^+_{J,\mathcal{M}}$ the set of selected optimal models will be $J^+_{LS\infty}(v^+, \mathcal{M}) = \{M_2\}$. There does not exist a data

sequence such that $J^{+}_{LS\infty}(v^{+},\mathcal{M}) = \{M_1\}$. □

As mentioned before in this thesis, the classical way of circumventing this problem of nondiscriminability is to restrict model sets by restricting the parameters $a_0/c_1=a_0/c_2=1$. For the purpose of constructing identifiable parametrization this restriction often is made effective by setting $a_0=c_1=c_2=1$; However one can imagine that this restriction of the model set is only one choice out of a very large set of possibilities, and what is more important: different ways of restricting the model set in order to arrive at discriminability, generally lead to the selection of different sets of optimal models.
Before we will start an analysis of this problem, we will first illustrate the previous remark with a simulation example.

Example 4.2.2.
Consider an input–output system $S\in\Sigma_{2,1}$, with $W=Y\times U$, $Y=\mathbb{R}^2$ and $U=\mathbb{R}$ (one input, two output system). $S=\tilde{M}_p(T^*)$, with $T^*=[P^*(z)|-Q^*(z)]$, and

$$P^*(z) = \begin{bmatrix} z^2-2.38z+0.2 & 1.16 \\ -1.24z+0.57 & z-0.327 \end{bmatrix}, \quad Q^*(z) = \begin{bmatrix} 0.10z + 0.16 \\ 0.676 \end{bmatrix}.$$

A data sequence $v^{+}\in\mathcal{B}^{+}_{obs,y}(S)$ is constructed by choosing the input u^{+} to be a zero mean white noise sequence, satisfying $\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} u^2(t) = 1$. The data sequence v^{+} is disturbed to $\tilde{v}^{+}=(\tilde{y}^{+},u^{+})$ by disturbing the output y^{+} with an additive zero mean and uncorrelated white noise sequence ξ^{+}, according to $\tilde{y}^{+}=y^{+}+\xi^{+}$, and ξ^{+} satisfies $\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \xi(t)\xi^{T}(t) = I_2$, and $\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \xi(t)u(t) = 0$.
The signal–to–noise ratios on the outputs $\tilde{y}^{+}$ are 37.8 dB for $\tilde{y}^{+}_1$ and 27.8 dB for $\tilde{y}^{+}_2$. This data sequence $\tilde{v}^{+}$ will be used in an identification procedure.
In order to construct a model set of (i/o/pr)–models, consider the following set of polynomial matrices: $\Theta \subset \mathbb{R}^{px(p+m+p)}[z]$, defined by:

$$T\in\Theta \rightarrow T=[P|-Q|-R] \text{ with } P(z)=\begin{bmatrix} \alpha_{113}z^2+\alpha_{112}z+\alpha_{111} & \alpha_{121} \\ \alpha_{212}z+\alpha_{211} & \alpha_{222}z+\alpha_{221} \end{bmatrix},$$

$$Q(z)=\begin{bmatrix} \beta_{112}z + \beta_{111} \\ \beta_{211} \end{bmatrix}, \text{ and } R(z)=\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

with all parameters $\alpha,\beta,c\in\mathbb{R}$, and $\alpha_{113},\alpha_{121},\alpha_{212},\alpha_{222},c_1,c_2$ unequal to 0.

Within this set of matrices we consider the following two subsets:

$$\Theta_1 = \{T\in\Theta|\ c_1=\alpha_{113},\ c_2=\alpha_{222}\}$$

$$\Theta_2 = \{T\in\Theta|\ c_1=\alpha_{121},\ c_2=\alpha_{212}\}$$

generating the following two model sets:

$$\mathcal{M}_1 = \{M\in\hat{\Sigma}_{2,1}|\ M=\tilde{M}_p(T) \text{ with } T\in\Theta_1\}$$

$$\mathcal{M}_2 = \{M\in\hat{\Sigma}_{2,1}|\ M=\tilde{M}_p(T) \text{ with } T\in\Theta_2\}.$$

We will now compare the selected optimal models $\hat{M}_1$ and $\hat{M}_2$, defined by $\{\hat{M}_1\}=J^+_{LS\infty}(\tilde{v}^+,\mathcal{M}_1)$ and $\{\hat{M}_2\}=J^+_{LS\infty}(\tilde{v}^+,\mathcal{M}_2)$. This comparison will be made by evaluating the Markov parameters of the i/o transfer functions $H_{yu}(z)$ of both optimal models, and by comparing these Markov parameters with the Markov parameters of the original process that has been used to generate the data sequence v^+.

Figure 4.2 shows the three sets of Markov parameters that are related to the first output y_1.

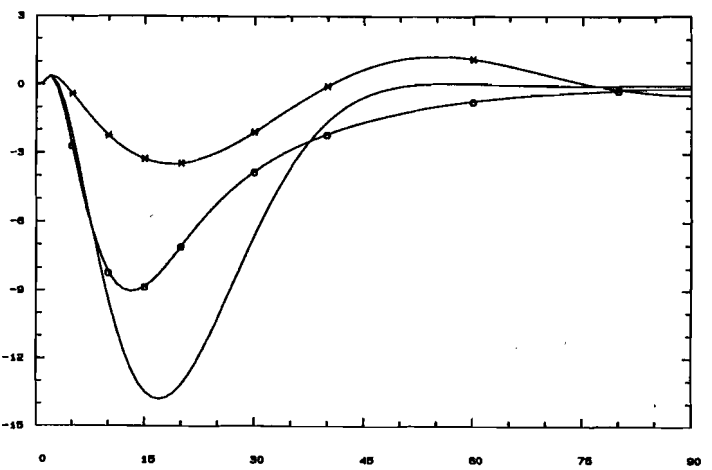

FIGURE 4.2. Markov parameters of the transfer function $H_{y_1u}(z)$ of the original process S (———), and of the two selected optimal models $\hat{M}_1\in\mathcal{M}_1$ (—×—) and $\hat{M}_2\in\mathcal{M}_2$ (—o—).

Figure 4.2. shows that the properties of the two selected optimal models are essen-

tially different, illustrating that the choice between model set $\mathcal{M}_1$ and $\mathcal{M}_2$ is a choice that really influences the dynamical properties of the selected optimal models.

Note that, without loss of generality, the restrictions on the sets of polynomial matrices Θ_1, Θ_2 can be formulated as $c_1=\alpha_{113}=c_2=\alpha_{222}=1$ for Θ_1, and $c_1=\alpha_{121}=c_2=\alpha_{212}=1$ for Θ_2. This specific choice only influences the parametrization of the model set and does not affect the model set itself, and leads to the restriction that in Θ_1 the polynomials on the diagonal of P(z) are monic, whereas in Θ_2 this holds for the polynomials on the antidiagonal of P(z). □

After these illustrations of the importance of the notion of discriminability, we can now start our analysis.

In all examples that we have presented we noticed that there is a problem of so called residual scaling. This aspect has been clearly isolated in example 4.2.1. Actually the problem of residual scaling comes down to the fact that in the model set $\mathcal{M}$ two models $M_1 \neq M_2$ exist, such that for all $v^+ \in \mathbb{L}V^+$:

$$M_1 \in J^+_{LS\infty}(v^+, \mathcal{M}) \Rightarrow M_2 \in J^+_{LS\infty}(v^+, \mathcal{M}). \tag{4.2.1}$$

This situation occurs e.g. if for all data sequences $v^+ \in \mathbb{L}V^+$, M_2 leads to a value of the residual function that is smaller than or equal to the corresponding value for M_1.

Note that for model sets that are discriminable by $J^+_{LS\infty}$, equation (4.2.1) by definition implies that $M_1=M_2$, leading to the conclusion that discriminable model sets do not have the problem of residual scaling.

A special cause for model sets to be not discriminable by $J^+_{LS\infty}$ is the occurrence of J^+–equivalent models (see definition 2.3.6), for which the expression (4.2.1) turns into an equivalence relation:

$$M_1 \in J^+_{LS\infty}(v^+, \mathcal{M}) \Leftrightarrow M_2 \in J^+_{LS\infty}(v^+, \mathcal{M}). \tag{4.2.2}$$

In this section 4.2. we will analyse the conditions that have to be imposed on model sets in order to be discriminable by the identification criterion $J^+_{LS\infty}$. For the different types of PE, OE and EE–model sets, as classified in chapter 3, different methods will be discussed to construct discriminable model sets. In subsection 4.2.4 it will also be shown to what extent different choices for arriving at discrimi-

nable model sets, do not affect the input/output properties of the optimal models obtained.

4.2.2. Conditions for discriminability

In this section a set of conditions is derived that will be shown to be sufficient conditions for the discriminability of model sets by the identification criterion $J^{+}_{LS\infty}$. These sufficient conditions will be first presented in terms of model behaviours and secondly in terms of the corresponding transfer functions. In the analysis of discriminability we will need a formal way of describing the set of admissible residual trajectories of two (i/o/pr)–models operating on the same measured data sequence v. This set is specified in the following definition.

Definition 4.2.3. Set of admissible residual trajectories of two models; $\mathcal{R}(M_1,M_2)$.

Let M_1, M_2 be two controllable (i/o/pr)–models, $M_1,M_2\in\hat{\Sigma}_{p,m}$.

We define the *set of admissible residual trajectories of two models* M_1,M_2 by:

$$\mathcal{R}(M_1,M_2) := \{(e_1,e_2)\in(E^2)^{\mathbb{Z}} \mid \exists v\in V^{\mathbb{Z}},\ (v,e_1)\in\mathcal{B}(M_1),\ (v,e_2)\in\mathcal{B}(M_2)\}; \qquad (4.2.3)\square$$

Actually $\mathcal{R}(M_1,M_2)$ reflects the combined admissible residual trajectories of two models, when driven by the same external data v. This set will be used when comparing two models in a model set with respect to their occurrence in the set of selected optimal models. A representation of this set $\mathcal{R}(M_1,M_2)$ in a polynomial matrix form is given in the following proposition. It will be shown that in this set of combined residual trajectories the effect of the output signal y can be eliminated, and that the relation between e_1 and e_2 can be described by using only the input signal u as an auxiliary signal.

Proposition 4.2.4

Let M_1,M_2 be two controllable (i/o/pr)–models $M_1,M_2\in\hat{\Sigma}_{p,m}$, $M_1=\tilde{M}_p(T_1)$, $M_2=\tilde{M}_p(T_2)$, with $T_i=[P_i|-Q_i|-R_i]\in\mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ for i=1,2.

a. $\mathcal{R}(M_1,M_2)=\{(e_1,e_2)\in(E^2)^{\mathbb{Z}} \mid \exists u\in U^{\mathbb{Z}},\ (u,e_1,e_2)\in\mathcal{B}_c(\tilde{M}_p(T_e))\}$, (4.2.4)

where $T_e\in\mathbb{R}^{px(m+p+p)}[z,z^{-1}]$ is defined by $T_e=[-Q_e|-V_1R_1|-V_2R_2]$, with $V_1,V_2\in\mathbb{R}^{pxp}[z,z^{-1}]$, nonsingular and left coprime such that $V_1P_1+V_2P_2=0$, and

$Q_e = V_1 Q_1 + V_2 Q_2$;

b. In matrix T_e, as defined above, $Q_e = 0$ if and only if $\mathcal{B}_c(M_1^{io}) = \mathcal{B}_c(M_2^{io})$, or equivalently $P_1^{-1} Q_1 = P_2^{-1} Q_2$. □

Proof. See Appendix A8.

The relation between e_1 and e_2 in $\mathcal{R}(M_1, M_2)$ is formulated, using the input signal u as an auxiliary signal. Part b of the proposition shows that the contribution of this auxiliary input signal is eliminated if and only if the two models are i/o transfer–equivalent.

In order to arrive at sufficient conditions for discriminability of model sets by $J^+_{LS\infty}$, the following philosophy will be pursued. A model set $\mathcal{M}$ is discriminable if for every model M in the set there exists a data sequence v^+ such that $\{M\} = J^+_{LS\infty}(v^+, \mathcal{M})$. We are going to require that this situation will occur for a very specific data sequence: a data sequence that has been constructed by the model M itself. To this end consider an uncorrelated input signal u^+, satisfying:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} u(t) u^T(t+\tau) = I_m \, \delta(\tau) \qquad \tau \in \mathbb{Z}_+ \tag{4.2.5}$$

where $\delta(\tau) = 0$ for $\tau \neq 0$, and $\delta(\tau) = 1$ for $\tau = 0$; I_m is the m×m identity matrix. Consider also an uncorrelated residual signal e^+, satisfying:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e(t) e^T(t+\tau) = I_p \, \delta(\tau) \qquad \tau \in \mathbb{Z}_+ \tag{4.2.6}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} u(t) e^T(t+\tau) = 0 \qquad \tau \in \mathbb{Z}_+ \tag{4.2.7}$$

Given such signals u^+ and e^+, every model $M \in \mathcal{M}$ can generate a trajectory $w^+_M = (y^+_M, u^+, e^+) \in \mathcal{B}^+_{obs,y}(M)$ that is considered to contain maximal information on this model M. Requiring that for this specific data sequence $v^+_M = (y^+_M, u^+)$ the set of selected optimal models $J^+_{LS\infty}(v^+, \mathcal{M})$ equals $\{M\}$, will lead to a set of sufficient conditions for discriminability of $\mathcal{M}$ by $J^+_{LS\infty}$. This situation is formulated in the following theorem.

Theorem 4.2.5.

Let $\mathcal{M}$ be a set of controllable (i/o/pr)–models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$, and let $M_1 \in \mathcal{M}$.

Let (u^+,e^+) be signals satisfying equations (4.2.5)–(4.2.7) and let $v^+_{M_1}=(y^+_{M_1},u^+)$ be a data sequence such that $(v^+_{M_1},e^+) \in \mathcal{B}^+_{obs,y}(M_1)$.

Consider the following conditions:

(i) For all $M_2 \in \mathcal{M}$: $(e^+,e^+_2) \in \mathcal{R}^+(M_1,M_2) \Rightarrow \ell^+_{LS\infty}(e^+_2) \geq \ell^+_{LS\infty}(e^+)$ (4.2.8a)

(ii) For all $M_2 \in \mathcal{M}$:

$$\{(e^+,e^+_2) \in \mathcal{R}^+(M_1,M_2) \wedge \ell^+_{LS\infty}(e^+_2) = \ell^+_{LS\infty}(e^+)\} \Rightarrow M_2=M_1 \qquad (4.2.8b)$$

If (i) and (ii) are satisfied for all $M_1 \in \mathcal{M}$ then $\mathcal{M}$ is discriminable by $J^+_{LS\infty}$. □

Proof. Condition (i) of the theorem guarantees that the residual function $\ell^+_{LS\infty}(e^+_2)$ of any model in the model set will always be greater than or equal to the residual function generated by M_1 itself. Condition (ii) guarantees that in the case of equality of these residual functions, both models M_1 and M_2 have to be the same, which ensures that $\{M_1\}=J^+_{LS\infty}(v^+,\mathcal{M})$. □

We have formulated sufficient conditions for general model sets to be discriminable by $J^+_{LS\infty}$. An alternative set of sufficient conditions can be obtained by requiring that each model M in the model set will be uniquely selected as an optimal model for a data sequence that has been generated with a zero residual. The corresponding restrictions on the model set will be more severe, as formulated in the following theorem.

Theorem 4.2.6.

Let $\mathcal{M}$ be a set of controllable (i/o/pr)–models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$. If for all $M_1,M_2 \in \mathcal{M}$:

$$M_1 \overset{t}{\underset{io}{\sim}} M_2 \Rightarrow M_1 = M_2 \qquad (4.2.9)$$

then $\mathcal{M}$ is discriminable by $J^+_{LS\infty}$. □

Proof. Let $M \in \mathcal{M}$, and consider a signal $w^+_M=(v^+_M,e^+) \in \mathcal{B}^+_{obs,y}(M)$, with $e^+=0$, $v^+_M=(y^+_M,u^+)$ and u^+ satisfies equation (4.2.5). Consequently $M_1 \in J^+_{LS\infty}(v^+_M,\mathcal{M}) \Leftrightarrow$

$\{\ell^+_{LS\infty}(e_1^+)=0$ for $(e_1^+,e^+)\in\mathcal{B}^+(M_1,M)\}$. Applying proposition 4.2.4 it follows that this is latter expression is equivalent with $M_1 \overset{t}{\underset{io}{\sim}} M_2$. Under the condition as formulated in the theorem, it follows that $\{M\}=J^+_{LS\infty}(v^+_M,\mathcal{M})$. □

Theorem 4.2.6 shows that absence of distinct models that are i/o transfer–equivalent, is sufficient for guaranteeing discriminability of the model set by $J^+_{LS\infty}$.

We will now specify the conditions as formulated in these theorems in terms of the transfer functions that are involved in the controllable (i/o/pr)–models.
In order to do so, we need an expression for the power amplification of dynamical systems for the specific (white) input signals that we have applied (eq. 4.2.5).
Consider a dynamical input–output system $S\in\Sigma_{p,p}$ with input variable e_1 and output variable e_2, with S controllable and determined by the transfer function $H(z)\in\mathbb{R}^{pxp}(z)$. If e_1^+ satisfies the condition of a white noise sequence (4.2.6), then $\ell^+_{LS\infty}(e_1^+)=p$, and for all e_2^+ such that $(e_2^+,e_1^+)\in\mathcal{B}^+(S)$ it can be verified that

$$\ell^+_{LS\infty}(e_2^+) = \mathrm{tr}\Big[\sum_{k=t}^{\infty} M^T(k)M(k)\Big]; \qquad (4.2.10)$$

with t and M(k) defined by the Laurent expansion of H(z) around $z=\infty$, written as:

$$H(z) = \sum_{k=t}^{\infty} M(k)z^{-k} \text{ for some } t\in\mathbb{Z}. \qquad (4.2.11)$$

If H(z) is not analytical for any finite $z\in\mathbb{C}$ with $|z|\geq 1$, then $\ell^+_{LS\infty}(e_2^+)$ is not bound.
A rational matrix or transfer function that is analytical for $|z|\geq 1$, except possibly in $z=\infty$, will be called *stable.*

Using these expressions we can now formulate an equivalent version of theorem 4.2.5 in terms of the transfer functions $H_e(z)=[H_{ey}(z)|H_{eu}(z)]$ that are involved.

Theorem 4.2.7.
Let $\mathcal{M}_{\Theta_t}$ be a set of controllable (i/o/pr)–models

$$\mathcal{M}_{\Theta_p}=\{M\in\hat{\Sigma}_{p,m} |\ M \text{ has a transfer function } H_e(z)\in\Theta_t\subset\mathbb{R}^{px(p+m)}(z)\}$$

and let $M_1\in\mathcal{M}_{\Theta_t}$ with a transfer function $H_e^{(1)}(z)=[H_{ey}^{(1)}|H_{eu}^{(1)}]\in\Theta_t$.
Consider the following condition:

(i) For all $M_2 \in \mathcal{M}_{\Theta_t}$ with $H_e^{(2)}(z)=[H_{ey}^{(2)}|H_{eu}^{(2)}]\in\Theta_t$, and $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}$ a stable rational matrix, having a Laurent expansion around $z=\infty$, written as $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}(z) = \sum_{k=t}^{\infty} M(k)z^{-k}$ for some $t\in\mathbb{Z}$, it holds that

a. $\mathrm{tr}\left[\sum_{k=t}^{\infty} M^T(k)M(k)\right] \geq p$; and (4.2.12)

b. If $M_1 \overset{t}{\underset{io}{\sim}} M_2$ and $\mathrm{tr}\left[\sum_{k=t}^{\infty} M^T(k)M(k)\right]=p$ then $M_1=M_2$. (4.2.13)

If condition (i) is satisfied for all $M_1\in\mathcal{M}_{\Theta_t}$, then $\mathcal{M}_{\Theta_t}$ is discriminable by $J^+_{LS\infty}$. □

Proof. See appendix A9.

For two models M_1,M_2 as represented in theorem 4.2.7, the rational matrix $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}(z)$ will be denoted by $H_{e_2e_1}(z)$.

Condition (i), as formulated in theorem 4.2.7, can be satisfied by requiring that, within a model set, any stable transfer function $H_{e_2e_1}(z)$ between two models can be written as:

$$H_{e_2e_1}(z) = I + \sum_{\substack{k=t \\ k\neq 0}}^{\infty} M(k)z^{-k} \qquad \text{for some } t\in\mathbb{Z}. \tag{4.2.14}$$

Taking into account that $H_{e_2e_1}(z)=H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}(z)$, it is a straightforward choice to realize this requirement by imposing restrictions on the transfer function $H_{ey}(z)$, or its inverse $H_{ye}(z)$, of the models in the set.

We have not been able to completely characterize the property of discriminability for general model sets. This is probably due to the fact that such a characterization will be dependent on the specific type of model set that is taken into account. In stead of this general characterization, sufficient conditions have been presented that will turn out to be satisfactory when dealing with common model sets.
The result of theorem 4.2.7 can similarly be formulated for model sets that are induced by (A,B,C,D) state space representations, as defined in section 2.4.

4.2.3. Discriminability of PE, OE and EE model sets

After having stated the general conditions for discriminability of model sets with respect to J^+_{LS}, we will now focus on specific ways of obtaining discriminability.
Considering the classification of models in terms of their residual properties, as presented in section 3.2, the following observation can be made.
K–step ahead prediction error models and output error models intrinsically contain a nonsingular matrix $L \in \mathbb{R}^{pxp}$ in their representation, as shown in definitions 3.2.1 and 3.2.3. This nonsingular matrix L is directed towards the relation between outputs and residuals in the corresponding models, reflected in the transfer function $H_{ye}(z)$. For both k–step ahead prediction error, and output error models, this nonsingular matrix is defined to be $L = \lim_{z \to \infty} H_{ye}(z)$. In view of theorem 4.2.7 and the remarks made above, this property advocates a special restriction of model sets in order to arrive at discriminability by $J^+_{LS\infty}$, by fixing this matrix L for all models in a model set on beforehand.

Definition 4.2.8.
Denote with $\mathcal{M}^{pe}_{p,m}(L)$, $\mathcal{M}^{oe}_{p,m}(L)$ any collection of k–step ahead prediction error models, c.q. output error models $M \in \hat{\Sigma}_{p,m}$, with a prespecified –and for all models equal– nonsingular matrix $L \in \mathbb{R}^{pxp}$ as defined in definitions 3.2.1. and 3.2.3. □

Theorem 4.2.9.
Any set of (i/o/pr)–models $\mathcal{M}^{pe}_{p,m}(L)$ or $\mathcal{M}^{oe}_{p,m}(L)$ is discriminable by $J^+_{LS\infty}$. □

Proof
The proof will be formulated by applying the result of theorem 4.2.7.
Let $M_1 \in \mathcal{M}^{pe}_{p,m}(L)$ or $\mathcal{M}^{oe}_{p,m}(L)$, with M_1 having transfer function $H^{(1)}_e(z) = [H^{(1)}_{ey} | H^{(1)}_{eu}] \in \mathbb{R}^{px(p+m)}(z)$. It follows from the definitions of prediction error and output error models that $H^{(1)}_{ye}(z) = [H^{(1)}_{ey}(z)]^{-1}$ is a proper rational matrix and $\lim_{z \to \infty} H^{(1)}_{ye}(z) = L$. Let M_2 be any other model from the same model set with transfer function $H^{(2)}_e(z)$. Then consequently also $\lim_{z \to \infty} H^{(2)}_{ye}(z) = L$. Now consider the rational matrix $H(z) = H^{(2)}_{ey}(H^{(1)}_{ey})^{-1} = (H^{(2)}_{ye})^{-1} H^{(1)}_{ye}(z)$; it follows

that $\lim_{z\to\infty} H(z) = I$, and $H(z) = \sum_{k=0}^{\infty} M(k)z^{-k}$, with $M(0)=I$. Consequently $\mathrm{tr}\left[\sum_{k=0}^{\infty} M^T(k)M(k)\right] \geq p$ with equality if and only if $H(z)=I$. Since $H(z)=I$ together with $M_1 \overset{t}{\underset{io}{\sim}} M_2$, or equivalently $H_{yu}^{(1)}(z) = H_{yu}^{(2)}(z)$, implies $M_1=M_2$, according to theorem 4.2.7 this proves the result. □

Considering model sets with a fixed nonsingular matrix $L=I$, is the most common situation in prediction error and output error modelling. We can compare this situation with the stochastic innovations representation in (A,B,C,D) state space form:

$$x(k+1) = Ax(k) + B_u u(k) + B_e e(k) \tag{4.2.15a}$$

$$y(k) = Cx(k) + D_u u(k) + e(k) \tag{4.2.15b}$$

where actually the matrix D_e $(= \lim_{z\to\infty} H_{ye}(z))$ has been fixed to I. Note that we have motivated the choice of a fixed matrix D_e by purely non–stochastic arguments.

One could ask what is the effect of the choice of a specific value of L. This matrix L operates as a weighting matrix in the least squares identification criterion; by weighting the residual signals one can express the relative importance that is attached to the different components of the residual e. Note that applying the identification criterion $J^+_{LS\infty}$ to a model set $\mathcal{M}^{pe}_{p,m}(L)$ or $\mathcal{M}^{oe}_{p,m}(L)$ is similar as applying:

$$J^+_{LS\infty}(v^+, \mathcal{M}(I)) = \arg \min_{M\in\mathcal{M}} \min_{e^+\in\mathcal{E}^+(v^+,M)} \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e^T(t)L^{-T}L^{-1}e(t) \tag{4.2.16}$$

This comes down to the application of a weighted least squares identification criterion to a model set with $L=I$.

For a model set $\mathcal{M}_{p,m}(L)$, as meant in definition 4.2.8, different choices of L are possible, and in general different choices of L will lead to essentially different identified models. In section 4.2.4 more attention will be paid to this aspect.

It has to be stressed that the result of theorem 4.2.9 shows that the only restriction on the transfer functions $H_{ye}(z)$ in the model set, is reflected by the condition $\lim_{z\to\infty} H_{ye}(z) = L$. This means that there are no restrictions imposed on the positions of the zeros of this rational matrix, e.g. there is no requirement that $H_{ye}(z)$ should be minimum–phase, as is a common restriction in identification methods (Ljung,

1987). This will be illustrated in the following example.

Example 4.2.10.

Consider two controllable models $M_1, M_2 \in \hat{\Sigma}_{1,1}$ with

$M_1 = \tilde{M}_p(T_1)$ with $T_1 = [z-a \mid -(z-b) \mid -(z-c)]$, and

$M_2 = \tilde{M}_p(T_2)$ with $T_2 = [z-a \mid -(z-b) \mid -(z-c^{-1})]$, with $a,b,c \in \mathbb{R}$ and $0<|c|<1$.

The corresponding transfer functions $H_{ye}(z)$ can be written as:

$H_{ye}^{(1)}(z) = \frac{z-c}{z-a}$, which is minimum–phase, and $H_{ye}^{(2)}(z) = \frac{z-c^{-1}}{z-a}$ which is the non-minimum–phase equivalent of $H_{ye}^{(1)}(z)$.

For any data sequence $v^+ \in V^{\mathbb{Z}_+}$, it follows that $(e_1^+, e_2^+) \in \mathscr{B}(M_1, M_2) \Leftrightarrow$ $(\sigma-c)e_1 = (\sigma-c^{-1})e_2$. The transfer function relating these two residuals, can be written as $H_{e_1e_2}(z) = \frac{z-c^{-1}}{z-c} = c^{-1}\left[c \frac{z-c^{-1}}{z-c} \right]$, with $c \frac{z-c^{-1}}{z-c}$ an all pass function, i.e. its power amplification is 1 for all possible residual signals e_2.

This implies that for all v^+ and $e_2^+ \in \mathscr{E}^+(v^+, M_2)$ such that $\ell^+_{LS\infty}(e_2^+)$ is finite and unequal to 0, it follows that $\ell^+_{LS\infty}(e_1^+) = c^{-1}\, \ell^+_{LS\infty}(e_2^+) > \ell^+_{LS\infty}(e_2^+)$, and consequently M_1 can not be selected. However for all v^+ such that $e_1^+ \in \mathscr{E}^+(v^+, M_1)$ is a white noise sequence, $\ell^+_{LS\infty}(e_2^+)$ will not be bound, since the corresponding transfer function $H_{e_2e_1}(z)$ is not stable. In this situation M_2 will not be selected. Consequently M_1 and M_2 can be distinguished from each other. □

We have shown how discriminability can be obtained for prediction error and output error model sets, by using the matrix L which is intrinsically defined within these models. The definition of equation error models does not point to a specific restriction of the corresponding model sets, as in the situations mentioned before. However also for equation error models discriminability can be achieved. One of the options is to choose prediction error or output error form restrictions, i.e. by requiring the EE model set to satisfy $\lim_{z\to\infty} H_{ye}(z) = L$ with the nonsingular $L \in \mathbb{R}^{p\times p}$ fixed over the model set (PE–form), or even $H_{ye}(z) = L$ (OE–form). Apart from these possibilities, equation error models give extra opportunities to restrict the corresponding model sets to be discriminable, by exploiting the fact that the residual e is observable from the external signals v. In the sequel of this

section this is going to be discussed.
In accordance with the definitions, as given in chapter 3, an (i/o/pr)–model $M\in\hat{\Sigma}_{p,m}$ is an equation error model if and only if the corresponding transfer function $H_e(z)=[H_{ey}(z)|H_{eu}(z)]$ is a polynomial matrix, i.e. $H_e(z)\in\mathbb{R}^{px(p+m)}[z,z^{-1}]$. This special property points to alternative ways for obtaining discriminability, through restrictions on the polynomial degrees in this transfer function.
In this line of thought we will present some results for discriminability of equation error model sets. Without loss of generality, as will be shown later on, we will restrict attention to models having a transfer function $H_e(z)$ that is polynomial in one indeterminate, $H_e(z)\in\mathbb{R}^{px(p+m)}[z]$.

Theorem 4.2.11.
Let Θ_t be a set of polynomial transfer function matrices: $\Theta_t\subset\mathbb{R}^{px(p+m)}[z]$ with $H_e\in\Theta_t$ written as $H_e=[H_{ey}|H_{eu}]$ and $\text{rank}_{\mathbb{R}(z)}H_{ey}=p$ for all $H_e\in\Theta_t$, and let L be a nonsingular matrix $L\in\mathbb{R}^{pxp}$.
Consider the set of equation error models:

$$\mathcal{M}^{ee}_{\Theta_t}(L) = \{M\in\hat{\Sigma}_{p,m} \mid M \text{ has transfer function } H_e(z)\in\Theta_t, \text{ and } \Gamma_{hr}(H_{ey})=L\}, \quad (4.2.17)$$

and consider the following conditions.

$$\text{(i)} \quad H_e^{(1)},H_e^{(2)}\in\Theta_t \Rightarrow \nu_i^{(u)}(H_{ey}^{(1)}) = \nu_i^{(u)}(H_{ey}^{(2)}) \text{ for } i\in\underline{p}; \quad (4.2.18)$$

$$\text{(ii)} \quad H_e^{(1)},H_e^{(2)}\in\Theta_t \Rightarrow \sum_{i=1}^{p} \nu_i^{(u)}(H_{ey}^{(1)}) = \sum_{i=1}^{p} \nu_i^{(u)}(H_{ey}^{(2)}); \text{ and} \quad (4.2.19)$$

$$H_e\in\Theta_t \Rightarrow \text{rank}[H_{ey}(0)|H_{eu}(0)]=p. \quad (4.2.20)$$

If Θ_t satisfies (i) or (ii) then $\mathcal{M}^{ee}_{\Theta_t}(L)$ is discriminable by $J^{+}_{LS\infty}$. □

Proof. See Appendix A11.

In theorem 4.2.11 discriminability of some equation error model sets has been shown. The restriction that is required for obtaining discriminability, is not presented in the form of a fixed Markov parameter of the transfer function $H_{ye}(z)$, but in the form of a fixed leading row coefficient matrix of the polynomial transfer function $H_{ey}(z)$. Some additional conditions have to be fulfilled, such as a fixed set of row degrees of $H_{ey}(z)$, or a fixed sum of row degrees of $H_{ey}(z)$ and absence of uncontrollable poles in the models. Note that equation (4.2.20) is equivalent with the condition that the i/o–parts of the models should have no uncontrollable poles

in z=0. Equation (4.2.20) together with (4.3.17) forces polynomial matrix H_e to be bilaterally row proper.
Under the additional condition on Θ_t, that $H_e \in \Theta_t \Rightarrow H_{yu}$ is a proper rational matrix, condition (ii) of theorem 4.2.11 guarantees that all models in $\mathcal{M}^{ee}_{\Theta_t}(L)$ will have model order $n(M)=\sum_{i=1}^{p} \nu_i^{(u)}(H_{ey})$. This can be understood by applying theorem 3.4.7d, considering that $M \in \mathcal{M}^{ee}_{\Theta_t}(L)$ can be written as $M=\tilde{M}_p(T)$ with $T=[H_{ey}|-H_{eu}|-I]$.
In theorem 4.2.11 a restriction has been made to model sets based on polynomial transfer functions $H_e(z)$ that are antiproper (i.e. $H_e(z) \in \mathbb{R}^{px(p+m)}[z]$). This is not a severe restriction, since any shift in the residual signals can be applied without changing the essential characteristics of the model sets. More attention will be paid to this in section 4.2.4.

In the previous part an EE model set has been restricted in terms of the row degrees of the polynomial transfer functions $H_e(z)$. In a similar situation the discriminability can be evaluated of model sets that are restricted in terms of their corresponding column degrees instead of row degrees. This situation is formulated in the following theorem.

Theorem 4.2.12.
Let Θ_t be a set of polynomial transfer function matrices: $\Theta_t \subset \mathbb{R}^{px(p+m)}[z]$ with $H_e \in \Theta_t$ written as $H_e=[H_{ey}|H_{eu}]$ and $\text{rank}_{\mathbb{R}(z)} H_{ey}=p$ for all $H_e \in \Theta_t$, and let L be a nonsingular matrix $L \in \mathbb{R}^{pxp}$.
Consider the set of equation error models:

$$\mathcal{M}^{ee}_{\Theta_t}(L) = \{M \in \hat{\Sigma}_{p,m} |\ M \text{ has transfer function } H_e(z) \in \Theta_t, \text{ and } \Gamma_{hc}(H_{ey})=L\}, \quad (4.2.21)$$

and consider the following conditions.

(i) $$H_e^{(1)}, H_e^{(2)} \in \Theta_t \Rightarrow \mu_i^{(u)}(H_{ey}^{(1)}) = \mu_i^{(u)}(H_{ey}^{(2)}) \quad \text{for } i \in \underline{p}; \quad (4.2.22)$$

(ii) $$H_e^{(1)}, H_e^{(2)} \in \Theta_t \Rightarrow \sum_{i=1}^{p} \mu_i^{(u)}(H_{ey}^{(1)}) = \sum_{i=1}^{p} \mu_i^{(u)}(H_{ey}^{(2)}); \text{ and} \quad (4.2.23)$$

$$H_e \in \Theta_t \Rightarrow \text{rank}[H_{ey}(0)]=p. \quad (4.2.24)$$

If Θ_t satisfies (i) or (ii) then $\mathcal{M}^{ee}_{\Theta_t}(L)$ is discriminable by $J^{+}_{LS\infty}$. □

Proof. See Appendix A12.

In theorem 4.2.12 a similar situation is dealt with as in theorem 4.2.11. One of the differences with theorem 4.2.11 is reflected in equation (4.2.24); when the sum of the column degrees of H_{ey} has been specified, the absence of uncontrollable poles in $z=0$ in the i/o–part of the models as formulated in equation (4.2.20), can not yet be proved to be sufficient for guaranteeing discriminability. However we have the strong impression that the same condition is also sufficient in theorem 4.2.12(ii). Nevertheless, at this moment the condition as reflected in equation (4.2.24) has to be imposed, requiring that $H_y(z)$ does not have any poles in $z=0$, or equivalently there are no delays in this transfer function. This condition establishes that the polynomial matrix H_{ey} is restricted to be bilaterally column proper.
Using the results of section 3.4, and posing the additional restriction that $H_e \in \Theta_t \Rightarrow$ {H_{yu} is a proper rational matrix}, it follows that condition (ii) of theorem 4.2.12 guarantees that the models in the model set $\mathcal{M}^{ee}_{\Theta_t}(L)$ have model order

$$n(M)=\sum_{i=1}^{p} \mu_i^{(u)}(H_{ey}).$$

Remark 4.2.13.
So far we have presented two specific ways of obtaining discriminability for equation error models. It has to be stressed that these are only two choices out of a great number of possibilities for obtaining discriminability of equation error model sets in general. Moreover, every other choice of arriving at discriminability will generally lead to different results of the identification procedure, i.e. to different sets of selected optimal models. □

The results as presented in this section 4.2.3 will be referred to when discussing the construction of identifiable parametrizations in section 4.3.

4.2.4. Some invariance properties in (i/o/pr) model sets

In all three situations of PE–, OE– and EE–model sets, discriminability is achieved by restricting the model set using a fixed nonsingular matrix L. It has been mentioned that different choices for this matrix L will generally lead to different properties of the selected models, and consequently different results of the identification procedure. However for specific relations between two different choices of L, some invariance properties exist with respect to the models finally obtained. Being more specific: the input/output part of a selected optimal model is invariant for certain operations on the matrix L.

A similar situation holds for a shift in the residual signals. It is quite straightforward that in terms of the identification criterion, it does not matter whether we consider the residual e as the signal to be applied to the residual function, or a shifted residual $\sigma^k e$ for any $k \in \mathbb{Z}$.
In this subsection we are going to characterize these invariance properties.
To this end we first have to define a notion of scaling–equivalent models and scaling–equivalent model sets.

Definition 4.2.14. Scaling–equivalent models with respect to $\ell^+_{LS\infty}$.
Two (i/o/pr)–models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ are *scaling–equivalent with respect to* $\ell^+_{LS\infty}$, denoted by $M_1 \overset{c}{\underset{\ell^+_{LS\infty}}{\sim}} M_2$, if there exists a constant $c \in \mathbb{R}\setminus\{0\}$ such that

$$\{(e_1^+, e_2^+) \in \mathscr{B}^+(M_1, M_2) \Rightarrow \ell^+_{LS\infty}(e_2^+) = c\, \ell^+_{LS\infty}(e_1^+)\}. \qquad (4.2.25)\ \square$$

Definition 4.2.15. Scaling–equivalent model sets with respect to $\ell^+_{LS\infty}$.
Two model sets $\mathscr{M}_1, \mathscr{M}_2 \subset \hat{\Sigma}_{p,m}$ are *scaling–equivalent with respect to* $\ell^+_{LS\infty}$, denoted by $\mathscr{M}_1 \overset{c}{\underset{\ell^+_{LS\infty}}{\sim}} \mathscr{M}_2$, if there exists a constant $c \in \mathbb{R}\setminus\{0\}$ such that

for all $M_1 \in \mathscr{M}_1$ there exists a $M_2 \in \mathscr{M}_2$, and
for all $M_2 \in \mathscr{M}_2$ there exists a $M_1 \in \mathscr{M}_1$, such that

$$\{(e_1^+, e_2^+) \in \mathscr{B}^+(M_1, M_2) \Rightarrow \ell^+_{LS\infty}(e_2^+) = c\, \ell^+_{LS\infty}(e_1^+)\}. \qquad (4.2.26)\ \square$$

Note that for scaling–equivalent model sets there exists a special bijection between $\mathscr{M}_1$ and $\mathscr{M}_2$, defined by scaling–equivalent models $M_1 \in \mathscr{M}_1$ and $M_2 \in \mathscr{M}_2$ that have the same constant $c \in \mathbb{R}\setminus\{0\}$ as meant in definition 4.2.14.

A characterization of the property of scaling–equivalence of two models with respect to $\ell^+_{LS\infty}$ is presented in the following proposition.

Proposition 4.2.16.
Two (i/o/pr)–models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ are scaling–equivalent with respect to the residual function $\ell^+_{LS\infty}$, if and only if there exists a constant $c \in \mathbb{R}\setminus\{0\}$ and there exist nonsingular polynomial matrices $R_{e_1}, R_{e_2} \in \mathbb{R}^{p\times p}[z,z^{-1}]$, such that the following conditions are satisfied:

(i) $(e_1,e_2)\in\mathscr{R}(M_1,M_2) \Leftrightarrow R_{e_1}(\sigma,\sigma^{-1})e_1 - R_{e_2}(\sigma,\sigma^{-1})e_2 = 0;$ (4.2.27)

(ii) the transfer functions $H_{e_1e_2}(z)=R_{e_1}^{-1}R_{e_2}$ and $H_{e_2e_1}(z)=R_{e_2}^{-1}R_{e_1}$ are polynomial and unimodular with respect to $\mathbb{R}[z,z^{-1}]$;

(iii) $H_{e_1e_2}^T(z^{-1})H_{e_1e_2}(z) = c^{-1}I$ and (4.2.28a)

$H_{e_2e_1}^T(z^{-1})H_{e_2e_1}(z) = cI.$ (4.2.28b)

Proof. See Appendix A13.

Interpreting the proposition it has to be noticed that the transfer functions between the two residuals e_1 and e_2 both have to be "all pass" with a constant scaling factor c, and moreover that they both have to be stable. Since both transfer functions are each others inverse, and all pass functions have their poles and zeros symmetric with respect to the unit circle in $\mathbb{C}$, it follows that both transfer functions should be analytic for all finite values of $z\in\mathbb{C}$ except possible in $z=0$. This requirement leads to the condition of being polynomial and unimodular with respect to $\mathbb{R}[z,z^{-1}]$.
The proposition can be formulated for (i/o/pr)–models represented in a polynomial matrix representation.

Proposition 4.2.17
Two controllable (i/o/pr)–models $M_1,M_2\in\hat{\Sigma}_{p,m}$ are scaling–equivalent with respect to $\ell^+_{LS\infty}$, if and only if there exist polynomial matrices $P,R,U\in\mathbb{R}^{pxp}[z,z^{-1}]$ and $Q\in\mathbb{R}^{pxm}[z,z^{-1}]$ such that

$$M_1 = \tilde{M}_p([P|-Q|-R]) \text{ and } M_2 = \tilde{M}_p([P|-Q|-RU]) \quad (4.2.29)$$

with U satisfying:

(i) U is unimodular with respect to $\mathbb{R}[z,z^{-1}]$; and

(ii) $U^T(z^{-1},z)U(z,z^{-1}) = cI$ with $c\in\mathbb{R}\setminus\{0\}$. (4.2.30) □

Proof. See Appendix A14.

The operation of a matrix U as introduced in proposition 4.2.17 both reflects the possibility of shifting the residuals over a specific time (e.g. $U(z)=\mathrm{diag}(z^{n_1}, . ,z^{n_p})$, with $n_i\in\mathbb{Z}$, i=1,.,p) and of changing the constant matrix L as meant in the introduction of this section 4.2.4., by applying a constant unitary matrix ($U^TU=I$).
Note that from the above proposition follows that $M_1 \underset{\ell^+_{LS\infty}}{\sim} M_2$ implies $M_1 \underset{io}{\sim} M_2$.

Proposition 4.2.17 can also be interpreted as a characterization of scaling– equivalent model sets, if the model sets are parametrized in a polynomial matrix parametrization.
We can wonder what are the consequences if we apply two scaling–equivalent model sets to the identification criterion $J^+_{LS\infty}$, for one and the same data sequence.

Proposition 4.2.18.
Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two model sets of controllable (i/o/pr)–models, $\mathcal{M}_1, \mathcal{M}_2 \subset \hat{\Sigma}_{p,m}$.
If $\mathcal{M}_1$ and $\mathcal{M}_2$ are scaling–equivalent with respect to $\ell^+_{LS\infty}$ then for all data sequences $v^+ \in \mathbb{V}^+$:

$$J^+_{LS\infty}(v^+, \mathcal{M}_1) \underset{\ell^+_{LS\infty}}{\overset{c}{\sim}} J^+_{LS\infty}(v^+, \mathcal{M}_2) \qquad (4.2.31)$$ □

Proof. This proposition follows directly from the appropriate definitions; if $M_1 \in J^+_{LS\infty}(v^+, \mathcal{M}_1)$ then automatically $M_2 \in J^+_{LS\infty}(v^+, \mathcal{M}_2)$ with $M_1 \underset{\ell^+_{LS\infty}}{\overset{c}{\sim}} M_2$, and vice versa. □

Since scaling–equivalence of two models with respect to $\ell^+_{LS\infty}$ has been shown to imply i/o–equivalence of the models, the input/output properties of the selected optimal models in $\mathcal{M}_1$ and $\mathcal{M}_2$, as meant in proposition 4.2.18, will always be the same, whether one has chosen $\mathcal{M}_1$ or $\mathcal{M}_2$ to apply the identification criterion to.
Applying the previous results to the situation of output error or prediction error model sets with a fixed nonsingular matrix L, we arrive at the following corollary.

Corollary 4.2.19.
Let $\mathcal{M}_{p,m}(L)$, be a prediction error or output error model set as meant in definition 4.2.8, and let $\mathcal{M}_{p,m}(L^*)$ be the same model set but with a different fixed matrix L^*.
If L and L^* satisfy $L=L^*U$, $U \in \mathbb{R}^{pxp}$ and $U^TU=cI$, $c \in \mathbb{R}\setminus\{0\}$, then the condition of proposition 4.2.18 is satisfied, and $J^+_{LS\infty}(v^+, \mathcal{M}_{p,m}(L))$ and $J^+_{LS\infty}(v^+, \mathcal{M}_{p,m}(L^*))$ are scaling–equivalent with respect to $\ell^+_{LS\infty}$ for all $v^+ \in \mathbb{V}^+$. □

Proof. Follows by inspection. □

For special types of model sets the invariance property as discussed in this section can be shown to hold for a larger class of operations on the model sets as discussed. In order to illustrate this, we will write the least squares residual function (with slight abuse of notation), componentwise decomposed over the components of the residual e:

$$\ell^+_{LS\infty}(e^+) = \sum_{i=1}^{p} \ell^+_{LS\infty}(e^{(i),+}) \tag{4.2.32}$$

with $\ell^+_{LS\infty}(e^{(i),+}) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} (e^{(i)}(t))^2$, where $e^{(i)}$ is the i^{th} residual component.

We now formulate the following property of a model set $\mathcal{M}$.

Definition 4.2.20. Componentwise decomposition property of $\mathcal{M}$ with respect to $\ell^+_{LS\infty}$.

A model set $\mathcal{M}$ of (i/o/pr)–models, $\mathcal{M}\subset\hat{\Sigma}_{p,m}$, is said to have the *componentwise decomposition property with respect to* $\ell^+_{LS\infty}$, if for all $\hat{M}\in\mathcal{M}$ and $v^+\in V^{\mathbb{Z}^+}$:

$$\{\hat{M} \in \underset{M\in\mathcal{M}}{\arg\min} \underset{e^+\in\mathcal{E}^+(v^+,M)}{} \ell^+_{LS\infty}(e^+)\} \Leftrightarrow \{\hat{M} \in \bigcap_i (\underset{M\in\mathcal{M}}{\arg\min} \underset{e^+\in\mathcal{E}^+(v^+,M)}{} \ell^+_{LS\infty}(e^{(i),+}))\} \tag{4.2.33}$$ □

If a model set $\mathcal{M}$ has the componentwise decomposition property, the identification criterion $J^+_{LS\infty}$ can be applied by considering the residual components $e^{(i)}$ separately. Model sets for which this property holds are e.g. equation error model sets that have independently parametrized residual components, as shown in the following example.

Example 4.2.21.

Consider a set of equation error models $\mathcal{M}_{\Theta_p}=\{M\in\hat{\Sigma}_{2,2} | M=\tilde{M}_p(T),\ T\in\Theta_p\subset\mathbb{R}^{2\times(2+2+2)}[z,z^{-1}]\}$, where Θ_p is the set of all matrices:

$$\left[\begin{array}{cc|cc|cc} 1+a_1z^{-1} & a_2+a_3z^{-1} & -b_0-b_1z^{-1} & -b_2-b_3z^{-1} & -1 & 0 \\ a_4+a_5z^{-1} & 1+a_6z^{-1} & -b_4-b_5z^{-1} & -b_6-b_7z^{-1} & 0 & -1 \end{array}\right]$$

with all parameters $\{a_i,b_j\}_{i=1,.,6;j=0,.,7}$ an element of $\mathbb{R}$.

Note that according to theorem 4.2.9 this model set is discriminable by $J^+_{LS\infty}$.
Since there is no relation between the parameters in the first and the second row of the matrices in Θ_p, the residuals e_1 and e_2 will independently take on their values for specific measurement data v^+. Minimization of $\ell^+_{LS\infty}(e^+)$ over $\mathcal{M}$, will come down to a separate minimization of $\ell^+_{LS\infty}(e^{(i),+})$, for i=1,2.
Note that the property is lost whenever there exists a relation between the two parameter sets $\{a_1,a_2,a_3,b_0,b_1,b_2,b_3\}$ and $\{a_4,a_5,a_6,b_4,b_5,b_6,b_7\}$. □

For a model set that satisfies this extra condition, a larger class of operations on this model set leads to invariance properties similar to the properties previously discussed. In order to characterize this larger class of operations, we need a reformulated definition of scaling–equivalence.

Definition 4.2.22. Componentwise scaling–equivalent models.
Two (i/o/pr)–models $M_1,M_2\in\hat{\Sigma}_{p,m}$ are *componentwise scaling–equivalent* with respect to $\ell^+_{LS\infty}$, if there exist constants $c_1, \ldots c_p\in\mathbb{R}\setminus\{0\}$ such that for all i=1,..,p:

$$\{(e_1^+,e_2^+)\in\mathcal{R}^+(M_1,M_2) \Rightarrow \ell^+_{LS\infty}(e_2^{(i),+}) = c_i\, \ell^+_{LS\infty}(e_1^{(i),+})\} \qquad (4.2.34)$$ □

The different components of the residual signal e are now considered separately. Definition 4.2.22 leads to a similar notion of componentwise scaling–equivalent model sets, as shown in definition 4.2.15. We will not repeat all this. At this position we only mention the result for an adapted version of proposition 4.2.17, showing that componentwise scaling–equivalence of models with respect to $\ell^+_{LS\infty}$ can be represented by postmultiplication of matrix R in the proposition mentioned above, by a polynomial matrix U, satisfying:

U is unimodular with respect to $\mathbb{R}[z,z^{-1}]$, and
$U^T(z^{-1},z)U(z,z^{-1}) = \mathrm{diag}(c_1, \ldots c_p)$ with $c_i\in\mathbb{R}\setminus\{0\}$, i=1,..,p.

For model sets satisfying this additional componentwise decomposition property with respect to $\ell^+_{LS\infty}$, a similar result can now be formulated for the output error and prediction error model sets as presented before.

Corollary 4.2.23.
Let $\mathcal{M}_{p,m}(L)$, be a prediction error or output error model set as meant in definition

4.2.8, and let $\mathcal{M}_{p,m}(L^*)$ be the same model set but with a different fixed matrix L^*.
If L and L^* satisfy $L=L^*U$, $U\in\mathbb{R}^{pxp}$ and $U^TU= \text{diag}(c_1, . c_p)$ with $c_i\in\mathbb{R}\setminus\{0\}$, then for all $v^+\in \mathbb{V}^+$, $J^+_{LS\infty}(v^+,\mathcal{M}_{p,m}(L))$ and $J^+_{LS\infty}(v^+,\mathcal{M}_{p,m}(L^*))$ are componentwise scaling–equivalent with respect to $\ell^+_{LS\infty}$. □

Proof. Follows by inspection. □

Application of this corollary to the example 4.2.21 shows that in the parametrization of model set $\mathcal{M}$, the matrix R, which actually equals I, may be replaced by any nonsingular matrix U, with U^TU diagonal, without altering the essential part of the identification result; again the input/output part M^{io} of the selected optimal models, is invariant under these kind of operations on the model set. Note that any other choice of R, not satisfying R^TR is diagonal, in this example will lead to a model set that generates optimal models with different dynamical properties with respect to their i/o behaviour.

4.3 IDENTIFIABLE POLYNOMIAL MATRIX PARAMETRIZATIONS

4.3.1. Introduction

In this section the results of section 4.2 on the discriminability of model sets by the identification criterion $J^+_{LS\infty}$ are going to be used for the construction of identifiable polynomial matrix parametrizations. As presented before in this thesis, the discriminability of model sets is a prerequisite for arriving at identifiable parametrizations. Once discriminability has been obtained, the creation of a strictly identifiable parametrization comes down to constructing a bijective parametrization $\tilde{M}$: $\Theta \rightarrow \mathcal{M}$, i.e. two distinct parameters $\theta_1,\theta_2\in\Theta$ represent two distinct models in $\mathcal{M}$ (see the definitions in section 2.4). Similar to the situation in the previous part of this chapter, the different model sets will be treated on the basis of their residual types, leading to a distinction between PE and OE model sets on the one hand (section 4.3.2), and EE model sets on the other (section 4.3.3). In both situations, first there will be discussed which requirements have to be laid upon a corresponding parametrization in order to be (almost) bijective. Afterwards, the results ob-

tained will be combined with the requirements of discriminability of the model set, in order to arrive at parametrizations that are identifiable by $J^+_{LS\infty}$. In both sections 4.3.2 and 4.3.3 special attention will be paid to the parametrization of model sets with models having a prespecified model order n(M). In section 4.3.4 the results of the previous sections will be illustrated. The different consequences for the use of identifiable parametrizations for OE or EE model sets will be clarified in a number of examples both with industrial and simulation data.

4.3.2. Identifiable parametrizations for PE and OE model sets

For the construction of identifiable parametrizations for PE and OE model sets, we will consider model sets $\mathcal{M}^{pe}_{p,m}(L)$, $\mathcal{M}^{oe}_{p,m}(L)$, as reported in section 4.2, that are shown to be discriminable by $J^+_{LS\infty}$. A polynomial matrix parametrization of model sets $\mathcal{M}^{pe}_{p,m}(L)$ and $\mathcal{M}^{oe}_{p,m}(L)$ is induced by a set of polynomial matrices $\Theta_p \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, according to

$$\tilde{M}_P: \Theta_p \rightarrow \mathcal{M} \tag{4.3.1}$$

The definitions of (i/o/pr) model sets $\mathcal{M}^{pe}_{p,m}(L)$ and $\mathcal{M}^{oe}_{p,m}(L)$ prescribe specific restrictions on a set Θ_p:

(i) $$T\in\Theta_p \Rightarrow T = [P|-Q|-R] \text{ with } \det_{\mathbb{R}(z)}P\neq 0,\ \det_{\mathbb{R}(z)}R\neq 0; \tag{4.3.2}$$

(ii) $$T\in\Theta_p \Rightarrow \operatorname{rank}_{\mathbb{C}} T(\lambda,\lambda^{-1}) = p \text{ for all } \lambda\in\mathbb{C}\backslash\{0\}, \text{ or equivalently}$$
$$P,Q,R \text{ are left coprime with respect to } \mathbb{R}[z,z^{-1}]. \tag{4.3.3}$$

(iii) $$(T\in\Theta_p \wedge \tilde{M}_p(T)\in\mathcal{M}^{pe}_{p,m}(L)) \Rightarrow \lim_{z\rightarrow\infty} P^{-1}R = L \tag{4.3.4a}$$
$$(T\in\Theta_p \wedge \tilde{M}_p(T)\in\mathcal{M}^{oe}_{p,m}(L)) \Rightarrow P^{-1}R = L. \tag{4.3.4b}$$

Condition (i) refers to the general properties of an (i/o/pr)–model, condition (ii) to its controllability, and condition (iii) refers to the discriminability of the model sets by $J^+_{LS\infty}$. Assuming that we are dealing with a discriminable model set, the question whether $\tilde{M}_p$ is identifiable by $J^+_{LS\infty}$ or not, is now represented by the question if $\tilde{M}_p$ is (almost) bijective, i.e.

$$(\tilde{M}_p(T_1) = \tilde{M}_p(T_2)) \Rightarrow T_1 = T_2 \text{ for (almost) all } T_1,T_2 \in \Theta_p. \tag{4.3.5}$$

In the first part of this section we will consider the conditions that have to be imposed on a set of polynomial matrices Θ_p, in order to induce an (almost) bijective parametrization. It has already been discussed before that for polynomial matrix representations of models: $\tilde{M}_p(T_1) = \tilde{M}_p(T_2)$ if and only if $T_1=UT_2$ with $U \in \mathbb{R}^{pxp}[z,z^{-1}]$ a polynomial matrix that is unimodular with respect to $\mathbb{R}[z,z^{-1}]$. Consequently, satisfying eq. (4.3.5) within a parametrization induced by Θ_p, comes down to taking care of the fact that within Θ_p there are (almost) no matrix pairs T_1,T_2 that are related through a premultiplication by a unimodular matrix. Note that the operation of premultiplication by a unimodular matrix U defines an equivalence relation on Θ_p.

Definition 4.3.1.
Let $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ be a set of polynomial matrices having full row rank. Two elements $T_1,T_2 \in \Theta_p$ are equivalent, denoted $T_1 \sim T_2$ if there exists a polynomial matrix $U(z,z^{-1}) \in \mathbb{R}^{pxp}[z,z^{-1}]$ that is unimodular with respect to $\mathbb{R}[z,z^{-1}]$, such that $T_2=UT_1$. □

Two polynomial matrices T_1,T_2 are equivalent if they represent the same model $\tilde{M}_p(T_1)=\tilde{M}_p(T_2)$. In order to arrive at a parametrization that is bijective, a set Θ_p of polynomial matrices has to be constructed that does not contain distinct matrices T_1 and T_2 that are equivalent.
To this end a general set of conditions to be imposed on Θ_p is formulated in the following theorem.

Theorem 4.3.2.
Consider a set Θ_p of full row rank polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, and let any two elements $T_1,T_2 \in \Theta_p$ be written as $T_1=[P_1|-Q_1|-R_1]$, $T_2=[P_2|-Q_2|-R_2]$. Consider the following conditions:

(i) For all $T_1,T_2 \in \Theta_p$: $\Gamma_{hc}(P_1) = \Gamma_{hc}(P_2)$, nonsingular; (4.3.6)

(ii) For all $T_1,T_2\in\Theta_p$: $\mu_i^{(u)}(P_1) = \mu_i^{(u)}(P_2)$; i=1,.,p; (4.3.7)

(iii) There is a set of integers $n_1, \ldots, n_{p+m+p}$ such that for all $T\in\Theta_p$:

a. $\mu_i^{(\ell)}(T) \geq n_i$, and (4.3.8a)

b. rank Γ = p, where $\Gamma\in\mathbb{R}^{px(p+m+p)}$ is defined by
$T = \Gamma\Lambda(z) + \bar{T}$, with $\Lambda(z)=\text{diag}(z^{n_1}, \ldots, z^{n_{p+m+p}})$, $\mu_i^{(\ell)}(\bar{T})>n_i$; (4.3.8b)

(iv) For all $T_1,T_2\in\Theta_p$: $\delta^{(\ell)}(T_1) = \delta^{(\ell)}(T_2) = \nu_i^{(\ell)}(T_j)$; i=1,.,p; j=1,2; (4.3.9a)

and rank $\Gamma_{lr}(T_1)$ = rank $\Gamma_{lr}(T_2)$ = p. (4.3.9b)

If conditions (i),(ii) and (iii) or (i),(ii) and (iv) are satisfied, then

$$\{T_1,T_2\in\Theta_p \wedge T_1\sim T_2\} \Rightarrow T_1=T_2,$$

and consequently Θ_p induces a bijective polynomial matrix parametrization. □

Proof. See Appendix A15.

Theorem 4.3.2 gives sufficient conditions for a parametrization to be bijective, by restricting the polynomial matrices in the set Θ_p by their column degrees. We will show that the conditions (i),(ii) and (iii), are generally used when dealing with BDE–parametrizations, whereas the conditions (i),(ii) and (iv) generally refer to the use of FDE–parametrizations.

The class of so called monic ARMA forms, as represented in equation (3.3.1), can be characterized by the following specifications:

- $\Gamma_{hc}(P) = I$ (4.3.10)
- $\mu_i^{(u)}(T) = 0$, i=1,.,p+m+p; (4.3.11)
- $\mu_i^{(\ell)}(T) \geq n_i$, with n_i a prespecified set of integers, i=1,.,p+m+p. (4.3.12)

With condition (iii), theorem 4.3.2 gives the additional restriction that has to be satisfied in order to achieve a bijective parametrization. The following example illustrates that this third condition indeed is required, or in other words, that (4.3.10)–(4.3.12) are not sufficient to obtain a unique representation.

Example 4.3.3.

Consider a set of matrices $\Theta_p\subset\mathbb{R}^{2x(2+1+2)}[z,z^{-1}]$, restricted by:

- $T\in\Theta_p \Rightarrow \Gamma_{hc}(P) = I$; and
- $T\in\Theta_p \Rightarrow \mu_i^{(u)}(T)=0$, i=1,.,5 and $\{\mu_i^{(\ell)}(T)\}_{i=1,.,5}=\{-1,-1,-2,-1,-1\}$.

Let $T_1 = \left[\begin{array}{cc|c|cc} 1+z^{-1} & z^{-1} & z^{-2} & 1+z^{-1} & z^{-1} \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$,

and consider the unimodular matrix $U(z,z^{-1}) = \left[\begin{array}{cc} 1 & -az^{-1} \\ 0 & 1 \end{array}\right]$ with $a \in \mathbb{R}$.

Then $T_2 = UT_1 = \left[\begin{array}{cc|c|cc} 1+z^{-1} & (1-a)z^{-1} & z^{-2}-az^{-1} & 1+z^{-1} & (1-a)z^{-1} \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \in \Theta_p$.

Consequently T_1 and T_2 are both an element of Θ_p, and they represent the same model $\tilde{M}_p(T_1) = \tilde{M}_p(T_2)$, while $T_1 \neq T_2$ for $a \neq 0$. □

The additional restriction as formulated in condition (iii) of theorem 4.3.2, is satisfied if the trailing column coefficient matrix $\Gamma_{lc}(T)$ is forced to have full row rank. In the literature this trailing coefficient matrix is often called the column end matrix. One can easily verify that this condition is not satisfied for the matrices T_1 and T_2 in example 4.3.3.

As an alternative, condition (iii) of theorem 4.3.2 could have been formulated as a requirement that $\mu_i^{(\ell)}(T_1) = \mu_i^{(\ell)}(T_2)$, i=1,.,p+m+p and rank $\Gamma_{lc}(T_1)$ = rank $\Gamma_{lc}(T_2)$ = p. However the presented formulation in (4.3.8) is less severe and allows that specific matrices in the set do not reach the prescribed minimum values of the column degrees $\mu_i^{(\ell)}(T)$. This leads to a zero column in the corresponding coefficient matrix Γ, but it does not necessarily result in a rank deficiency of Γ.

Based on the results of theorem 4.3.2, we will now formulate two corollaries, in which specific sets Θ_p will be formulated that generate bijective parametrizations. The first corollary will be directed towards specification of the lower column degrees (often used in BDE–parametrizations); the second one will be directed towards the specification of lower row degrees (often used in FDE–forms).

Corollary 4.3.4.

Consider a set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ and let any element $T \in \Theta_p$ be written as $T=[P|-Q|-R]$.

Consider a set of integers $\{(n_i^{(u)})_{i=1,.,p}, (n_j^{(\ell)})_{j=1,.,p+m+p}\}$, with $n_i^{(u)}, n_j^{(\ell)} \in \mathbb{Z}$, (i=1,.,p; j=1,.,p+m+p), and let Θ_p be defined to contain all matrices $T \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$,

satisfying the following conditions.

(i) $\Gamma_{hc}(P) = I$ (4.3.13)

(ii) $\mu_i^{(u)}(P) = n_i^{(u)}$; i=1,.,p; (4.3.14)

(iii) $\mu_j^{(\ell)}(T) \geq n_j^{(\ell)}$, j=1,.,p+m+p; (4.3.15)

(iv) rank Γ = p, where $\Gamma \in \mathbb{R}^{p \times (p+m+p)}$ is defined by

$T = \Gamma\Lambda(z) + \bar{T}$, with $\Lambda(z)=\text{diag}(z^{n_1^{(\ell)}}, . , z^{n_{p+m+p}^{(\ell)}})$, $\mu_j^{(\ell)}(\bar{T}) > n_j^{(\ell)}$; (4.3.16)

(v) $\text{rank}_{\mathbb{C}}\, T(\lambda,\lambda^{-1}) = p$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. (4.3.17)

Then Θ_p induces a bijective parametrization. If Θ_p is restricted to satisfy conditions (i)–(iii) only, then the induced parametrization is almost bijective. □

Proof. This corollary follows from conditions (i),(ii) and (iii) of theorem 4.3.2 and the consideration that conditions (iv) and (v) in this corollary are generic properties of matrices T within Θ_p, as defined by the conditions (i)–(iii). □

Note that the parametrization that follows from conditions (i)–(iii) of the corollary is a differentiable parametrization. The specific choice $n_i^{(u)}=0$, $i \in \underline{p}$, corresponds with a BDE–parametrization. The monic ARMA forms with prescribed column degrees (cf. equations 4.3.10–4.3.12), are an example of this situation. For these forms, the results formulated in the corollary correspond with the identifiability results as presented in Hannan (1971).

Condition (v) in corollary 4.3.4 is not necessary for obtaining a bijective parametrization. However, this condition has been incorporated in view of the fact that we are going to use the set of matrices Θ_p in order to represent a set of controllable (i/o/pr)–models. The condition mentioned reflects this controllability.

Under the additional condition that the model set to be parametrized contains causal models only, in the sense that $P^{-1}[Q|R]$ is a proper rational matrix, it follows from proposition 3.4.14 that the specific choice $n_j^{(\ell)}=0$, j=1,.,p+m+p, leads to parametrized models with a model order $n(M)=\sum_{i=1}^{p} n_i^{(u)}$. This specific situation actually forms the bridge to the second corollary, that is based on conditions (i),(ii) and (iv) of theorem 4.3.2.

Corollary 4.3.5.

Consider a set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ and let any element $T \in \Theta_p$ be written as $T=[P|-Q|-R]$.

Consider a set of integers $\{(n_i^{(u)})_{i=1,..,p}, n^{(\ell)}\}$, and let Θ_p be defined by all $T \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ that satisfy the following conditions.

(i) $\Gamma_{hc}(P) = I$ (4.3.18)

(ii) $\mu_i^{(u)}(P) = n_i^{(u)}$; i=1,..,p; (4.3.19)

(iii) $\nu_j^{(\ell)}(T) = \delta^{(\ell)}(T) = n^{(\ell)}$, j=1,..,p+m+p; (4.3.20)

(iv) rank $\Gamma_{lr}(T) = p$; (4.3.21)

(v) $\text{rank}_{\mathbb{C}}\, T(\lambda,\lambda^{-1}) = p$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. (4.3.22)

Then Θ_p induces a bijective parametrization. If Θ_p is restricted to satisfy conditions (i)–(iii) only, then the induced parametrization is almost bijective. □

Proof. This corollary follows with similar reasoning as in the proof of corollary 4.3.4, from conditions (i),(ii) and (iv) in theorem 4.3.2. □

Again as in the previous case, the parametrization defined by the conditions (i)–(iii) in the corollary, is a differentiable parametrization.

Corollary 4.3.5 shows a similar result as corollary 4.3.4. However in this situation the lower row degrees of T have to be specified to be equal. Consequently this type of parametrization is more appropriate when dealing with FDE–parametrizations, where all lower row degrees of T are fixed to be 0. Referring to proposition 3.4.14 it follows again that in this situation with the lower row degrees of T fixed to 0 and $P^{-1}[Q|R]$ a proper rational matrix, the parametrization as given in corollary 4.3.5 generates models with model order $n(M)=\sum_{i=1}^{p} n_i^{(u)}$.

Remark 4.3.6.

In theorem 4.3.2 and corollaries 4.3.4 and 4.3.5, bijective parametrizations have been discussed by fixing the leading column coefficient matrix of the matrix P, and by restricting the corresponding upper column degrees. A dual formulation for restrictions on the leading row coefficient matrix and its corresponding row degrees can only lead to bijective forms if all upper row degrees are fixed to be equal. It is for this reason that the parametrization that induces the model set of causal models, as mentioned in proposition 3.3.8, in general is not identifiable. □

We will now combine the results on bijective parametrizations, as presented in this section, with the discriminability of the model sets $\mathcal{M}^{pe}_{p,m}(L)$ and $\mathcal{M}^{oe}_{p,m}(L)$, in order to obtain parametrizations for these model sets that are identifiable by $J^{+}_{LS\infty}$. This step is straightforward, since we have to add only one condition on the parametrizations presented in corollaries 4.3.4 and 4.3.5. This is formulated in the following proposition.

Proposition 4.3.7.
Let $\tilde{M}_p$ be a polynomial matrix parametrization induced by a set of full row rank matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, and let $\tilde{M}_p$ be bijective (almost bijective) as formulated in corollary 4.3.4 or 4.3.5.
Then $\tilde{M}_p$ is a strictly identifiable (identifiable) parametrization for $\mathcal{M}^{pe}_{p,m}(L)$ or $\mathcal{M}^{oe}_{p,m}(L)$ if Θ_p additionally satisfies:

$$\lim_{z\to\infty} P^{-1}R = L \quad \text{(PE), or} \tag{4.3.23}$$

$$P^{-1}R = L \quad \text{(OE).} \tag{4.3.24}$$ □

Proof. Follows by applying the definitions. □

For the specific situation of BDE–parametrizations, with $\mu_i^{(u)}(P)=0$, $i\in\underline{p}$, satisfying the conditions of corollary 4.3.4 or 4.3.5, the PE–condition (4.3.23) can simply be incorporated by the restriction $\mu_i^{(u)}(R)=0$, $i\in\underline{p}$, and $\Gamma_{hc}(R)=L$. In other situations the additional conditions (4.3.23) and (4.3.24) are much harder to incorporate.

Remark 4.3.8.
Note that there is an essential difference between the roles of the matrix L as meant in the previous proposition, and the specific choice $\Gamma_{hc}(P)=I$ in the corollaries 4.3.4 and 4.3.5. A different choice of L leads to a different model set that is parametrized, and generally to different properties of selected optimal models (see section 4.2.4). However, choosing a specific nonsingular matrix for $\Gamma_{hc}(P)$, is only a matter of representation of the models in the set. A different choice for $\Gamma_{hc}(P)$ will not influence the model set, but only the specific parametrization. □

In the sequel of this subsection we will focus on the construction of identifiable

parametrizations for prediction error and output error model sets containing all models with a prespecified value of the model order n(M). For the most important part of this discussion, it is not relevant what kind of (i/o/pr)–models are involved (PE,OE). For this reason we will deal with a general set of input–output models with a fixed model order. Later on we will return to the specific implications that follow for corresponding PE and OE model sets.

Definition 4.3.9.
Denote with:
$\Sigma_{p,m}(n) := \{M \in \Sigma_{p,m} |$ M is a causal input–output system, M is controllable having model order $n(M)=n\}$. □

Remark 4.3.10.
Note that since we are dealing with controllable models that are causal when considering y as the output signal of the system, the model order as defined in chapter 3 coincides with the McMillan degree of the corresponding transfer function $H_y(z)$, as well as with the minimal state space dimension in any minimal (A,B,C,D) state space representation. □

Remark 4.3.11.
Because of the use of controllable models there is no difference in finding unique representations of models in terms of behaviours, compared to constructing unique representations of transfer functions, as is a common practice in the literature on identifiable parametrizations. In both situations the equivalence relation that is involved when dealing with polynomial matrix representations, is defined by a premultiplication with a unimodular matrix. The main difference is that in our framework the polynomial matrix parametrization can be represented in the two shift operators σ and σ^{-1}. □

The problem of constructing bijective polynomial matrix parametrizations for the model set $\Sigma_{p,m}(n)$ comes down to the problem of finding a set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m)}[z,z^{-1}]$, such that Θ_p induces a parametrization $\tilde{M}_p$ that satisfies:

$$\Sigma_{p,m}(n) = Im(\tilde{M}_p), \tag{4.3.25}$$

$$\text{and } \tilde{M}_p(T_1) = \tilde{M}_p(T_2) \text{ implies } T_1 = T_2 \text{ for almost all } T_1, T_2 \in \Theta_p. \tag{4.3.26}$$

It has been shown by Hazewinkel and Kalman (1976) that for p>1 there does not exist a single differentiable parametrization that is able to uniquely parametrize

the set $\Sigma_{p,m}(n)$ of all proper dynamical systems (transfer functions) with a prescribed McMillan degree. Consequently in this situation there will not exist a single differentiable polynomial matrix parametrization that satisfies (4.3.25) and (4.3.26).

However it has also been shown that the model set concerned can be represented as a union of a finite number of model sets that indeed can be uniquely parametrized with differentiable parametrizations. In other words: there exists a finite number of parametrizations $\tilde{M}^{(i)}$, such that

$$\Sigma_{p,m}(n) = \bigcup_i \mathcal{M}_i \tag{4.3.27}$$

$$\text{with } \mathcal{M}_i = Im(\tilde{M}^{(i)}) \tag{4.3.28}$$

and $\tilde{M}^{(i)}$ a differentiable parametrization that is almost bijective.

Two philosophies can be followed by constructing such parametrizations, by applying canonical (non–overlapping) forms or pseudo–canonical (overlapping, or multi–structural) forms. In both situations the model set $\Sigma_{p,m}(n)$ is written as the union of a finite number of model sets as in equation (4.3.27); the terms "canonical" (non–overlapping) and "pseudo–canonical" (overlapping) straightforwardly then refer to the situation that the model sets $\mathcal{M}_i$ are disjoint ($\mathcal{M}_i \cap \mathcal{M}_j$ is empty for $i \neq j$) or overlapping ($\mathcal{M}_i \cap \mathcal{M}_i$ is non–empty for $i \neq j$).

This problem has been considered for both (A,B,C,D) state space parametrizations, and for polynomial matrix parametrizations in the shift operator σ.

A solution can be found in terms of a set of observability canonical forms, establishing that

$$\Sigma_{p,m}(n) = \bigcup_{\substack{p \\ \sum_{i=1} \gamma_i = n}} Im(\tilde{M}^{(\gamma_1,..,\gamma_p)}) \tag{4.3.29}$$

where – $Im(\tilde{M}^{(\gamma_1,..,\gamma_p)})$ contains all models $M \in \Sigma_{p,m}(n)$ that can be represented with an (A,B,C,D) state space representation having observability indices $(\gamma_1,..,\gamma_p)$;

– every single parametrization $\tilde{M}^{(\gamma_1,..,\gamma_p)}$ is almost bijective and differentiable;

– $Im(\tilde{M}^{(\gamma_1,..,\gamma_p)}) \cap Im(\tilde{M}^{(\tilde{\gamma}_1,..,\tilde{\gamma}_p)}) = \emptyset$ for $(\gamma_i)_{i=1,..,p} \neq (\tilde{\gamma}_i)_{i=1,..,p}$;

After having specified the set of observability indices, one is able to parametrize

this set of models with a differentiable and almost bijective parametrization. The specifications of this set of canonical forms in terms of a polynomial matrix parametrization is given in table 4.1, where the parametrization $\tilde{M}^{(\gamma_1,..,\gamma_p)}$ is defined through a specification of the corresponding set of polynomial matrices $\Theta_p^{(\gamma_1,..,\gamma_p)}$.

Observability canonical forms	Overlapping forms
$\Gamma_{hc}(P) = I$ $\mu_i^{(u)}(P) = n_i$, i=1,.,p $\nu_i^{(\ell)}(T) = 0$, i=1,.,p $\text{rank}_{\mathbb{C}}\ T(\lambda,\lambda^{-1}) = p \quad \forall\lambda\in\mathbb{C}$	$\Gamma_{hc}(P) = I$ $\mu_i^{(u)}(P) = n_i$, i=1,.,p $\nu_i^{(\ell)}(T) = 0$, i=1,.,p $\text{rank}_{\mathbb{C}}\ T(\lambda,\lambda^{-1}) = p \quad \forall\lambda\in\mathbb{C}$
$\nu_i^{(u)}(P) = \mu_i^{(u)}(P)$, i=1,.,p $\nu_i^{(u)}(P) = \nu_i^{(u)}(T)$, i=1,.,p $\Gamma_{hr}(P) = I + F$ with $F\in\mathbb{R}^{p\times p}$, lower triangular	$P^{-1}Q$ proper

TABLE 4.1 Specification of polynomial matrix parametrizations of $\Sigma_{p,m}(n)$ in terms of canonical observability forms ($n_i=\gamma_i$), and overlapping forms ($n_i=\zeta_i$), induced by $\Theta_p^{(n_1,..,n_p)}\subset\mathbb{R}^{px(p+m)}[z,z^{-1}]$ with $\sum_{i=1}^{p} n_i=n$ and $T\in\Theta_p$ written as $T=[P|-Q]$.

An analysis of these forms can be found in more detail in Luenberger (1967), Popov (1972) and Denham (1974) for state space parametrizations, and Boniventi and Guidorzi (1971), Guidorzi (1975), Beghelli and Guidorzi (1976) and Guidorzi

(1981) for polynomial matrix parametrizations.
The use of so called pseudo–canonical or overlapping forms is an alternative way of solving the problem, establishing that in a similar way as described above:

$$\Sigma_{p,m}(n) = \bigcup_{\substack{\sum_{i=1}^{p} \zeta_i = n}} Im(\tilde{M}^{(\zeta_1,..,\zeta_p)}) \qquad (4.3.30)$$

where – the set of integers $(\zeta_i)_{i=1,..,p}$ are now the so called *pseudo–structure indices*, which are not related anymore to any observability indices as in the case of the canonical forms;
– every single parametrization $\tilde{M}^{(\zeta_1,..,\zeta_p)}$ is almost bijective and differentiable;
– $Im(\tilde{M}^{(\zeta_1,..,\zeta_p)}) \cap Im(\tilde{M}^{(\zeta_1^*,..,\zeta_p^*)}) \neq \emptyset$ for $(\zeta_i)_{i=1,..,p} \neq (\zeta_i^*)_{i=1,..,p}$;

The specification of these forms has also been given in table 4.1.
More details on these forms can be found in Glover and Willems (1974), Ljung and Rissanen (1976), van Overbeek and Ljung (1982) and Gevers and Wertz (1984) for state space representations, and Guidorzi and Beghelli (1982), Corrêa and Glover (1984a), and Gevers and Wertz (1984) for polynomial matrix representations.
A survey of both canonical and overlapping forms is given in Gevers and Wertz (1987).

Remark 4.3.12.
Note that the conditions on Θ_p that are common for both parametrizations in table 4.1 are exactly the conditions as presented in corollary 4.3.5 on bijective polynomial matrix parametrizations. Because of the specific choice $\nu_i^{(\ell)}(T)=0$, i=1,..,p, the combination of both conditions (iv) and (v) in this corollary lead to the requirement that $\text{rank}_{\mathbb{C}} T(\lambda,\lambda^{-1})=p$ for all $\lambda \in \mathbb{C}$, or equivalently: P and Q are left coprime with respect to $\mathbb{R}[z]$. □

Remark 4.3.13.
The rank condition that is present in table 4.1 causes the corresponding parametrization to be nondifferentiable and bijective. When this rank condition is neglected, the resulting parametrizations become differentiable and almost bijective. □

Remark 4.3.14.

In the situation of the observability canonical forms, apart from the column degrees of P also its row degrees have been specified, together with the row properness of P. Causality of the concerning models can now simply be achieved by requiring that $\nu_i^{(u)}(P)=\nu_i^{(u)}(T)$. This follows directly from proposition 3.3.8. Contrary to this, in the situation of the overlapping forms, the row degrees of P have not been specified, and consequently the causality of the models can not be achieved by simple restrictions on the degrees of polynomials in T. The (nonlinear) restrictions on T required for the causality constraints are investigated in Beghelli and Guidorzi (1983) and Corrêa and Glover (1984a). Extensions to incorporate specific time delays are discussed in Janssen (1987) and Janssen and Damen (1987). The direct consequence of these nonlinear constraints on T is that the polynomial matrix parametrization actually looses its property of being differentiable. No additional constraints are required in a corresponding (A,B,C,D) state space parametrization, since the causality of the models in that parametrization is guaranteed. □

Remark 4.3.15.

Every model $M\in\Sigma_{p,m}(n)$ has its unique representation in the parametrization that is defined by the set of canonical forms, whereas it has several different representations in the parametrization with overlapping forms. Moreover for any choice of pseudo–structure indices satisfying $\sum_{i=1}^{p}\zeta_i=n$, $Im(\tilde{M}^{(\zeta_1,..,\zeta_p)})$ is an open and dense subset of $\Sigma_{p,m}(n)$; consequently for any such choice of pseudo–structure indices, almost all models within $\Sigma_{p,m}(n)$ can be represented.

This property has been a motivation for the use of overlapping forms. Contrary to the situation of canonical forms, where first the proper observability indices have to be estimated, in the overlapping forms any set of pseudo structure indices generically will be appropriate. However, note that, despite of this generic property, for each set of pseudo–structure indices chosen, specified classes of systems are excluded from the model set. □

In order to apply these results to identifiable parametrizations for PE and OE model sets, let us first formally denote the model sets to be considered.

Definition 4.3.16.

Denote with:

$\mathcal{M}^{pe}_{p,m}(n,L) := \{M \in \hat{\Sigma}_{p,m} \mid M \in \mathcal{M}^{pe}_{p,m}(L)$, with $L \in \mathbb{R}^{pxp}$ non–singular, M a causal system when y considered as output, and model order $n(M)=n\}$.

$\mathcal{M}^{oe}_{p,m}(n,L) := \{M \in \hat{\Sigma}_{p,m} \mid M \in \mathcal{M}^{oe}_{p,m}(L)$, with $L \in \mathbb{R}^{pxp}$ non–singular, M a causal system when y considered as output, and model order $n(M)=n\}$. □

Since the models in these model sets $\mathcal{M}^{pe}_{p,m}(n,L)$ and $\mathcal{M}^{oe}_{p,m}(n,L)$ can be considered as input–output models with u and e as inputs and y as output, they can be parametrized by applying the same observability canonical forms and overlapping forms as specified in table 4.1. However one has to take into account that in this new situation the set of polynomial matrices is extended to $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$,

	Observability canonical forms	Overlapping forms
PE	$\nu_i^{(u)}(P) = \mu_i^{(u)}(R)$, i=1,..,p $\Gamma_{hr}(R) = \Gamma_{hr}(P)\ L$	$\lim_{z\to\infty} P^{-1}R = L$
OE	R = PL	R = PL

TABLE 4.2 Restrictions on the set of polynomial matrices $\Theta_p^{(n_1,..,n_p)} \in \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ with $\sum_{i=1}^{p} n_i = n$ and $T=[P|-Q|-R] \in \Theta_p$, as additional requirements on parametrizations in observability canonical forms and overlapping forms for the model sets $\mathcal{M}^{pe}_{p,m}(n,L)$ and $\mathcal{M}^{oe}_{p,m}(n,L)$, in order to obtain identifiability by $J^{+}_{LS\infty}$.

corresponding to the addition of a residual part, reflected by the polynomial matrix R. Because of the specific properties of the model sets, the conditions as formulated in table 4.1 now have to be extended with the straightforward requirements:

$$\lim_{z\to\infty} P^{-1}R = L \qquad \text{for } \mathcal{M}^{pe}_{p,m}(n,L) \tag{4.3.31}$$

and

$$P^{-1}R = L \qquad \text{for } \mathcal{M}^{oe}_{p,m}(n,L). \tag{4.3.32}$$

The additional requirements are listed in table 4.2.
With these additional requirements as formulated in table 4.2 the two parametrizations discussed, are parametrizations of $\mathcal{M}^{pe}_{p,m}(n,L)$ and $\mathcal{M}^{oe}_{p,m}(n,L)$ that are strictly identifiable by the identification criterion $J^{+}_{LS\infty}$.

Remark 4.3.17.
Since the identification criterion $J^{+}_{LS\infty}$ regularly partitions any set of (i/o/pr)-models $\mathcal{M}$, it follows that for a given data sequence v^{+}, the set of selected optimal models within $\mathcal{M}$, $J^{+}_{LS\infty}(v^{+},\mathcal{M})$, can always be evaluated by considering any union of subsets of $\mathcal{M}$ that covers $\mathcal{M}$. Consequently the partitioning of $\mathcal{M}^{pe}_{p,m}(n,L)$ and $\mathcal{M}^{oe}_{p,m}(n,L)$ that is created by the finite number of (canonical or overlapping) parametrizations is permitted, and does not influence the final result of the identification procedure, in terms of the sets $J^{+}_{LS\infty}(v^{+},\mathcal{M}^{pe}_{p,m}(n,L))$ and $J^{+}_{LS\infty}(v^{+},\mathcal{M}^{oe}_{p,m}(n,L))$.

4.3.3. Identifiable parametrizations for EE model sets

When discussing parametrizations that are identifiable by the identification criterion $J^{+}_{LS\infty}$, the situation of equation error model sets is essentially different from the situation as described in the previous section. For PE and OE model sets it has been noticed that, after the choice for a model set that is discriminable by $J^{+}_{LS\infty}$, like $\mathcal{M}^{pe}_{p,m}(n,L)$ and $\mathcal{M}^{oe}_{p,m}(n,L)$, a parametrization problem remains. In a polynomial matrix parametrization, an equivalence relation between polynomial matrices is involved that is represented by a premultiplication with a polynomial matrix that is unimodular with respect to $\mathbb{R}[z,z^{-1}]$. The corresponding equivalence classes have to be uniquely parametrized in order to arrive at identifiable parametrizations. In order to illustrate that the situation for equation error model sets is quite different, consider the following.

Let $\mathcal{M}^{ee}$ be a set of equation error models that is discriminable by $J^{+}_{LS\infty}$ and let $\mathcal{M}^{ee}$ be represented by a parametrization $\tilde{M}_p$ that is induced by the set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, with $T\in\Theta_p$ written as $T=[P|-Q|-R]$.
By definition an equation error model in a polynomial matrix representation satisfies that R is unimodular with respect to $\mathbb{R}[z,z^{-1}]$. Now using the property that the induced model is invariant for a premultiplication of T by a unimodular matrix, there evidently exists a set of matrices $\Theta'_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$ that induces $\mathcal{M}^{ee}$ satisfying:

$$T'\in\Theta'_P \Rightarrow T'=[P'|-Q'|-I]. \tag{4.3.33}$$

This form has been reached by premultiplication of all elements in Θ_p by $R(z,z^{-1})^{-1}$. Note that within Θ'_p there does not exist a parametrization problem anymore, since all different elements in Θ'_p refer to different models $M\in\mathcal{M}^{ee}$:

$$\{T_1,T_2\in\Theta'_P \wedge \tilde{M}_p(T_1)=\tilde{M}_p(T_2)\} \Rightarrow T_1=T_2 \tag{4.3.34}$$

The consequence of this property is that all EE model sets that are discriminable by $J^{+}_{LS\infty}$, directly yield a parametrization that is strictly identifiable by $J^{+}_{LS\infty}$. This is formalized in the following theorem.

Theorem 4.3.18.
Let $\mathcal{M}_{\Theta_p}$ be a set of equation error (i/o/pr)–models that is induced by a set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, and that is discriminable by $J^{+}_{LS\infty}$.
If Θ_p satisfies:

$$\{T_1=[P_1|-Q_1|-R_1]\in\Theta_p \wedge T_2=[P_2|-Q_2|-R_2]\in\Theta_p\} \Rightarrow R_1=R_2 \tag{4.3.35}$$

then the parametrization $\tilde{M}_p: \Theta_p \rightarrow \mathcal{M}_{\Theta_p}$ is bijective, and consequently it is strictly identifiable by $J^{+}_{LS\infty}$. □

Proof. Since $R_1=R_2$ it follows directly that $\{T_1=UT_2$ with U polynomial$\} \Rightarrow U=I$, showing that there do not exist distinct elements in Θ_P that induce the same model. □

In the situation that has been discussed in equation (4.3.33), the fixed matrix R as meant in the above theorem has been chosen to be equal to I. Note that any

choice for Θ_p will be appropriate if it satisfies that R is a fixed polynomial matrix that is unimodular with respect to $\mathbb{R}[z,z^{-1}]$.

The consequence of this theorem is that for equation error model sets there actually does not exist a parametrization problem, but there only exists a problem of achieving discriminability.
This situation is caused by the very specific way in which the residuals are connected to the input–output signals within an equation error (i/o/pr)–model, i.e. the fact that the residuals are observable from these input–output signals.

In order to present parametrizations for EE model sets that are discriminable by $J^+_{LS\infty}$, we necessarily have to return to the results of section 4.2 concerning discriminability. As reported in section 4.2.3, discriminability of equation error model sets by $J^+_{LS\infty}$ can be obtained in several ways. One of the options is to choose prediction error or output error form restriction, i.e. by requiring that the concerning set of polynomial matrices Θ_p that induces the model set, satisfies:

$$\lim_{z\to\infty} P^{-1}R = L \quad \text{(PE)} \tag{4.3.36}$$

or

$$P^{-1}R = L \quad \text{(OE)} \tag{4.3.37}$$

with $L\in\mathbb{R}^{p\times p}$, nonsingular, and fixed over the set of models.
Identifiability results for these model sets are formulated in the following corollary.

Corollary 4.3.19.
Let $\mathcal{M}_{\Theta_p}$ be a set of equation error (i/o/pr)–models induced by $\Theta_p\subset\mathbb{R}^{p\times(p+m+p)}[z,z^{-1}]$. Let $L\in\mathbb{R}^{p\times p}$ be a given nonsingular matrix, and let an element of Θ_p be written as $T=[P|-Q|-R]$.
Consider the following conditions:

(i) For all $T\in\Theta_p$: $\lim_{z\to\infty} P = I$ and $R=L$ (4.3.38)

(ii) For all $T\in\Theta_p$: $P = I$ and $R=L$. (4.3.39)

If Θ_p satisfies (i) or (ii) then the parametrization $\tilde{M}_p\colon \Theta_p\to\mathcal{M}_{\Theta_p}$ is strictly identifiable by $J^+_{LS\infty}$. □

Proof. The result follows directly from theorems 4.2.9 and 4.3.18. □

Situation (i) of corollary 4.3.19 represents a parametrization in monic ARMA form,

since matrix P is restricted to be of the form:

$$P(z^{-1}) = I + P_1 z^{-1} + P_2 z^{-2} + \cdot \cdot + P_k z^{-k}. \tag{4.3.40}$$

Note that this monic ARMA form is strictly identifiable by $J^{+}_{LS\infty}$ without any additional constraints, whereas for the similar form in a general prediction error or output error model set, additional conditions are required in terms of:

- the trailing column coefficient matrix (column end matrix) having full row rank, and
- the matrices (P,Q,R) being left coprime with respect to $\mathbb{R}[z,z^{-1}]$;

(see corollary 4.3.4 and concerning remarks thereafter).
For equation error models the left coprimeness of (P,Q,R), or equivalently the controllability of the corresponding models, is guaranteed because of the fact that R is unimodular; consequently no additional conditions are required.

Situation (ii) of corollary 4.3.19 refers to parametrizations in a moving average (MA) form. The coefficient matrices of the polynomial matrix $Q(z,z^{-1})$ now represent the Markov parameters of the transfer function between input u and output y within the model.

We will now turn to equation error model sets for which discriminability is obtained by employing the specific properties of equation error models. For this situation two classes of discriminable model sets have been presented in theorems 4.2.11 and 4.2.12, directed towards restrictions on row degrees (theorem 4.2.11), or restrictions on column degrees (theorem 4.2.12) of the polynomial transfer function $H_e(z)=[H_{ey}|H_{eu}]\in\mathbb{R}^{pxp}[z]$. As mentioned in section 4.2, the two forms presented are only very specific choices out of a great number of possibilities for which discriminability by $J^{+}_{LS\infty}$ can be obtained. We will reformulate the results of the theorems mentioned above, in the following corollary.

Corollary 4.3.20.

Let Θ_p be a set of polynomial matrices $\Theta_p \subset \mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, such that $T\in\Theta_p \Rightarrow T=[P|-Q|-I]$, and Θ_p induces a model set $\mathcal{M}$.

Let $L\in\mathbb{R}^{pxp}$ be nonsingular, $(n_i)_{i=1,.,p}$ a sequence of integers, and n an integer.

Consider the following conditions:

(i) For all $T\in\Theta_p$: $\Gamma_{hr}(P)=L$ and $\nu_i^{(u)}(P)=n_i$, i=1,.,p; (4.3.41a)

(ii) For all $T\in\Theta_p$: $\Gamma_{hc}(P)=L$ and $\mu_i^{(u)}(P)=n_i$, i=1,.,p; (4.3.41b)

(iii) For all $T \in \Theta_p$: $\Gamma_{hr}(P)=L$, $\sum_{i=1}^{p} \nu_i^{(u)}(P)=n$, and $\text{rank}[P(0)|-Q(0)]=p$; (4.3.41c)

(iv) For all $T \in \Theta_p$: $\Gamma_{hc}(P)=L$, $\sum_{i=1}^{p} \mu_i^{(u)}(P)=n$, and $\text{rank}[P(0)]=p$. (4.3.41d)

If Θ_p satisfies one of the conditions (i)–(iv) then the parametrization $\tilde{M}_p:\Theta_p \to \mathcal{M}$ is strictly identifiable by $J^+_{LS\infty}$. □

Proof. The corollary follows directly from theorems 4.2.11, 4.2.12 and 4.3.18. □

Note that specification of the leading row (column) coefficient matrix together with the corresponding row (column) degrees of the matrix P in Θ_p is sufficient to create a parametrization for an EE model set that is strictly identifiable by $J^+_{LS\infty}$.

Comparing the results of this corollary with the conditions that are required for obtaining identifiable parametrizations for PE and OE model sets, it is remarkable that the required conditions for EE model sets are much weaker than their counterparts in the PE and OE situation (cf. corollaries 4.3.4 and 4.3.5).

Remark 4.3.21.

In the case of restrictions on the column degrees of P in Θ_p, as formulated in conditions (ii) and (iv) of corollary 4.3.20, the situation

$$T=[P|-Q|-I] \text{ and } \Gamma_{hc}(P)=L$$

can be equivalently formulated by

$$T=[P|-Q|-L^{-1}] \text{ and } \Gamma_{hc}(P)=I.$$

This is due to the fact that premultiplication of T with a constant nonsingular matrix, directly operates on the leading column coefficient matrix of P, and does not change the corresponding model behaviour.

A similar statement for the leading row coefficient matrix, as applied in conditions (i) and (iii) of the corollary, can not be made, since a direct operation on the leading row coefficient matrix of P is represented by a postmultiplication of P. □

The identifiable parametrizations as presented in conditions (iii) and (iv) of the corollary, point to discriminable equation error model sets, which we will formally define, and in which we will add the condition that the input/output relation within the models is restricted to be causal.

Definition 4.3.22.
Denote the following equation error model sets that are discriminable by $J^{+}_{LS\infty}$;

$\mathcal{M}^{r,ee}_{p,m}(n,L) := \{M \in \hat{\Sigma}_{p,m} |$ M is an EE model, $H_{ey}(z) \in \mathbb{R}^{pxp}[z]$, $\Gamma_{hr}(H_{ey}(z))=L$,
$H_y(z)$ is a proper rational matrix, $n(M)=n\}$;

$\mathcal{M}^{c,ee}_{p,m}(n,L) := \{M \in \hat{\Sigma}_{p,m} |$ M is an EE model, $H_{ey}(z) \in \mathbb{R}^{pxp}[z]$, $\Gamma_{hc}(H_{ey}(z))=L$,
$H_y(z)$ is a proper rational matrix analytic in $z=0$,
$n(M)=n\}$; □

The question whether we can find identifiable parametrizations for model sets of models having the same model order n(M), can now be answered straightforwardly: all models within the discriminable model sets already have the same model order n(M)=n. Parametrizations of $\mathcal{M}^{r,ee}_{p,m}(n,L)$ and $\mathcal{M}^{c,ee}_{p,m}(n,L)$ that are strictly identifiable by the identification criterion $J^{+}_{LS\infty}$ are given by the sets of polynomial matrices Θ_p as defined in conditions (iii) and (iv) of corollary 4.3.20, under the additional constraint that $P^{-1}Q$ is a proper matrix, in order to ensure the requested causality.

Before we come to a formulation of the specific parametrizations for the model sets discussed above, we will first pay attention to the i/o properties of the models within the two model sets.
As mentioned before, all models in the model sets $\mathcal{M}^{r,ee}_{p,m}(n,L)$ and $\mathcal{M}^{c,ee}_{p,m}(n,L)$ by definition have model order n(M)=n. An interesting question now is: what are the properties of the i/o parts of the models in both model sets; or more specific: are all input–output models with a prespecified model order n(M)=n present in both model sets?
In the situation of OE and PE model sets, we have first parametrized all input-output models with model order n(M)=n, and afterwards the presence of residuals has been accounted for by adding restrictions on the matrix R in $T \in \Theta_p$. In the situation of EE model sets, these two steps can not be separated, because of the fact that the discriminable model sets $\mathcal{M}^{r,ee}_{p,m}(n,L)$ and $\mathcal{M}^{c,ee}_{p,m}(n,L)$ already are model sets with a fixed model order.
When we consider the i/o parts of the models in these model sets by neglecting the residual signal, we can wonder if all input–output models with model order n are present in the i/o parts of $\mathcal{M}^{r,ee}_{p,m}(n,L)$ and $\mathcal{M}^{c,ee}_{p,m}(n,L)$.

For $\mathscr{M}^{r,ee}_{p,m}(n,L)$ it can be verified that not all such input–output models are present, since not all bilaterally row proper polynomial matrices $T = [P|-Q] \in \mathbb{R}^{px(p+m)}[z]$ can be brought into the specific form $\Gamma_{hr}(P)=L$ by only premultiplication of T with a unimodular matrix. A postmultiplication of P with a constant nonsingular matrix is required for arriving at the situation that all input–output models with model order n can be represented in a form satisfying $\Gamma_{hr}(P)=L$, for L a general nonsingular matrix. A consequence of this is formulated in the following remark.

Remark 4.3.23.
For all causal input–output models with input u and output y, having model order n, there exists a nonsingular constant transformation of the output signals ($y'=Ty$ with $T\in\mathbb{R}^{pxp}$) such that the resulting model with input u and output y' is present in the i/o part of the model set $\mathscr{M}^{r,ee}_{p,m}(n,L)$. □

For $\mathscr{M}^{c,ee}_{p,m}(n,L)$ the situation is different. This is caused by the fact that by unimodular premultiplication all polynomial matrices $T = [P|-Q] \in \mathbb{R}^{px(p+m)}[z]$ can be brought to P being column proper and $\Gamma_{hc}(P)=L$. In this statement unimodularity and column properness should be considered with respect to $\mathbb{R}[z]$. Consequently the following remark can be made.

Remark 4.3.24.
All causal input–output models that have model order n, and that have no delays in the transfer function $H_{yu}(z)$, i.e. $H_{yu}(z)$ is analytical in z=0, are present in the i/o part of the model set $\mathscr{M}^{c,ee}_{p,m}(n,L)$. □

Returning to the construction of identifiable parametrizations, we will now formulate parametrizations for the two model sets $\mathscr{M}^{r,ee}_{p,m}(n,L)$ and $\mathscr{M}^{c,ee}_{p,m}(n,L)$.
Similar to the situation for PE and OE model sets, both equation error model sets can be parametrized by considering a finite number of identifiable parametrizations. This is formulated in the following corollaries.

Corollary 4.3.25.

$$\mathcal{M}^{r,ee}_{p,m}(n,L) = \bigcup_{\substack{p \\ \sum_{i=1} \rho_i = n}} Im(\tilde{M}^{(\rho_1,..,\rho_p)}_p) \tag{4.3.42}$$

with $\tilde{M}^{(\rho_1,..,\rho_p)}_p$ is induced by $\Theta^{(\rho_1,..,\rho_p)}_p \subset \mathbb{R}^{px(p+m+p)}[z]$, with $\Theta^{(\rho_1,..,\rho_p)}_p$ defined by $T=[P|-Q|-I] \in \Theta^{(\rho_1,..,\rho_p)}_p$, and

(i) $\Gamma_{hr}(P) = L$ (4.3.43a)

(ii) $\nu^{(u)}_i(P) = \nu^{(u)}_i(T) = \rho_i$, i=1,..,p (4.3.43b)

(iii) $\text{rank}[P(0)|-Q(0)]=p$ (4.3.43c)

Every single parametrization $\tilde{M}^{(\rho_1,..,\rho_p)}_p$ is strictly identifiable by $J^+_{LS\infty}$, and the collection of model sets (4.3.42) is nonoverlapping, i.e.

$$Im(\tilde{M}^{(\rho_1,..,\rho_p)}_p) \cap Im(\tilde{M}^{(\tilde{\rho}_1,..,\tilde{\rho}_p)}_p) = \emptyset \quad \text{for } (\rho_i)_{i=1,..,p} \neq (\tilde{\rho}_i)_{i=1,..,p}. \qquad \square$$

Proof. This corollary follows directly from corollary 4.3.20 with the additional constraint that causality of the models in $\mathcal{M}^{r,ee}_{p,m}(n,L)$ can simply be achieved by the condition $\nu^{(u)}_i(P) = \nu^{(u)}_i(T)$, i=1,..,p. $\square$

Remark 4.3.26.
Each of the model sets $Im(\tilde{M}^{(\rho_1,..,\rho_p)}_p)$ in (4.3.42) contains all models within $\mathcal{M}^{r,ee}_{p,m}(n,L)$ that have left structure indices $(\rho_i)_{i=1,..,p}$. $\square$

Remark 4.3.27.
Every single parametrization $\tilde{M}^{(\rho_1,..,\rho_p)}_p$ in (4.3.42) is strictly identifiable by $J^+_{LS\infty}$. Without the rank condition (iii) as formulated in the corollary, the parametrizations become identifiable by $J^+_{LS\infty}$ and differentiable. $\square$

A similar corollary can be given for the model set $\mathcal{M}^{c,ee}_{p,m}(n,L)$.

Corollary 4.3.28.

$$\mathcal{M}^{c,ee}_{p,m}(n,L) = \bigcup_{\substack{p \\ \sum_{i=1} \zeta_i = n}} Im(\tilde{M}_p^{(\zeta_1,..,\zeta_p)}) \tag{4.3.44}$$

with $\tilde{M}_p^{(\zeta_1,..,\zeta_p)}$ is induced by $\Theta_p^{(\zeta_1,..,\zeta_p)} \subset \mathbb{R}^{p \times (p+m+p)}[z]$, with $\Theta_p^{(\zeta_1,..,\zeta_p)}$ defined by $T=[P|-Q|-L^{-1}] \in \Theta_p^{(\zeta_1,..,\zeta_p)}$, and

(i) $\Gamma_{hc}(P) = I$ (4.3.45a)

(ii) $\mu_i^{(u)}(P) = \zeta_i$, i=1,..,p (4.3.45b)

(iii) rank[P(0)]=p (4.3.45c)

(iv) $P^{-1}Q$ is a proper rational matrix (4.3.45d)

Every single parametrization $\tilde{M}_p^{(\zeta_1,..,\zeta_p)}$ is strictly identifiable by $J^+_{LS\infty}$, and the collection of model sets (4.3.44) is non–overlapping, i.e.

$$Im(\tilde{M}_p^{(\zeta_1,..,\zeta_p)}) \cap Im(\tilde{M}_p^{(\zeta^*_1,..,\zeta^*_p)}) = \emptyset \quad \text{for } (\zeta_i)_{i=1,..,p} \neq (\zeta^*_i)_{i=1,..,p}. \qquad \square$$

Proof. This corollary follows directly from corollary 4.3.20 with the additional constraint of causality of the i/o part of the model. Note that in this situation this causality can not simply be achieved by restrictions on the polynomial degrees of the entries in T. $\square$

Remark 4.3.29.

Each of the model sets $Im(\tilde{M}_p^{(\zeta_1,..,\zeta_p)})$ in (4.3.44) contains all models within $\mathcal{M}^{c,ee}_{p,m}(n,L)$ that have pseudo–structure indices $(\zeta_i)_{i=1,..,p}$. Note that in contrast with the situation for PE and OE model sets, every model within $\mathcal{M}^{c,ee}_{p,m}(n,L)$ has a unique set of pseudo–structure indices. $\square$

Remark 4.3.30.

Every single parametrization $\tilde{M}_p^{(\zeta_1,..,\zeta_p)}$ in (4.3.44) is strictly identifiable by $J^+_{LS\infty}$; however the parametrization is not differentiable because of the causality restriction: $P^{-1}Q$ being a proper rational matrix. $\square$

Remark 4.3.31.

Concerning the restrictions on the polynomial matrices P and Q, the parametriza-

tion presented for $\mathcal{M}^{c,ee}_{p,m}(n,L)$ comes very close to the overlapping parametrization for the model set $\Sigma_{p,m}(n)$ as presented in eq. (4.3.30) and table 4.1. The parametrization for $\mathcal{M}^{c,ee}_{p,m}(n,L)$ only contains the additional condition that rank[P(0)]=p, or in other words, that the transfer function $H_y(z)$ should have no poles in z=0 (no delays). It has to be stressed that concerning the model set $\Sigma_{p,m}(n)$ this parametrization is overlapping (even stronger: the image of every single parametrization is generic in $\Sigma_{p,m}(n)$) whereas concerning $\mathcal{M}^{c,ee}_{p,m}(n,L)$ the parametrization is nonoverlapping and in fact canonical.
Moreover note that for the parametrization discussed in corollary 4.3.28, left coprimeness of (P,Q) is not required, whereas in the situation of $\Sigma_{p,m}(n)$, this left coprimeness is required in order to satisfy the controllability of the (i/o/pr)–models. □

In the following section there will be paid some more attention to this model set $\mathcal{M}^{c,ee}_{p,m}(n,L)$, its parametrization, and its relation with the overlapping parametrization of output error model sets. Some consequences of the results as presented in this section will be illustrated in a few examples.

4.3.4 Some consequences of using identifiable parametrizations for OE and EE model sets

The consequences of the results presented in sections 4.3.2 and 4.3.3 are best illustrated by comparing equation error and output error model sets.
As a first illustration, a comparison will be made between the set of overlapping forms for $\mathcal{M}^{oe}_{p,m}(n,L)$, cf. eq. (4.3.30) and tables 4.1 and 4.2, and the very similar identifiable parametrization of the model set $\mathcal{M}^{c,ee}_{p,m}(n,L)$, presented in corollary 4.3.28. It will be illustrated that for output error model sets, the parametrization concerned is overlapping, whereas for equation error model sets it is nonoverlapping.
To this end we consider the following set of polynomial matrices:

$$\Theta^{(\zeta_1,..,\zeta_p)}_{io} \subset \mathbb{R}^{px(p+m)}[z] \tag{4.3.46}$$

with $T_{io} \in \Theta^{(\zeta_1,..,\zeta_p)}_{io}$ written as $T_{io}=[P|-Q]$, and $\Theta^{(\zeta_1,..,\zeta_p)}_{io}$ defined to contain all

matrices $T_{io} \in \mathbb{R}^{px(p+m)}[z]$ that satisfy the following conditions:

(i) $\Gamma_{hc}(P) = I$ (4.3.47a)

(ii) $\mu_i^{(u)}(P) = \zeta_i$, $i \in \underline{p}$ (4.3.47b)

(iii) (P,Q) left coprime with respect to $\mathbb{R}[z]$ (4.3.47c)

(iv) rank[P(0)] = p (4.3.47d)

(v) $P^{-1}Q$ proper (4.3.47e)

Note that for all causal and controllable input–output dynamical systems with order n and without any delays, there exists a sequence of integers $(\zeta_i)_{i=1,..,p}$ such that $\sum_{i=1}^{p} \zeta_i = n$, and such that the model is induced by an element $T_{io} \in \Theta_{io}^{(\zeta_1,..,\zeta_p)}$.
Based on this set of polynomial matrices $\Theta_{io}^{(\zeta_1,..,\zeta_p)}$, two model sets can be constructed:

$$\mathscr{M}_{pco}^{ee}(n,L) = \bigcup_{\substack{\sum_{i=1}^{p} \zeta_i = n}} \mathscr{M}_{pco}^{ee}(\Theta_{io}^{(\zeta_1,..,\zeta_p)},L) \tag{4.3.48}$$

with

$$\mathscr{M}_{pco}^{ee}(\Theta_{io}^{(\zeta_1,..,\zeta_p)},L) = \{M \in \hat{\Sigma}_{p,m} |\ M = \tilde{M}_p(T) \text{ with } T=[T_{io}|-L],\ T_{io} \in \Theta_{io}^{(\zeta_1,..,\zeta_p)}\} \tag{4.3.49}$$

and

$$\mathscr{M}_{pco}^{oe}(n,L) = \bigcup_{\substack{\sum_{i=1}^{p} \zeta_i = n}} \mathscr{M}_{pco}^{oe}(\Theta_{io}^{(\zeta_1,..,\zeta_p)},L) \tag{4.3.50}$$

with

$$\mathscr{M}_{pco}^{oe}(\Theta_{io}^{(\zeta_1,..,\zeta_p)},L) = \{M \in \hat{\Sigma}_{p,m} |\ M = \tilde{M}_p(T) \text{ with } T=[T_{io}|-P_{io}L],\ T_{io}=[P_{io}|-Q_{io}] \in \Theta_{io}^{(\zeta_1,..,\zeta_p)}\} \tag{4.3.51}$$

The results of section 4.3.3 show that the union of model sets (4.3.50) is overlapping and moreover that a model in $\mathscr{M}_{pco}^{oe}(n,L)$ generically is present in every set $\mathscr{M}_{pco}^{oe}(\Theta_{io}^{(\zeta_1,..,\zeta_p)},L)$ satisfying $\sum_{i=1}^{p} \zeta_i = n$.

Consequently one set of matrices $\Theta_{io}^{(\zeta_1,..,\zeta_p)}$ with $\sum_{i=1}^{p} \zeta_i = n$ can both construct a set of overlapping forms as well as a set of nonoverlapping forms, dependent on the type of (i/o/pr)–models that are going to be considered.
The difference in the two parametrizations concerned, is that for the output error

model set, two elements $T_1, T_2 \in \Theta_{io}^{(\zeta_1,..,\zeta_P)}$ are equivalent if $T_1 = UT_2$ with U a unimodular polynomial matrix, whereas for the equation error model set two elements $T_1, T_2 \in \Theta_{io}^{(\zeta_1,..,\zeta_P)}$ are equivalent if and only if $T_1 = T_2$.

When models are considered as polynomial matrices, and not as behaviours including a residual, this property would lead to a situation that the equivalence of two models depends on the residual type of model set that is considered.

This approach has led to the introduction of a criterion based notion of model equivalence, and was followed in Van den Hof (1987), (1988), (1989) and Janssen (1988a). It is illustrated by the block diagrams in figure 4.3.

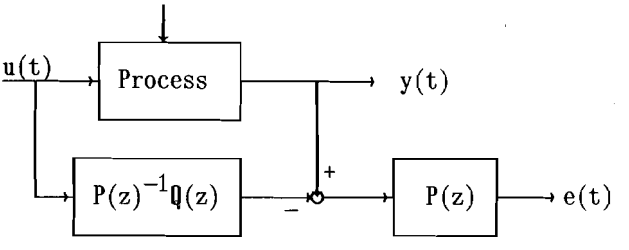

FIGURE 4.3a Block diagram of equation error model with L=I

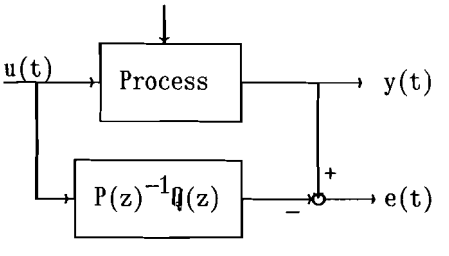

FIGURE 4.3b Block diagram of output error model with L=I.

The diagrams show that premultiplication of [P|-Q] with a unimodular polynomial matrix will not change the output error model, but will definitely affect the equation error model.

In the following example it will be illustrated that overlapping forms actually are nonoverlapping when applied to an equation error model set. It will be shown that choosing different sets of pseudo-structure indices $(\zeta_i)_{i=1,..,p}$ satisfying $\sum_{i=1}^{p} \zeta_i = n$,

will lead to different selected optimal models in an identification procedure.

Example 4.3.32

A data sequence is considered that has been obtained from measurements of a multivariable industrial process as described in Backx (1987). The process is the tube shaping part of a tube glass production process, and is considered to be represented by a dynamical system with 2 inputs (pulling speed and air pressure) and 2 outputs (tube diameter and tube wall thickness).

The least squares identification criterion has been applied to two model sets, both of the equation error type (cf. 4.3.49):

$$\mathcal{M}_1 = \mathcal{M}^{ee}_{pco}(\Theta^{(\zeta_1,\zeta_2)}_{io},L) \text{ with } L=I \text{ and } (\zeta_1,\zeta_2) = (4,2)$$

and $$\mathcal{M}_2 = \mathcal{M}^{ee}_{pco}(\Theta^{(\zeta_1,\zeta_2)}_{io},L) \text{ with } L=I \text{ and } (\zeta_1,\zeta_2) = (2,4).$$

A measured data sequence v^N has been used with N=500.

The selected optimal models $\{\hat{M}_1\}=J^N_{LS}(v^N,\mathcal{M}_1)$ and $\{\hat{M}_2\}=J^N_{LS}(v^N,\mathcal{M}_2)$ will be compared by showing the Markov parameters of the transfer functions $H_{y_2u_1}(z)$ of both selected optimal models.

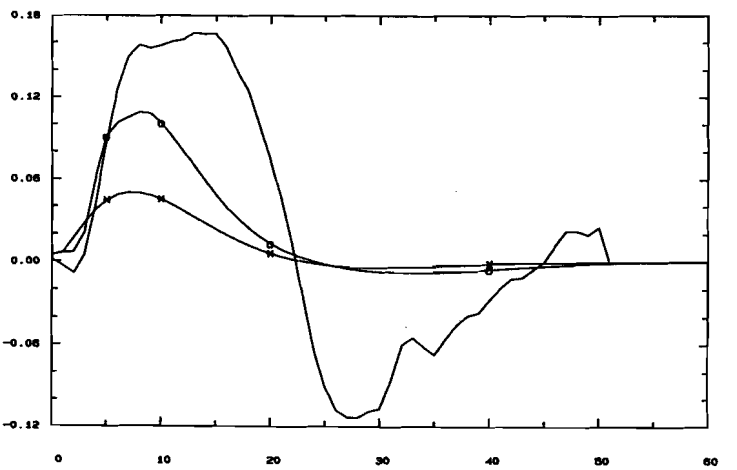

FIGURE 4.4 Markov parameters of the transfer functions $H_{y_2u_1}(z)$ of a high order moving average model (———), and of the selected optimal models $\hat{M}_1 \in \mathcal{M}_1$ (—×—) and $\hat{M}_2 \in \mathcal{M}_2$ (—o—).

In figure 4.4 these Markov parameters are presented together with the Markov parameters of a very high order moving average model, which has been shown to give a good representation of the underlying industrial process (Backx, 1987). Clearly the two models $\hat{M}_1$ and $\hat{M}_2$ show different dynamical properties; in order to find the set of selected optimal models within $\mathcal{M}^{ee}_{pco}(n,L)$ with n=6 and L=I, all different sequences of pseudo–structure indices that sum up to 6 have to be evaluated. □

The results as shown in this section 4.3 and as illustrated in this example, are supported by the results in Van den Hof and Janssen (1986), (1987), where asymptotic properties of selected optimal equation error models within the model set $\mathcal{M}^{ee}_{pco}(n,L)$ are derived in terms of its corresponding Markov parameters, and where these results are shown to be dependent on the specific set of pseudo–structure indices chosen.

As a second illustration of the results presented in this section 4.3, we will consider the situation that an equation error model set has been constructed in which all input–output models concerned are represented exactly once. In other words, we will consider a set of equation error models that is generated by a set of polynomial matrices Θ_{io} with $T_{io}=[P|-Q]\in\Theta_{io}$, such that Θ_{io} does not contain distinct elements that induce the same input–output model.
In this situation, Θ_{io} is a set of canonical forms with respect to the equivalence relation $T^{(1)}_{io}\sim T^{(2)}_{io} \Leftrightarrow T^{(1)}_{io}=UT^{(2)}_{io}$ with U polynomial and unimodular.
Such a set of canonical forms has been discussed in section 4.3.2, and similar to this set of observability canonical forms we will consider the following set of polynomial matrices which is taken from the results as presented in table 4.1:

$$\Theta^{(\gamma_1,..,\gamma_p)}_{io} \subset \mathbb{R}^{p\times(p+m)}[z] \tag{4.3.52}$$

with $T_{io}\in\Theta^{(\gamma_1,..,\gamma_p)}_{io}$ written as $T_{io}=[P|-Q]$, and $\Theta^{(\gamma_1,..,\gamma_p)}_{io}$ defined to contain all matrices $T_{io}\in\mathbb{R}^{p\times(p+m)}[z]$ that satisfy the following conditions:

(i) $\Gamma_{hc}(P) = I$ (4.3.53a)

(ii) $\mu^{(u)}_i(P) = \gamma_i, \quad i\in\underline{p}$ (4.3.53b)

(iii) $\nu^{(u)}_i(P) = \mu^{(u)}_i(P) = \nu^{(u)}_i(T_{io}), \quad i=1,..,p$ (4.3.53c)

(iv) $\Gamma_{hr}(P) = I + F$ with $F \in \mathbb{R}^{pxp}$ lower triangular (4.3.53d)

(v) (P,Q) left coprime with respect to $\mathbb{R}[z]$ (4.3.53e)

(vi) $\text{rank}[P(0)] = p$ (4.3.53f)

Similar to the first part of this section we can construct:

$$\mathcal{M}^{ee}_{can}(n,L) = \bigcup_{\substack{\sum_{i=1}^{p} \gamma_i = n}} \mathcal{M}^{ee}_{can}(\Theta^{(\gamma_1,..,\gamma_p)}_{io},L) \tag{4.3.54}$$

with

$$\mathcal{M}^{ee}_{can}(\Theta^{(\gamma_1,..,\gamma_p)}_{io},L) = \{M \in \hat{\Sigma}_{p,m} |\ M=\tilde{M}_p(T) \text{ with } T=[T_{io}|-L],\ T_{io} \in \Theta^{(\gamma_1,..,\gamma_p)}_{io}\} \tag{4.3.55}$$

and

$$\mathcal{M}^{oe}_{can}(n,L) = \bigcup_{\substack{\sum_{i=1}^{p} \gamma_i = n}} \mathcal{M}^{oe}_{can}(\Theta^{(\gamma_1,..,\gamma_p)}_{io},L) \tag{4.3.56}$$

with

$$\mathcal{M}^{oe}_{can}(\Theta^{(\gamma_1,..,\gamma_p)}_{io},L) = \{M \in \hat{\Sigma}_{p,m} |\ M=\tilde{M}_p(T) \text{ with } T=[T_{io}|-P_{io}L],\ T_{io}=[P_{io}|-Q_{io}] \in \Theta^{(\gamma_1,..,\gamma_p)}_{io}\} \tag{4.3.57}$$

From the discussion in section 4.3.2 it follows that the union of model sets that constitutes $\mathcal{M}^{oe}_{can}(n,L)$ in (4.3.56) is nonoverlapping; moreover $\mathcal{M}^{oe}_{can}(n,L)$ equals $\mathcal{M}^{oe}_{pco}(n,L)$. The union of model sets that constitutes $\mathcal{M}^{ee}_{can}(n,L)$ is also nonoverlapping, but the model set has changed; in other words $\mathcal{M}^{ee}_{can}(n,L) \neq \mathcal{M}^{ee}_{pco}(n,L)$ and actually:

$$\mathcal{M}^{ee}_{can}(n,L) \subset \mathcal{M}^{ee}_{pco}(n,L).$$

The consequence of this reduction of the model set is that in an identification procedure, the selected optimal models will depend on the specific subset of the model set $\mathcal{M}^{ee}_{pco}(n,L)$ that has been chosen, i.e. it will become dependent on the specific set of "canonical" forms that is applied.

$\mathcal{M}^{ee}_{can}(n,L)$ has been based on the set of observability canonical forms for $\Sigma_{p,m}(n)$ as represented by Θ_{io} (eqs. 4.3.52–53). Every other set of canonical forms for $\Sigma_{p,m}(n)$ will lead to another equation error model set $\mathcal{M}^{ee*}_{can}(n,L)$ and consequently will lead to a different set of selected optimal models during identification.

Two sets of matrices $\Theta^{(1)}_{io}, \Theta^{(2)}_{io}$ that induce the same set of input–output models,

will generally lead to different equation error model sets, and different selected optimal models. This is illustrated in the following simulation example.

Example 4.3.33.

Consider an input–output system $S\in\Sigma_{2,1}$, with $W=Y\times U$, $Y=\mathbb{R}^2$ and $U=\mathbb{R}$ (one input, two output system). $S=\tilde{M}_p(T^*)$, with $T^*=[P^*(z)|-Q^*(z)]$, and $T^*(z)$ being represented in a canonical observability form with observability indices $\gamma_1=2$, $\gamma_2=1$.

$$P^*(z)=\begin{bmatrix} z^2-2.7z+2.12 & -0.16 \\ 2.0z-4.6 & z \end{bmatrix}, \quad Q^*(z)=\begin{bmatrix} -0.7z+1.49 \\ -2.2 \end{bmatrix}.$$

The poles of the corresponding transfer function $H^*(z) = P^*(z)^{-1}Q^*(z)$ are positioned at $z=0.9$ and $z=0.9\pm 0.1j$.

A data sequence $v^+\in\mathcal{B}^+_{obs,y}(S)$ is constructed by choosing the input u^+ to be a zero mean white noise sequence, satisfying $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} u^2(t) = 1$. The data sequence v^+ is disturbed to $\tilde{v}^+=(\tilde{y}^+,u^+)$ by disturbing the output y^+ with an additive zero mean and uncorrelated white noise sequence ξ^+, according to $\tilde{y}^+=y^++\xi^+$, and ξ^+ satisfying $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\xi(t)\xi^T(t) = I_2$, and $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\xi(t)u(t) = 0$.

The signal–to–noise ratios on the outputs $\tilde{y}^+$ are 40 dB.

This data sequence $\tilde{v}^+$ will be used for identification.

In this identification procedure we are going to apply two equation error model sets, whose i/o parts contain exactly the same sets of input–output models.

In order to construct these model sets, consider the following sets of polynomial matrices Θ_1 and $\Theta_2 \subset \mathbb{R}^{p\times(p+m)}[z]$, defined by:

$T\in\Theta_1 \rightarrow T=[P_1|-Q_1]$ with

$$P_1(z)=\begin{bmatrix} z^2+\alpha_{112}z+\alpha_{111} & \alpha_{121} \\ \alpha_{212}z+\alpha_{211} & z \end{bmatrix}, \quad Q_1(z)=\begin{bmatrix} \beta_{112}z+\beta_{111} \\ \beta_{211} \end{bmatrix}$$

with all parameters $\alpha,\beta\in\mathbb{R}$, and $\alpha_{212}\neq 0$.

$T\in\Theta_2 \rightarrow T=[P_2|-Q_2]$ with

$$P_2(z)=\begin{bmatrix} z+\lambda_{111} & \lambda_{122}z \\ \lambda_{211} & z^2+\lambda_{222}z+\lambda_{221} \end{bmatrix}, \quad Q_2(z)=\begin{bmatrix} \delta_{111} \\ \delta_{212}z+\delta_{211} \end{bmatrix}$$

with all parameters $\lambda,\delta\in\mathbb{R}$, and $\lambda_{122}\neq 0$.

Note that for any element $T_1\in\Theta_1$ there exists a polynomial matrix $U(z)$ that is unimodular with respect to $\mathbb{R}[z]$, such that $T_1'=UT_1\in\Theta_2$. Similarly for any element $T_2\in\Theta_2$ there exists a polynomial matrix $V(z)$ that is unimodular with respect to $\mathbb{R}[z]$, such that $T_2'=VT_2\in\Theta_1$. These unimodular matrices can be verified to be:

$$U(z)=\begin{bmatrix} 0 & 1/\alpha_{212} \\ -\alpha_{212} & z+\alpha_{112}-\alpha_{211}/\alpha_{212} \end{bmatrix} \text{ and } V(z)=\begin{bmatrix} z+\lambda_{222} & -\lambda_{122} \\ 1/\lambda_{122} & 0 \end{bmatrix}$$

Consequently both sets of polynomial matrices Θ_1 and Θ_2 induce exactly the same set of dynamical input–output systems.
For identification purposes we are going to deal with the following two equation error model sets:

$$\mathcal{M}_1=\{M\in\hat{\Sigma}_{2,1} \mid M=\tilde{M}_p(T) \text{ with } T=[T_{io}|-I] \text{ and } T_{io}\in\Theta_1\}$$
$$\mathcal{M}_2=\{M\in\hat{\Sigma}_{2,1} \mid M=\tilde{M}_p(T) \text{ with } T=[T_{io}|-I] \text{ and } T_{io}\in\Theta_2\}.$$

In the model sets $\mathcal{M}_1$ and $\mathcal{M}_2$ an equation error residual term has been added to the models. Note that, by theorem 4.2.6, both model sets are discriminable by $J^+_{LS\infty}$.
We will compare the selected optimal models $\hat{M}_1$ and $\hat{M}_2$, defined by

$$\{\hat{M}_1\}=J^+_{LS\infty}(\tilde{v}^+,\mathcal{M}_1) \text{ and } \{\hat{M}_2\}=J^+_{LS\infty}(\tilde{v}^+,\mathcal{M}_2).$$

For both situations the asymptotic results of the least squares identification method are calculated analytically, without really generating the data, based on covariance information on the process data considered. The applied algorithm is a straightforward extension of the method described by Mullis and Roberts (1976) for the scalar case.
A comparison of the two models $\hat{M}_1$ and $\hat{M}_2$ will be made by evaluating the Markov parameters of the i/o part of both optimal models, and by comparing

these Markov parameters with the Markov parameters of the original process that has been used to generate the data sequence v^+.
Figure 4.5 shows the three sets of Markov parameters that are related to the second output y_2.

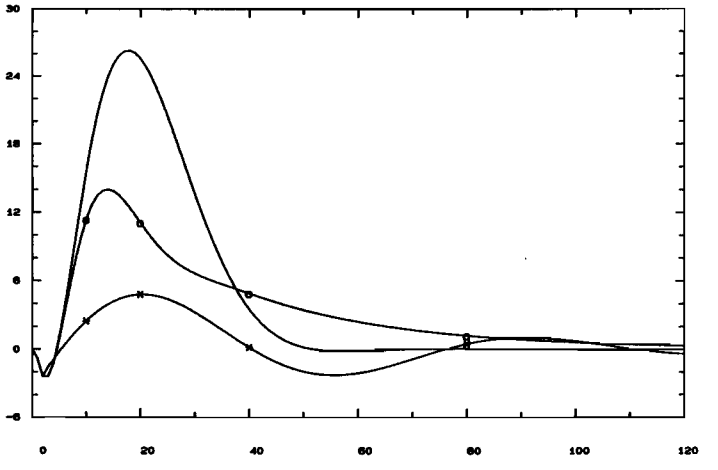

FIGURE 4.5. Markov parameters of the transfer function $H_{y_2u}(z)$ of the original process S (———), and of the i/o part of the two selected optimal models $\hat{M}_1 \in \mathcal{M}_1$ (—×—) and $\hat{M}_2 \in \mathcal{M}_2$ (—o—).

The properties of the two selected optimal models apparently are essentially different, illustrating that the choice between model set $\mathcal{M}_1$ and $\mathcal{M}_2$ is a choice that really influences the dynamical properties of the optimal models. This phenomenon can also be evaluated on the basis of the poles of the transfer function $H_{yu}(z)$ of the selected optimal models, as illustrated in table 4.3.

Poles of process and identified models		
S	$\hat{M}_1$	$\hat{M}_2$
0.900 ± 0.100j	0.975 ± 0.087j	0.831 ± 0.195j
0.9000	-0.102	0.966

TABLE 4.3 Poles of the input/output transfer function $H_{yu}(z)$ of the original process S and of the two selected optimal models $\hat{M}_1$ and $\hat{M}_2$.

This example 4.3.33 shows that the identification result depends on the specifically parametrization chosen for the input/output part of the model set. Restricting oneself to one specific form (e.g. the canonical observability form) is a rather arbitrary choice, and therefore leads to rather arbitrary sets of selected optimal models.

4.4 DISCUSSION

In this chapter the identification framework as presented in the previous chapters, has been elaborated upon by dealing with the problem of parametrizing model sets for identification. The concept of discriminability has been shown to play an important role for the choice of a combination of model set and identification criterion, in order to guarantee that the model set does not contain "trivial" models. The occurrence of such trivial models may lead to a situation in which the identification criterion can only yield selected optimal models for data sequences that can be modelled exactly. In such a situation no optimal models will be selected if the data can not be modelled exactly.
Sufficient conditions for discriminability have been derived for the common least squares identification criterion $J^{+}_{LS\infty}$ and for different model sets.

The different residual types of (i/o/pr)–models PE, OE and EE, as characterized in chapter 3, are shown to require a different treatment in this respect. Contrary to the situation for prediction error and output error models, the equation error type of models do not intrinsically point to a specific way of obtaining discriminability. Therefore, a great number of choices for discriminable equation error model sets is possible, most of them leading to different optimal models obtained in an identification procedure.
For prediction error and output error model sets a parametrization problem remains when a discriminable model set has been specified. This is illustrated for polynomial matrix parametrizations, where (strict) identifiability has to be achieved by removing polynomial matrices that induce equal models. For equation error model sets, the only concern is the discriminability problem. When a discriminable equation error model set is specified, the parametrization problem is inherently solved. A discriminable equation error model set directly leads to a parametrization that is strictly identifiable by $J^{+}_{LS\infty}$.

It has to be stressed that, in contrast with the literature on the parametrization

problem, the (strict) identifiability of parametrizations should be considered in relation to the residual–type of models that is taken into account, and not only in relation to the input–output properties of these models.
An immediate consequence of this result is that nonexpected situations can occur when parametrizing equation error model sets, based on considerations with respect to the input–output properties of the models only; overlapping forms become nonoverlapping and the use of canonical forms leads to a situation where the choice for a specific set of canonical forms essentially influences the set of optimal models obtained in an identification procedure.

5. ADDITIONAL REMARKS ON IDENTIFICATION METHODS

In this short chapter some remarks on system identification are collected concerning items that are not within the central scope of this thesis, but that are worthwhile to pay attention to, because of their relation with the subjects as discussed in the previous chapters.

Remark 5.1

In the previous part of this thesis, we have been discussing different identification methods, mainly due to the choice of different residual–based model sets. However nothing has been said about the related identification algorithms that are required for obtaining the selected optimal models. As stated in chapter two, this set of optimal models is determined by the choice of a model set $\mathcal{M}$ and an identification criterion J. The properties of the corresponding identification algorithm are determined by the identification criterion J and the parametrization $\tilde{M}$. This algorithm is of practical importance when applying system identification methods, since it determines also the effort that is required for obtaining the optimal models. Properties of algorithms are very often a reason for choosing specific parametrizations and consequently specific model sets.

Polynomial matrix parametrizations of equation error model sets in general show the property that the residual signal within a model can be expressed as a linear function of the parameters (the parametrized models are linear–in–the–parameters). When quadratic functions of these residuals are considered, like in a least squares identification criterion, the optimization of these functions over the parameters, can be executed analytically with simple algebraic operations. This favorable property is characteristic for these so called linear regression models. Note that this is actually one of the important reasons for considering equation error models in combination with a quadratic residual function. In this respect the equation error residual does not have a direct physical interpretation with respect to the external signals of the model, but it is a mathematical tool for representing errors in equations. Equation error type models are very popular, especially by the favorable properties of the resulting algorithms. □

Remark 5.2

In this work hardly any attention has been paid to validation of the selected optimal models. It has been mentioned that models obtained by identification meth-

ods, should preferably be validated in a situation that is comparable with the ultimate use of the model; this refers to a proper choice of the validation criterion. Note that there may be an important difference between criteria for identification and for validation. A selected optimal model whose equation error residual function is very small, i.e. represents a good fit in terms of this residual function, may behave badly in terms of another criterion (e.g. an output error validation criterion). Consequently undesired situations can occur if the validation criterion is different from the identification criterion. Some simple examples of such situations for equation error identification and output error validation can be found in Damen *et al.* (1985), (1986) and Tomita *et al.* (1986).
It is a natural requirement that the criteria for identification and validation should be tuned to each other, in order to arrive at well–defined selected optimal models. This is especially important for equation error model sets, since in general no direct physical interpretation can be given to the equation errors. On the contrary, output error and prediction errors are more directly related to specific model applications, such as simulation and prediction, and therefore are appropriate measures for validation purposes in several situations. □

Remark 5.3
Very often the simulation error (output error) of the i/o part of a selected optimal model is used as a criterion for validation. This originates from the consideration that, in the case of a white input signal and input–independent disturbances on the signals, a quadratic measure of this simulation error corresponds with a quadratic measure of deviation between the transfer function of the original (linear) process and that of the selected optimal models. For a SISO (single input single output) situation, this comes down to:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e_{oe}^2(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{process}(e^{j\omega}) - H_{model}(e^{j\omega})|^2 \, d\omega \qquad (5.1)$$

$$= \sum_{k=t_1}^{\infty} [h_{process}(k) - h_{model}(k)]^2 \qquad (5.2)$$

with $\{h(k)\}_{k=t_1,..,\infty}$ the Markov parameters (impulse response) of a transfer function. The quadratic functions considered in (5.1), (5.2) give a good insight in the deviations between a linear process to be identified and a model, measured over the whole frequency range. Actually this validation criterion is also used in the examples as presented in chapters one and four of this thesis, where identification methods are evaluated by comparing the Markov parameters of a selected optimal

model with the corresponding Markov parameters of the original process.

However, unlike the equation error type model, simulation error or output error type models have the disadvantage that in general they yield complex identification algorithms, due to the fact that the residual is not linear–in–the–parameters. In these situations the identification problem has to be solved by applying nonlinear minimization techniques with time–consuming algorithms and problems of possible local minima.
Special parametrizations and corresponding model sets can reduce these kinds of problems; to this end a model set has been proposed that is based on a sequence of Markov parameters and a set of coefficients of the minimal polynomial (see Backx, 1987, and section 3.3.4 in this thesis). This parametrized model set has the property that the output error residual is nonlinear in a relatively small number of parameters, which simplifies the corresponding output error identification algorithm, at the cost of a reduction in the freedom of choosing model sets. □

Remark 5.4
The complexity of identification algorithms that have to generate selected optimal models by nonlinear minimization techniques, has also led to an alternative approach to the system identification problem. In order to arrive at compact optimal models with a low complexity a two–step method can be followed:

1. Apply a relatively simple identification method, e.g. on the basis of equation error models, to a model set with models of very high order; and subsequently
2. Reduce the order of the selected optimal models by applying a method of model reduction such as e.g. balanced model reduction (Moore, 1981; Pernebo and Silverman, 1982), optimal Hankel norm approximation (Adamjan, Arov and Krein, 1971; Glover, 1983).

In the first step we do not have to arrive directly at a proper model which satisfies the requirements of the intended application, which means that there is a freedom of choosing model sets and parametrizations that yield favorable algorithms.
An essential part of the identification method now is moved to the model reduction step, in which model reduction criteria have to be chosen that fit the intended application of the ultimately selected optimal models.
As a possible third step, the appropriate nonlinear minimization algorithm can still be applied to the measured data sequence, while the model obtained so far, can act as a proper initial estimate for starting the algorithm.
This two–step (three–step) method for solving an identification problem is very often successfully applied in practice, and shows the importance of the relation

between the problem of constructing approximate models on the basis of data and the problem of model reduction. This approach has been followed for instance by Backx (1987) and Wahlberg (1986). □

Remark 5.5

It has been shown in chapter four that there exists a problem of lack of discriminability with respect to a least squares identification criterion, when using the very popular class of equation error type models in identification problems. In other words, a choice has to be made for arriving at discriminable model sets. Since there are many possible choices for arriving at discriminability and since each different choice generally leads to different selected optimal models, apparently the experimenter has a great influence on the optimal models finally obtained, by choosing a proper discriminable model set. Moreover it is not clear what kind of effect different choices have on the properties of the selected models. In view of the well–definedness of the identification results, this is not a very favorable situation. It has been partly this argument that has brought some people to the opinion that these identification methods are non–scientific (Los, 1988).

Despite of the fact that in this way selected optimal models to some extent may become arbitrary, it has to be noted that for a specific class of discriminable equation error model sets, remarkable properties of asymptotic optimal models have been shown (Van den Hof and Janssen, 1986, 1987). For this specific class of methods it has been derived that, under the weak conditions of white input signals and input–independent disturbances on the signals, the start sequence of Markov parameters of the (linear) process at hand, is asymptotically equal to the corresponding start sequence of the Markov parameters of the selected optimal model. This shows that the discriminability question in equation error model sets can also be treated in view of these kind of properties of selected optimal models. □

6. CONCLUSIONS

In this thesis the problem of system identification is considered as a problem of deterministic approximate modelling on the basis of measurement data. In contrast with the classical approach to the problem of modelling systems on the basis of observed time series, the philosophy in this work shows two important aspects:

- the models finally obtained are accepted to be only approximations of the dynamical processes underlying the data; and
- in the methodology of constructing models from time series, no use is made of any statistical assumptions on the data.

This philosophy of deterministic approximate modelling has been followed in order to arrive at well–defined models, in the sense that these models proceed from the available data sequence and from explicit users' choices, and not from implicit (statistical) and hardly verifiable assumptions on the data and the underlying process. The implicit assumptions on the data sequence, required in a statistical approach to be able to solve the modelling problem, here have been replaced by explicit users' choices. Consequently it is important to state clearly the various choices that have to be made in an identification procedure.

In order to bring these choices to the surface, a fundamental treatment is required of the problem of constructing models from time series. To this end in this thesis fruitful use has been made of the system theoretic concepts as developed and advocated by Willems (1986a),(1988). Representing a dynamical system in terms of its behaviour, i.e. the space of admissible signal trajectories, we have introduced a framework in which the identification problem as considered above, can be formulated properly. In this framework the various components of an identification method (model set $\mathscr{M}$, parametrization $\tilde{M}$ and identification criterion J) are defined in a fundamental and natural way, and their role in an identification method can consequently be distinguished clearly.

In the identification framework presented, models are used that exhibit three types of external signals: inputs, outputs and residuals, where the fictitious residual signals are incorporated to deal with modelling errors. Existing identification methods have been characterized within this framework with special attention being paid to the type of residuals (prediction error, output error and equation

error type of residuals) that are incorporated in the models.

In the approach presented, models have been used that are slightly more general than the commonly applied standard models, that mainly originate from statistical considerations. Because of this generalization, questions have emerged that do not occur when dealing with the standard models. The main consequence of this approach is reflected in the introduction of the notion of discriminability, as a property of a model set in relation with an identification criterion. It can be achieved by imposing proper restrictions on the model sets and identification criteria taken into account. Discriminability of model sets has appeared to be an important aspect for several reasons. The property has been shown to be required for arriving at a nontrivial identification problem; besides different choices of obtaining discriminable model sets influence the optimal models that are finally obtained. Consequently the way in which discriminability is achieved has appeared to be one of the users' choices that has to be specified.

For output error and prediction error identification methods, discriminability has been shown to be obtained by restricting the model sets in a specific way. However for equation error identification methods, the formulation of discriminable model sets has given rise to interesting contemplations. For these methods, the aspect of discriminability is generally not distinguished as an explicit users' choice, leading to optimal models that are determined partly by the (arbitrary) way in which discriminability was achieved. This means that the experimenter – without notice – can have an essential influence on the optimal models finally obtained.
In view of the fact that the identification methods discussed are required to be based on explicit users' choices, the statement made above makes the equation error approach rather questionable. This is especially remarkable if we take into account that equation error methods are by far the most popular methods in the application of system identification techniques.

Discriminability has also been shown to play a role in the problem of parametrization. In the framework presented, the problems of identification and parametrization have been clearly distinguished from each other; however through a redefinition of the concept of identifiability it has been shown that the two problems can not be considered independently from each other. The identifiability of a parametrization $\tilde{M}$ has been shown to be dependent on the question whether the identification criterion J can discriminate between different models in the model set $\mathcal{M}$, i.e. whether $\mathcal{M}$ is discriminable by J. Different parameter values can only be dis-

tinguished from each other if the identification criterion can distinguish between the two corresponding (distinct) models. In the current literature on identifiability this latter situation actually is implicitly assumed to be satisfied.

In this thesis the discriminability of model sets has been investigated for a least squares identification criterion, and it has been shown that the way in which discriminability of model sets is achieved generally will influence the optimal models finally obtained. For prediction error (PE) and output error (OE) type model sets it has been shown that discriminability by a least squares identification criterion can be obtained quite easily; a representation of model sets by a unique parametrization is the central problem in the construction of identifiable parametrizations. For equation error (EE) type model sets this situation has been shown to be just the other way around. The central problem is to obtain discriminability, and once this has been obtained, the construction of identifiable parametrizations is straightforward.
In this respect the different situations for PE and OE models on the one side and EE models on the other side, are specifically remarkable when considering model sets with models having a fixed model order. What is known as the set of overlapping forms in terms of PE and OE model sets, under minor restrictions has turned out to be a set of nonoverlapping forms in terms of EE model sets.

The identification framework has been shown to give good possibilities to consider the identification problem in a fundamental way. A clear distinction of the basic concepts has been obtained, not in the least by the treatment of dynamical systems and models in terms of the well–defined system theoretical notion of "behaviour" (Willems, 1986a). This notion of model behaviour has appeared to present a unifying approach to all existing definitions of linear, time–invariant and finite-dimensional models, such as transfer functions, state space representations and difference equations. These different representations can now be related to each other clearly; the theories on models described in the forward or backward shift operator (sometimes called MFD and ARMA models) have been generalized to one theory on representations in terms of difference equations. Expressions have been derived for the model order and corresponding structure indices of these models.

Looking over the results obtained so far, the following remarks have to be made, partially pointing to questions that have to be faced.

* Discriminability has been shown to play an important role in the modelling of time series. The notion has emerged because of a treatment of the identification problem as a problem of deterministic approximate modelling. This philosophy is more general than the common approach based on statistical considerations and needs further elaboration. Especially attention has to be paid to a characterization of the specific approximations that are involved; this topic has not been paid attention to in this thesis.
* For output error and prediction error methods discriminability has been obtained by restricting the model sets in a specific way. One could wonder if it is worthwhile to consider alternative ways of obtaining discriminability in order to explicitly influence the models obtained.
* For equation error methods there definitely exists a problem of obtaining discriminability. For a proper application of these methods it is required that the consequences of different choices of discriminable model sets are clarified, in order to be able to use these choices as explicit users' choices.
* In this thesis we have restricted attention to models and identification criteria that are very closely related to the commonly applied identification methods originating from statistical considerations. Basically there is no reason for such a restriction, and consequently more general models (e.g. (i/o/r)–models instead of (i/o/pr)–models) as well as more general identification criteria need be taken into account and need a further development.
* An important question that remains is the question which kind of models and identification criteria are best suited for incorporating intended model applications into the construction of optimal models.

The framework presented in this thesis is believed to give good opportunities for a formal treatment of identification methods and for developing future methods in which intended model applications (such as control system design) should be considered directly within the identification stage.

APPENDIX

(A1) Proof of proposition 3.2.6b

Because of the controllability of the model, the unimodularity of R in part (a) of the proposition is equivalent with $H_e(z) = [H_{ey}(z) | H_{eu}(z)] = R^{-1}[P|Q]$ being polynomial in $\mathbb{R}[z,z^{-1}]$. Consequently the transfer function $H_e(z)$ should be analytical in $\mathbb{C}$ except possibly in z=0; in other words: $H_e(z)$ should not contain any poles for finite values of $z\in\mathbb{C}\backslash\{0\}$. In order to evaluate this we will use a polynomial matrix descriptions of $H_e(z)$ according to Rosenbrock (1970) as described in Kailath (1980). Considering the state space representation:

$$x(k+1) = Ax(k) + B_u u(k) + B_e e(k)$$
$$y(k) = Cx(k) + D_u u(k) + D_e e(k)$$

a polynomial matrix description, representing $H_e(z)$ can be written as:

$$\left[\begin{array}{cc:cc} zI-A & B_e & 0 & B_u \\ -C & D_e & -I & D_u \\ \hdashline 0 & I & 0 & 0 \end{array}\right] \left[\begin{array}{c} x(z) \\ -e(z) \\ -y(z) \\ -u(z) \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ -e(z) \end{array}\right] \qquad \text{(A1.1)}$$

$$\text{with } H_e(z) = [0|I]\begin{bmatrix} zI-A & B_e \\ -C & D_e \end{bmatrix}^{-1}\begin{bmatrix} 0 & B_u \\ -I & D_u \end{bmatrix}. \qquad \text{(A1.2)}$$

The finite poles of $H_e(z)$ are characterized by those values of $\lambda\in\mathbb{C}$ that satisfy the following three conditions:

$$\text{(i): } \det_{\mathbb{C}} \begin{bmatrix} \lambda I-A & B_e \\ -C & D_e \end{bmatrix} = 0; \quad \text{(ii): } \operatorname{rank}_{\mathbb{C}} \left[\begin{array}{cc} \lambda I-A & B_e \\ \hdashline -C & D_e \\ \hdashline 0 & I \end{array}\right] = n+p; \text{ and}$$

$$\text{(iii): } \operatorname{rank}_{\mathbb{C}} \left[\begin{array}{cc:cc} \lambda I-A & B_e & 0 & B_u \\ -C & D_e & -I & D_u \end{array}\right] = n+p.$$

Because of the fact that (C,A) is an observable pair, due to the minimality of the realization, condition (ii) will always be satisfied. Because of the controllability of the model, $\operatorname{rank}_{\mathbb{C}} [\lambda I-A \; B_e \; B_u] = n$ for all $\lambda\in\mathbb{C}\backslash\{0\}$, and due to the minimality of the realization this also holds for $\lambda=0$. Consequently all finite poles of $H_e(z)$ are represented by condition (i) formulated above. □

(A2) Proof of Proposition 3.3.8

For this proof the representations of prediction error models will be used as formulated in proposition 3.2.2 (a).

(a) Properness of the transfer function $P(z)^{-1}Q(z)$ can be imposed by writing

$P(z) = \Lambda_p(z)\Gamma_{hr}(P) + \bar{P}(z)$, and $Q(z) = \Lambda_q(z)\Gamma_{hr}(Q) + \bar{Q}(z)$

with $\Lambda_p(z) = \mathrm{Diag}\{z^{n_p^{(1)}}, \ldots, z^{n_p^{(p)}}\}$, $\Lambda_q(z) = \mathrm{Diag}\{z^{n_q^{(1)}}, \ldots, z^{n_q^{(p)}}\}$,

and $\delta_{ij}^{(u)}(\bar{P}) < n_p^{(i)}$, $i,j\in\underline{p}$; $\delta_{ij}^{(u)}(\bar{Q}) < n_q^{(i)}$, $i\in\underline{p};j\in\underline{m}$, (see Kailath, 1980).

Then $P(z)^{-1}Q(z) = [\Lambda_p(z)^{-1}P(z)]^{-1}[\Lambda_q(z)^{-1}Q(z)]$

$= [\Gamma_{hr}(P)+\Lambda_p(z)^{-1}\bar{P}(z)]^{-1}[\Lambda_p(z)^{-1}][\Lambda_q(z)\Gamma_{hr}(Q)+\bar{Q}(z)]$. (A2.1)

Since $\Gamma_{hr}(P)$ is nonsingular, $[\Gamma_{hr}(P)+\Lambda_p(z)^{-1}\bar{P}(z)]^{-1} = \Gamma_{hr}(P)^{-1} + \tilde{P}(z)$ with $\tilde{P}(z)$ proper, and consequently $P(z)^{-1}Q(z)$ is proper if and only if $\Lambda_p(z)^{-1}\Lambda_q(z)$ is proper, which is equivalent to $n_p^{(i)} \geq n_q^{(i)}$, $i\in\underline{p}$.

(b) The second condition of proposition 3.2.2.a is reflected by the properness of $[P(z)^{-1}R(z)-L]z^k$, with $L\in\mathbb{R}^{pxp}$, nonsingular. This condition incorporates properness of $P(z)^{-1}R(z)$. Writing $R(z)=\Lambda_r(z)\Gamma_{hr}(R)+\bar{R}(z)$ with $\Lambda_r(z)=\mathrm{Diag}\{z^{n_r^{(1)}}, \ldots, z^{n_r^{(p)}}\}$, and $\delta_{ij}^{(u)}(\bar{R}) < n_r^{(i)}$, $i,j\in\underline{p}$, it follows that

$P(z)^{-1}R(z) = [\Gamma_{hr}(P)+\Lambda_p(z)^{-1}\bar{P}(z)]^{-1}[\Lambda_p(z)^{-1}\Lambda_r(z)][\Gamma_{hr}(R)+\Gamma_r(z)^{-1}\bar{R}(z)]$

$= [\Gamma_{hr}(P)^{-1}+\tilde{P}(z)][\Lambda_p(z)^{-1}\Lambda_r(z)][\Gamma_{hr}(R)+\tilde{R}(z)]$ with $\tilde{P}(z)$ and $\tilde{R}(z)$ strictly proper.

$P(z)^{-1}R(z)$ is proper if and only if $\Lambda_p(z)^{-1}\Lambda_r(z)$ is proper, and from $\lim_{z\to\infty} P^{-1}R=L$, nonsingular, it follows that $\Lambda_p(z)=\Lambda_r(z)$. In this situation $P(z)^{-1}R(z)$ equals $\Gamma_{hr}(P)^{-1}\Gamma_{hr}(R) + H(z)$, with $H(z)$ strictly proper, and the second condition of proposition 3.2.2.a consequently is represented by properness of the rational matrix $[P(z)^{-1}R(z)-\Gamma_{hr}(P)^{-1}\Gamma_{hr}(R)]z^k$. □

(A3) Proof of theorem 3.4.7

In order to prove the parts a and c, first there will be shown that $\delta_M(P^{-1}Q) \leq \pi_p^{(u)}(T) - \pi_p^{(\ell)}(T)$, with equality if and only if $\mathrm{rank}_{\mathbb{C}} T(\lambda,\lambda^{-1})=p$ for all $0\neq\lambda\in\mathbb{C}$. Secondly part a will be proved.

The following additional notation will be required; let $G(z)\in\mathbb{R}^{p\times m}(z)$, then denote:

$\delta_a^p(G)$:=the degree of a pole in $z=a$; (A3.1)

$\delta_a^z(G)$:=the degree of a zero in $z=a$; (A3.2)

$\text{exc}_a(G):=\delta_a^p(G)-\delta_a^z(G)$; $\text{exc}_{tot}(G):=\sum_{a\in\mathbb{C}^*}\delta_a^p(G)-\delta_a^z(G)$, with $\mathbb{C}^*=\mathbb{C}\cup\{\infty\}$; (A3.3)

$c_a^{(i)}(G)$:=the maximum of $\text{exc}_a(g^{(i)}(z))$ over all i×i–minors $g^{(i)}(z)$ of G. (A3.4)

By definition:

$$\delta_M(G)=\sum_{a\in\mathbb{C}^*}\delta_a^p(G)=\sum_{a\in\mathbb{C}^*}\max_{i\geq 0}[c_a^{(i)}(G)] \text{ with } c_a^{(0)}(G):=0 \text{ for all } a\in\mathbb{C}^*. \quad (A3.5)$$

For $T=[P|-Q]$ polynomial in z and $G=P^{-1}Q$, it has been shown by Janssen (1988a) that $\delta_M(G)=\text{exc}_{tot}(T)$. It can easily be verified that this result remains valid for $T\in\mathbb{R}^{p\times(p+m)}[z,z^{-1}]$. Since T has only poles in $z=0$ and $z=\infty$, it follows that $\delta_M(G)=\text{exc}_{tot}(T)\leq\text{exc}_0(T)+\text{exc}_\infty(T)$ with equality if and only if T has no finite zeros $\neq 0$, or equivalently $\text{rank}_{\mathbb{C}}T(\lambda,\lambda^{-1})=p$ for all $\lambda\in\mathbb{C}\setminus\{0\}$.

Using a result of Verghese and Kailath (1981), showing that $\text{exc}_a(H)=c_a^{(r)}(H)$ for any rational matrix H with $\text{rank}_{\mathbb{R}(z)}H=r$, it follows that $\delta_M(G)=c_0^{(p)}(T)+c_\infty^{(p)}(T)$. Let any p×p–minor of T be equal to $t(z)=a_1z^{n_1}+\ .\ +a_{n_1-n_2+1}z^{n_2}$ with $n_1,n_2\in\mathbb{Z}$ and $n_1\geq n_2$, then $\text{exc}_\infty(t(z))=n_1$ and $\text{exc}_0(t(z))=-n_2$;

consequently $\delta_M(G)\leq\pi_p^{(u)}(T)-\pi_p^{(\ell)}(T)$, with equality if and only if $\text{rank}_{\mathbb{C}}T(\lambda,\lambda^{-1})=p$ for all $0\neq\lambda\in\mathbb{C}$. This proves the relation between the statements a and c of the theorem.

If the polynomial matrix T is bilaterally row proper, it can be written as:
$T(z,z^{-1})=\text{diag}(z^{u_1},\ \cdot\ ,z^{u_p})\Gamma_{hr}+\bar{T}(z,z^{-1})+\text{diag}(z^{\ell_1},\ \cdot\ ,z^{\ell_p})\Gamma_{lr}$, with $u_i=\nu_i^{(u)}(T)$, $\ell_i=\nu_i^{(\ell)}(T)$, $i\in\underline{p}$, and $\bar{T}(z,z^{-1})$ satisfying $\nu_i^{(u)}(\bar{T})<u_i$ and $\nu_i^{(\ell)}(\bar{T})>\ell_i$, $i\in\underline{p}$.

Since Γ_{hr} and Γ_{lr} have full row rank, $\pi_p^{(u)}(T)=\Sigma_i u_i$, and $\pi_p^{(\ell)}(T)=\Sigma_i\ell_i$, leading to

$$\pi_p^{(u)}(T)-\pi_p^{(\ell)}(T)=\Sigma_i(u_i-\ell_i)=\sum_{i=1}^{p}\nu_i(T)=n(M).$$

If T is not bilaterally row proper, it can always be brought to a bilaterally row proper form by unimodular premultiplication. Let $T'(z,z^{-1})=U(z,z^{-1})T(z,z^{-1})$ with U unimodular with respect to $\mathbb{R}[z,z^{-1}]$, then any p×p–minor of T' equals $\det(U)\cdot$(the corresponding p×p–minor of T). Since $\det(U)=cz^d$, $0\neq c\in\mathbb{R}$, $d\in\mathbb{Z}$, it follows that $\pi_p^{(u)}(T')-\pi_p^{(\ell)}(T')=\pi_p^{(u)}(T)-\pi_p^{(\ell)}(T)$, showing that $\pi_p^{(u)}(\cdot)-\pi_p^{(\ell)}(\cdot)$ is representation independent, which proves statement a of the theorem. □

(A4) Proof of proposition 3.4.8

Let $G(z)=P(z,z^{-1})^{-1}Q(z,z^{-1})$ and $T = [P|-Q]$, then, using the notation of (A3),

a) G proper $\Leftrightarrow \delta_\infty^p(G)=0 \Leftrightarrow \text{exc}_\infty(T)=\text{exc}_\infty(P)$ by using lemma P4A–4 in Janssen (1988a), stating that $\delta_\alpha^p(G)=\text{exc}_\alpha(T)-\text{exc}_\alpha(P)$ for $\alpha\in\mathbb{C}^*$. Consequently G proper $\Leftrightarrow c_\infty^{(p)}(T)=c_\infty^{(p)}(P) \Leftrightarrow \pi_p^{(u)}(T)=\pi_p^{(u)}(P)=\delta^{(u)}\{\det(P)\}$.

b) Using the same lemma as mentioned above, it follows that $\delta_0^p(G)=0 \Leftrightarrow \text{exc}_0(T)=\text{exc}_0(P) \Leftrightarrow c_0^{(p)}(T)=c_0^{(p)}(P) \Leftrightarrow \pi_p^{(\ell)}(T)=\pi_p^{(\ell)}(P)=\delta^{(\ell)}\{\det(P)\}$. □

(A5) Proof of Corollary 3.4.9

a) If $\pi_p^{(\ell)}(T_f)\neq 0$ then $\pi_p^{(\ell)}(T_f)>0$ and consequently $T_f(\lambda)$ has a zero in $\lambda=0$, which is conflicting with the condition $\text{rank}_{\mathbb{C}}T_f(\lambda)=p$ for $\lambda=0$.

b) If $\pi_p^{(u)}(T_b)\neq 0$ then $\pi_p^{(u)}(T_b)<0$ and consequently $T_b(\lambda)$ has a zero in $\lambda=0$, which is conflicting with the condition $\text{rank}_{\mathbb{C}}T_b(\lambda)=p$ for $\lambda=0$.

c) The statement follows directly from theorem 3.4.7c, proposition 3.4.8, and part a of this corollary, taking into account that left coprimeness with respect to $\mathbb{R}[z]$ is equivalent with the condition $\text{rank}_{\mathbb{C}}T(\lambda)=p$ for all $\lambda\in\mathbb{C}$.

d) The proof is similar as the proof of c, applying theorem 3.4.7c, proposition 3.4.8 and part b of this corollary. □

(A6) Proof of theorem 3.4.12

Parts a and b of this theorem are extensions of results of Forney (1975) and Kailath (1980); for the situation that $T\in\mathbb{R}^{p\times(p+m)}[z]$, it has been shown that the set of row degrees of T is equal to the set of left dynamical indices of $P^{-1}Q$ if and only if (P,Q) is left coprime and T is row reduced with respect to $\mathbb{R}[z]$. Extension of this result to polynomials in $\mathbb{R}[z,z^{-1}]$ follows from the consideration that {left coprimeness and row properness with respect to $\mathbb{R}[z]$} $\Leftrightarrow$ {left coprimeness and bilaterally row properness with respect to $\mathbb{R}[z,z^{-1}]$}.
Part c follows from results of Willems (1986a), and the consideration that the set of row degrees of T is invariant for postmultiplication of T with a constant nonsingular matrix. □

(A7) Proof of theorem 3.4.17

The model set $\mathcal{M}_{p,m,\gamma_1,.,\gamma_p}$, as defined, can be parametrized in a set of canonical forms by a minimal parameter set $\{d_{ij}\}_{i=1,.,p;j=1,.,m}$, $\{b_{ijk}\}_{i=1,.,p;j=1,.,\gamma_i;k=1,.,m}$ and $\{a_{ijk}\}_{i=1,.,p;j=1,.,p;k=1,.,\gamma_{ij}}$ with the integers $\{\gamma_{ij}\}_{i=1,.,p;j=1,.,p}$ defined by:

$\gamma_{ij} = \min(\gamma_i+1,\gamma_j)$ for $j\leq i$, and $\gamma_{ij} = \min(\gamma_i,\gamma_j)$ for $j\geq i$. (A7.1)

(see Guidorzi, 1981). Consequently the total number of parameters equals:

$$pm + mn + \sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_{ij}, \tag{A7.2}$$

with $\sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_{ij} = \sum_{i=1}^{p}\sum_{j=1}^{i-1} \min(\gamma_i+1,\gamma_j) + \sum_{i=1}^{p}\sum_{j=i}^{p} \min(\gamma_i,\gamma_j)$. (A7.3)

For evaluation of this expression we will first consider the situation that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p$, and afterwards generalize the result for the general case.

a. In the situation $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p$ it follows directly that

$$\sum_{i=1}^{p}\sum_{j=i}^{p} \min(\gamma_i,\gamma_j) = \sum_{i=1}^{p}\sum_{j=i}^{p} \gamma_j = \sum_{i=1}^{p} i\gamma_i. \tag{A7.4}$$

For analysis of the expression $\sum_{i=1}^{p}\sum_{j=1}^{i-1} \min(\gamma_i+1,\gamma_j)$ we will consider the sequence $(\gamma_i)_{i=1,.,p}$ to be represented as:

$$\gamma_1,\cdot,\gamma_{t_1},\gamma_{t_1+1},\cdot,\gamma_{t_1+t_2},\cdots,\gamma_{t_1+.+t_{d-1}+1},\cdot,\gamma_{t_1+.+t_d}$$

with $\gamma_1 = . = \gamma_{t_1} > \gamma_{t_1+1} = . = \gamma_{t_1+t_2} > \cdots$; so $(\gamma_i)_{i=1,.,p}$ consists of d groups of equal indices. Now we can write:

$$\sum_{i=1}^{p}\sum_{j=1}^{i-1} \min(\gamma_i+1,\gamma_j) = \sum_{k=1}^{d}\sum_{\ell=1}^{t_k}\sum_{i=t_1+.+t_{k-1}+\ell+1}^{p} \min(\gamma_{t_1+.+t_{k-1}+\ell},\gamma_i+1) =$$

$$\sum_{k=1}^{d}\sum_{\ell=1}^{t_k}\Big\{\sum_{i=t_1+.+t_{k-1}+\ell+1}^{t_1+.+t_k} \gamma_i + \sum_{i=t_1+.+t_{k+1}}^{p} (\gamma_i+1)\Big\} =$$

$$\sum_{k=1}^{d}\sum_{\ell=1}^{t_k}\Big\{\sum_{i=t_1+.+t_{k-1}+\ell+1}^{p} \gamma_i + p - (t_1+.+t_k+1) + 1\Big\} =$$

$$\sum_{j=1}^{p}\sum_{i=j+1}^{p} \gamma_i + \sum_{j=1}^{p} p - \sum_{k=1}^{d} t_k(t_1+.+t_k) = \sum_{k=1}^{p} (k-1)\gamma_k + p^2 - \sum_{k=1}^{d} t_k(t_1+.+t_k).$$

Since $p = \sum_{k=1}^{d} t_k$ it follows that $p^2 = \sum_{k=1}^{d} t_k^2 + 2\sum_{k=1}^{d} t_k(t_1+.+t_{k-1})$, leading to

$\sum_{k=1}^{d} t_k(t_1+.+t_k) = \frac{1}{2}p^2 + \frac{1}{2}\sum_{k=1}^{d} t_k^2$. Substitution of this expression gives

$\sum_{i=1}^{p}\sum_{j=1}^{i-1}\min(\gamma_i+1,\gamma_j) = \sum_{k=1}^{p}(k-1)\gamma_k + \frac{1}{2}p^2 - \frac{1}{2}\sum_{k=1}^{d}t_k^2$. Together with (A7.2)–(A7.4) this leads to $\tau(\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}) = 2\sum_{k=1}^{p}k\gamma_k + (m-1)n + mp + \frac{1}{2}(p^2 - \sum_{k=1}^{d}t_k^2)$.

b. Consider a set of reordered observability indices $(\tilde{\gamma}_i)_{i=1,..,p}$ with $\tilde{\gamma}_j=\gamma_j$ for $j\neq k,k+1$, $\tilde{\gamma}_k=\gamma_{k+1}$, $\tilde{\gamma}_{k+1}=\gamma_k$ for any fixed $k\in\underline{p-1}$. The corresponding integers $(\tilde{\gamma}_{ij})_{i=1,..,p;j=1,..,p}$ are determined by $\tilde{\gamma}_{ij}=\gamma_{ij}$ for $i,j\notin\mathbb{Z}\cap[k,k+1]$, $\tilde{\gamma}_{kk}=\gamma_{k+1,k+1}$, $\tilde{\gamma}_{k,k+1}=\min(\gamma_k,\gamma_{k+1})$, $\tilde{\gamma}_{k+1,k}=\min(\gamma_k+1,\gamma_{k+1})$.

It follows that $\sum_{i=1}^{p}\sum_{j=1}^{p}\tilde{\gamma}_{ij} - \sum_{i=1}^{p}\sum_{j=1}^{p}\gamma_{ij}$

$= 1$ for $\gamma_{k+1}>\gamma_k$;

$= -1$ for $\gamma_{k+1}<\gamma_k$;

$= 0$ for γ_{k+1},γ_k.

Since $\gamma_{k+1}<\gamma_k$ every adjacent permutation of two indices in $(\gamma_i)_{i=1,..,p}$ causes a subtraction of one parameter. □

(A8) Proof of proposition 4.2.4

Consider two models $M_1,M_2\in\hat{\Sigma}_{p,m}$ with $M_i=\tilde{M}_p(T_i)$, $T_i=[P_i|-Q_i|-R_i]$, and $T_i\in\mathbb{R}^{px(p+m+p)}[z,z^{-1}]$, for i=1,2. Define the combined dynamical system $M_{1,2}$ with variables (y,u,e_1,e_2) such that

$$(v,e_1,e_2)\in\mathcal{B}(M_{1,2}) \Leftrightarrow \{(v,e_1)\in\mathcal{B}(M_1) \wedge (v,e_2)\in\mathcal{B}(M_2)\} \qquad (A8.1)$$

Consequently $\mathcal{R}(M_1,M_2)=\{(e_1,e_2)\in(E^2)^{\mathbb{Z}} \mid \exists v\in V^{\mathbb{Z}},\ (v,e_1,e_2)\in\mathcal{B}(M_{1,2}),$ (A8.2)

and $M_{1,2}:=\tilde{M}_p(T_3)$ with $T_3=\begin{bmatrix} P_1|-Q_1|-R_1| & 0 \\ P_2|-Q_2| & 0 |-R_2\end{bmatrix}$.

Now consider two matrices $V_1,V_2\in\mathbb{R}^{pxp}[z,z^{-1}]$ with V_1,V_2 nonsingular and left coprime with respect to $\mathbb{R}[z,z^{-1}]$, such that $V_1P_1+V_2P_2=0$. By premultiplication of T_3 with the matrix $\begin{bmatrix} I & 0 \\ V_1 & V_2\end{bmatrix}$ it follows that $\mathcal{B}(M_{1,2})=\mathcal{B}_c(\tilde{M}_p(T_4))$ with

$$T_4=\begin{bmatrix} P_1|-Q_1| & -R_1 & | & 0 \\ 0 & |-Q_e|-V_1R_1|-V_2R_2 \end{bmatrix}, \text{ and } Q_e=V_1Q_1+V_2Q_2. \tag{A8.3}$$

This T_4 shows that

$$\{(y,u,e_1,e_2)\in\mathcal{B}(M_{1,2})\} \Leftrightarrow \{(y,u,e_1)\in\mathcal{B}(M_1) \wedge (u,e_1,e_2)\in\mathcal{B}_c(M_e)\} \tag{A8.4}$$

$$\text{with } M_e:=\tilde{M}_p([-Q_e|-V_1R_1|-V_2R_2]). \tag{A8.5}$$

Since e_1 is free in $\mathcal{B}(M_1)$, it follows that $(e_1,e_2)\in\mathcal{R}(M_1,M_2)$ if and only if there exists a $u\in U^{\mathbb{Z}}$ such that $(u,e_1,e_2)\in\mathcal{B}_c(M_e)$. This proves the result of part (a). The result of part (b) follows directly by considering that $Q_e=V_1Q_1+V_2Q_2$ and $V_1P_1+V_2P_2=0$, which leads to $P_1^{-1}Q_1=P_2^{-1}Q_2$. □

(A9) Proof of Theorem 4.2.7

The proof of this theorem will be given by considering polynomial matrix representations of the models concerned.

Consider two models $M_1,M_2\in\mathcal{M}_{\Theta_t}$ with $M_i=\tilde{M}_p(T_i)$, $T_i=[P_i|-Q_i|-R_i]$, and $T_i\in\mathbb{R}^{p\times(p+m+p)}[z,z^{-1}]$, for i=1,2, and consider the combined dynamical system $M_{1,2}$ defined by:

$$(v,e_1,e_2)\in\mathcal{B}(M_{1,2})\} \Leftrightarrow \{(v,e_1)\in\mathcal{B}(M_1) \wedge (v,e_2)\in\mathcal{B}(M_2)\}. \tag{A9.1}$$

or equivalently:

$$\{(v,e_1,e_2)\in\mathcal{B}(M_{1,2})\} \Leftrightarrow \{(y,u,e_1)\in\mathcal{B}(M_1) \wedge (u,e_1,e_2)\in\mathcal{B}_c(\tilde{M}_p(T_e))\} \tag{A9.2}$$

$$\text{with } T_e=[-Q_e|-V_1R_1|-V_2R_2], \tag{A9.3}$$

$Q_e=V_1Q_1+V_2Q_2$, $V_1P_1+V_2P_2=0$, and $V_1,V_2\in\mathbb{R}^{p\times p}[z,z^{-1}]$ nonsingular and left coprime. Denote $H_{e_2e_1}(z) = H_{ey}^{(2)}(H_{ey}^{(1)})^{-1} = -R_2^{-1}V_2^{-1}V_1R_1(z)$.

Now consider any $M_1\in\mathcal{M}$ and construct a data sequence v^+ that satisfies $(v^+,e_1^+)\in\mathcal{B}^+(M_1)$, with u^+ and e_1^+ satisfying:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} u(t)u^T(t+\tau) = I_m\delta(\tau), \quad \tau\in\mathbb{Z}_+, \tag{A9.4}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} e_1(t)e_1^T(t+\tau) = I_p\delta(\tau), \quad \tau\in\mathbb{Z}_+, \tag{A9.5}$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} u(t)e_1^T(t+\tau) = 0 \quad \tau\in\mathbb{Z}_+. \tag{A9.6}$$

Since u and e_1 are free in M_1, such a data sequence can always be constructed.

Now considering (A9.2) and (A9.3), and taking account of (4.2.10), it follows from the special character of the signals u^+ and e_1^+ that if $H_{e_2e_1}(z)$ is stable, and $H_{e_2e_1}(z) = \sum_{k=-\infty}^{\infty} M(k)z^{-k}$ with $\mathrm{tr}\left[\sum_{k=-\infty}^{\infty} M^T(k)M(k)\right] \geq p$, then automatically $\ell^+_{LS\infty}(e_2^+) \geq \ell^+_{LS\infty}(e_1^+)$.

If $P_1^{-1}Q_1 = P_2^{-1}Q_2$, or equivalently $M_1 \overset{t}{\underset{io}{\sim}} M_2$, then $Q_e = 0$ in (A9.3) and consequently $\ell^+_{LS\infty}(e_2^+) = \ell^+_{LS\infty}(e_1^+)$ if and only if $\mathrm{tr}\left[\sum_{k=-\infty}^{\infty} M^T(k)M(k)\right] = p$.

Since in this situation $\mathrm{tr}\left[\sum_{k=-\infty}^{\infty} M^T(k)M(k)\right] = p$ implies $M_1 = M_2$, it follows that $J^+_{LS\infty}(v^+, \mathcal{M}) = \{M_1\}$.

If this result holds for all $M_1 \in \mathcal{M}$, then consequently $\mathcal{M}$ is discriminable by $J^+_{LS\infty}$, which proves condition (i) to be sufficient. □

(A10) Lemma A10

Consider polynomial matrices $P_1, P_2 \in \mathbb{R}^{p\times p}[z]$ and $Q_1, Q_2 \in \mathbb{R}^{p\times m}[z]$, with $\det_{\mathbb{R}(z)} P_1 \neq 0$ and $\det_{\mathbb{R}(z)} P_2 \neq 0$, such that $P_1^{-1}Q_1 = P_2^{-1}Q_2$.

If $\mathrm{rank}[P_1(0)|Q_1(0)] = \mathrm{rank}[P_2(0)|Q_2(0)] = p$ then $P_1(z)P_2^{-1}(z)$ is analytic in $z=0$. □

Proof.

From $P_1^{-1}Q_1 = P_2^{-1}Q_2$ it follows that there exist polynomial matrices $V_1, V_2 \in \mathbb{R}^{p\times p}[z]$, with V_1, V_2 left coprime, i.e. $\mathrm{rank}_{\mathbb{C}}[V_1(\lambda)|V_2(\lambda)] = p$ for all $\lambda \in \mathbb{C}$, such that $V_1T_1 + V_2T_2 = 0$ with $T_1 = [P_1|Q_1]$ and $T_2 = [P_2|Q_2]$. Consequently $V_1(0)T_1(0) + V_2(0)T_2(0) = 0$, or $[V_1(0)|V_2(0)]\begin{bmatrix} T_1(0) \\ T_2(0) \end{bmatrix} = 0$. Since $\mathrm{rank}[V_1(0)|V_2(0)] = \mathrm{rank}[T_1(0)] = \mathrm{rank}[T_2(0)] = p$, it follows with Sylvester's inequality that $\mathrm{rank}\begin{bmatrix} T_1(0) \\ T_2(0) \end{bmatrix} = p$, and consequently $V_1(0)$ and $V_2(0)$ both have to be nonsingular.

Since $P_1(z)P_2^{-1}(z) = V_1^{-1}(z)V_2(z)$ it follows that $P_1(z)P_2^{-1}(z)$ is analytic in $z=0$, i.e. $P_1(z)P_2^{-1}(z)$ has no poles in the origin. □

(A11) Proof of theorem 4.2.11

In order to prove this theorem, the result of theorem 4.2.7 will be applied. Consider two models $M_1, M_2 \in \mathcal{M}^{ee}_{\theta_t}(L)$ with M_1, M_2 having transfer functions $H_e^{(1)}, H_e^{(2)} \in \Theta_t$. Since $\Gamma_{hr}(H_{ey}^{(1)}) = \Gamma_{hr}(H_{ey}^{(2)}) = L$, nonsingular, there can be written:

$$H_{ey}^{(j)}(z) = \Lambda_j(z)L + \bar{H}_{ey}^{(j)}(z) \quad j=1,2 \tag{A11.1}$$

$$\text{with} \quad \Lambda_j(z) = \text{diag}(z^{n_1^{(j)}}, \ldots, z^{n_p^{(j)}}), \tag{A11.2}$$

$$n_i^{(j)} = \nu_i^{(u)}(H_{ey}^{(j)}) \text{ and } \nu_i^{(u)}(\bar{H}_{ey}^{(j)}) < \nu_i^{(u)}(H_{ey}^{(j)}), \; i=1,\ldots,p. \tag{A11.3}$$

Now considering $H_{e_2e_1}(z) = H_{ey}^{(2)}(z)\{H_{ey}^{(1)}(z)\}^{-1}$, it follows that

$$H_{e_2e_1}(z) = [\Lambda_2(z)L + \bar{H}_{ey}^{(2)}(z)][\Lambda_1(z)L + \bar{H}_{ey}^{(1)}(z)]^{-1} = \Lambda_2(z)G(z)\Lambda_1(z)^{-1}, \tag{A11.4}$$

$$\text{with } G(z) = [L + \Lambda_2(z)^{-1}\bar{H}_{ey}^{(2)}(z)][L + \Lambda_1(z)^{-1}\bar{H}_{ey}^{(1)}(z)]^{-1}. \tag{A11.5}$$

Since $\Lambda_j(z)^{-1}\bar{H}_{ey}^{(j)}(z)$ is strictly proper for j=1,2, it follows that G(z) is proper with $\lim_{z\to\infty} G(z) = I$.

Writing $H_{e_2e_1}(z) = \sum_{k=t}^{\infty} M(k)z^{-k}$ for some $t \in \mathbb{Z}$, and $G(z) = \sum_{k=0}^{\infty} N(k)z^{-k}$ with N(0)=I, it follows with (A11.4) that $\text{tr}\left[\sum_{k=-\infty}^{\infty} M^T(k)M(k)\right] \geq p$ if and only if $\text{tr}\left[\sum_{k=0}^{\infty} N^T(k)N(k)\right] \geq p$. Since N(0)=I this condition is fulfilled, which satisfies part (a) of theorem 4.2.7.

For satisfying part (b), note that $\text{tr}\left[\sum_{k=-\infty}^{\infty} M^T(k)M(k)\right] = p$ if and only if $\text{tr}\left[\sum_{k=0}^{\infty} N^T(k)N(k)\right] = p$, which is equivalent with G(z)=I. For the proof of discriminability of model sets we now have to formulate conditions such that $H_{e_2e_1}(z) = \Lambda_2(z)\Lambda_1(z)^{-1} = I$ in situations that $H_{yu}^{(1)}(z) = H_{yu}^{(2)}(z)$, with $H_{yu}(z) = H_{ey}^{-1}(z)H_{eu}(z)$.

(i) If $\nu_i^{(u)}(H_{ey}^{(1)}) = \nu_i^{(u)}(H_{ey}^{(2)})$ for i=1,..,p, then $\Lambda_2(z) = \Lambda_1(z)$ and $H_{e_2e_1}(z) = I$, which proves discriminability.

(ii) If $\sum_{i=1}^{p} \nu_i^{(u)}(H_{ey}^{(1)}) = \sum_{i=1}^{p} \nu_i^{(u)}(H_{ey}^{(2)})$ then $\det(\Lambda_2(z)\Lambda_1(z)^{-1}) = 1$. Under the additional condition that $H_{e_2e_1}(z)$ does not contain any poles in z=0 it

follows that $H_{e_2e_1}(z)=I$. The equivalent condition $\text{rank}_{\mathbb{C}}\,[H_{ey}^{(1)}(0)|H_{eu}^{(1)}(0)]=\text{rank}_{\mathbb{C}}\,[H_{ey}^{(2)}(0)|H_{eu}^{(2)}(0)]=p$ is formulated in Lemma A10 of this appendix. □

(A12) Proof of theorem 4.2.12

This theorem is proved along similar lines as followed in the proof of theorem 4.2.11. Consider two models $M_1,M_2\in\mathcal{M}_{\theta_t}^{ee}(L)$ with M_1,M_2 having transfer functions $H_e^{(1)},H_e^{(2)}\in\Theta_t$. Since $\Gamma_{hc}(H_{ey}^{(1)})=\Gamma_{hc}(H_{ey}^{(2)})=L$, nonsingular, there can be written:

$$H_{ey}^{(j)}(z)=L\Lambda_j(z)+\bar{H}_{ey}^{(j)}(z)\quad j=1,2 \tag{A12.1}$$

$$\text{with}\quad \Lambda_j(z)=\text{diag}(z^{n_1^{(j)}},..,z^{n_p^{(j)}}), \tag{A12.2}$$

$$n_i^{(j)}=\mu_i^{(u)}(H_{ey}^{(j)})\quad\text{and}\quad \mu_i^{(u)}(\bar{H}_{ey}^{(j)})<\mu_i^{(u)}(H_{ey}^{(j)}),\ i=1,..,p. \tag{A12.3}$$

Now considering $H_{e_2e_1}(z)=H_{ey}^{(2)}(z)\{H_{ey}^{(1)}(z)\}^{-1}$, it follows that

$$H_{e_2e_1}(z)=[L\Lambda_2(z)+\bar{H}_{ey}^{(2)}(z)][L\Lambda_1(z)+\bar{H}_{ey}^{(1)}(z)]^{-1}=G_2(z)\Lambda_2(z)\Lambda_1(z)^{-1}G_1(z), \tag{A12.4}$$

$$\text{with}\ G_2(z)=[L+\bar{H}_{ey}^{(2)}(z)\Lambda_2(z)^{-1}]\ \text{and}\ G_1(z)=[L+\bar{H}_{ey}^{(1)}(z)\Lambda_1(z)^{-1}]^{-1}. \tag{A12.5}$$

Since $\bar{H}_{ey}^{(j)}(z)\Lambda_j(z)^{-1}$ is strictly proper for $j=1,2$, it follows that $G_1(z)$ and $G_2(z)$ are proper with $\lim_{z\to\infty} G_1(z)=L^{-1}$ and $\lim_{z\to\infty} G_2(z)=L$, and consequently $G_2(z)G_1(z)$ is proper with $\lim_{z\to\infty} G_2(z)G_1(z)=I$.

In the situations (i) and (ii) of the theorem, it follows that $\Lambda_2(z)\Lambda_1(z)^{-1}=I$, leading to $H_{e_2e_1}(z)=G_2(z)G_1(z)$. In situation (ii) this is caused by the fact that $H_{e_2e_1}(z)$ is not allowed to have any poles in $z=0$. Since $\lim_{z\to\infty} H_{e_2e_1}(z)=I$, and consequently $H_{e_2e_1}(z)=I+\sum_{k=0}^{\infty}M(k)z^{-k}$, it follows that $\text{tr}\left[\sum_{k=-\infty}^{\infty}M^T(k)M(k)\right]\geq p$ with equality if and only if $H_{e_2e_1}(z)=I$. This proves the results. □

(A13) Proof of proposition 4.2.16.

The proof of this proposition is based on the following considerations.
Let M_e be a dynamical system $M_e\in\Sigma$ with variables $e_1,e_2\in(\mathbb{R}^p)^{\mathbb{Z}}$, and $M_e=\tilde{M}_p(T)$

with $T=[R_{e_1}|R_{e_2}]$, $R_{e_1},R_{e_2}\in\mathbb{R}^{p\times p}[z,z^{-1}]$, nonsingular. If $H(z)=R_{e_2}^{-1}R_{e_1}(z)$ is stable, and if $(e_1^+,e_2^+)\in\mathcal{B}^+(M_e)$ then

$$(\|H^{-1}(z)\|_{L^\infty})^{-1}\ell^+_{LS\infty}(e_1^+)\le\ell^+_{LS\infty}(e_2^+)\le\|H(z)\|_{L^\infty}\cdot\ell^+_{LS\infty}(e_1^+) \tag{A13.1}$$

where $$\|H(z)\|_{L^\infty}=\sup_{\omega\in(-\pi,\pi]}\bar{\sigma}(|H(e^{j\omega})|), \text{ and} \tag{A13.2}$$

$$(\|H^{-1}(z)\|_{L^\infty})^{-1}=\inf_{\omega\in(-\pi,\pi]}\underline{\sigma}(|H(e^{j\omega})|), \tag{A13.3}$$

with $\bar{\sigma}$, $\underline{\sigma}$ the maximum, minimum, singular value of a real constant matrix.

If in such a situation $\ell^+_{LS\infty}(e_2^+)=c\,\ell^+_{LS\infty}(e_1^+)$ for all $e_1^+\in\mathcal{L}^+$, then consequently $(\|H^{-1}(z)\|_{L^\infty})^{-1}=\|H(z)\|_{L^\infty}=c$, which leads to the situation that $\sigma_i(|H(e^{j\omega})|)=c$ for all singular values σ_i, $i\in\underline{p}$ and for all $\omega\in(-\pi,\pi]$. Consequently $H^T(e^{-j\omega})H(e^{j\omega})=cI$. These expressions will be used in the sequel of this proof.

Conditions (i),(ii) and (iii) are sufficient for the scaling–equivalence of M_1 and M_2 since (i) guarantees that there indeed exists a transfer function between e_1 and e_2; (ii) guarantees that the transfer functions $H_{e_1e_2}(z)$ and $H_{e_2e_1}(z)$ both are stable, and (iii) guarantees the proper constant quotient of $\ell^+_{LS\infty}(e_2^+)$ and $\ell^+_{LS\infty}(e_1^+)$.

The necessity of condition (i) follows from the representation of $\mathcal{R}(M_1,M_2)$ as presented in proposition 4.2.4. The influence of the free variable u has to be eliminated in $\mathcal{R}(M_1,M_2)$, in order to be able to satisfy the requirement that $\ell^+_{LS\infty}(e_2^+)=c\,\ell^+_{LS\infty}(e_1^+)$ for all possible data. In the notation of proposition 4.2.4 this means that $Q_e=0$, which is equivalent with $\mathcal{B}_c(M_1^{io})=\mathcal{B}_c(M_2^{io})$, and equivalent with condition (i) in proposition 4.2.16.

The necessity of condition (ii) follows from the fact that in order to satisfy $\ell^+_{LS\infty}(e_2^+)=c\,\ell^+_{LS\infty}(e_1^+)$, both transfer functions $H_{e_1e_2}(z)$ and $H_{e_2e_1}(z)$ have to satisfy the requirement that they are stable, and moreover that $H^T_{e_2e_1}(z^{-1})H_{e_2e_1}(z)=cI$ and $H^T_{e_1e_2}(z^{-1})H_{e_1e_2}(z)=c^{-1}I$. From these requirements it follows that both transfer functions are not allowed to have any finite poles, except in $z=0$.

Necessity of condition (iii) follows directly from the remarks made in the beginning of this proof. □

(A14) Proof of proposition 4.2.17

This proposition is going to be proved on the basis of the results of proposition 4.2.16. Condition (i) of proposition 4.2.16 can simply be shown to be equivalent with $\mathcal{B}_c(M_1^{io}) = \mathcal{B}_c(M_2^{io})$ (see also the proof of proposition 4.2.16).
We will show that the condition $\mathcal{B}_c(M_1^{io})=\mathcal{B}_c(M_2^{io})$, together with condition (ii) of proposition 4.2.16, implies that $\mathcal{B}^{io}(M_1) = \mathcal{B}^{io}(M_2)$. If $\mathcal{B}^{io}(M_1) = \mathcal{B}^{io}(M_2)$ then condition (ii) of proposition 4.2.16 can simply be shown to imply eq. (4.2.29), which proves the result.

Consider two controllable models $M_1,M_2\in\hat{\Sigma}_{p,m}$, with $M_1=\tilde{M}_p(T_1)$ and $M_2=\tilde{M}_p(T_2)$, where $T_1=[P_1|-Q_1|-R_1]$ and $T_2=[P_2|-Q_2|-R_2]$. If $\mathcal{B}_c(M_1^{io}) = \mathcal{B}_c(M_2^{io})$ and condition (ii) of proposition 4.2.16 is satisfied then the following equations hold true:

$$(P_1,Q_1,R_1) \text{ left coprime with respect to } \mathbb{R}[z,z^{-1}] \tag{A14.1}$$

$$(P_2,Q_2,R_2) \text{ left coprime with respect to } \mathbb{R}[z,z^{-1}] \tag{A14.2}$$

$$V_1P_1 + V_2P_2 = 0 \text{ with } V_1,V_2\in\mathbb{R}^{pxp}[z,z^{-1}], \text{ nonsingular} \tag{A14.3}$$

$$V_1Q_1 + V_2Q_2 = 0 \tag{A14.4}$$

$$V_1R_1 + V_2R_2U = 0 \text{ with } U\in\mathbb{R}^{pxp}[z,z^{-1}] \text{ a unimodular matrix} \tag{A14.5}$$

It follows from (A14.3), (A14.4) and (A14.5) that

$$[P_1|-Q_1|-R_1] = -V_1^{-1}V_2\,[P_2|-Q_2|-R_2U] \tag{A14.6}$$

Using (A14.1) and (A14.2) it follows that $V_1^{-1}V_2$ has to be a unimodular polynomial matrix, showing that $\mathcal{B}^{io}(M_1) = \mathcal{B}^{io}(M_2)$. □

(A15) Proof of theorem 4.3.2

The proof of this theorem is given along the following lines. Suppose that there are two matrices $T_1,T_2\in\Theta_p$ such that $T_2=UT_1$ with $U\in\mathbb{R}^{pxp}[z,z^{-1}]$ and U unimodular

with respect to $\mathbb{R}[z,z^{-1}]$; then show that U=I.
If $T_2=UT_1$ then $P_2=UP_1$.

Now write $P_j(z)=\Gamma_{hc}(P_j)\Lambda_j(z)+\bar{P}_j(z)$ j=1,2 (A15.1)

with $\Lambda_j(z) = \mathrm{diag}(z^{n_1^{(j)}}, \ldots, z^{n_p^{(j)}})$, (A15.2)

$n_i^{(j)} = \mu_i^{(u)}(P_j)$ and $\mu_i^{(u)}(\bar{P}_j)<\mu_i^{(u)}(P_j)$, i=1,..,p.; j=1,2; (A15.3)

Then $U(z,z^{-1}) = P_2(z)P_1^{-1}(z)$, leading to

$U(z,z^{-1})=[\Gamma_{hc}(P_2)\Lambda_2(z)+\bar{P}_2(z)][\Gamma_{hc}(P_1)\Lambda_1(z)+\bar{P}_1(z)]^{-1}$. (A15.4)

Because of conditions (i) and (ii) it follows that $\Gamma_{hc}(P_1)=\Gamma_{hc}(P_2)$ and $\Lambda_1(z)=\Lambda_2(z)$.

Consequently $U(z,z^{-1})=[\Gamma_{hc}(P_2)+\bar{P}_2(z)\Lambda_2(z)^{-1}][\Gamma_{hc}(P_1)+\bar{P}_1(z)\Lambda_1(z)^{-1}]^{-1}$, (A15.5)

with $\bar{P}_j(z)\Lambda_j(z)^{-1}$ strictly proper for j=1,2.

It follows that $U(z,z^{-1})=I+G(z)$ with G(z) strictly proper. (A15.6)

Now let $T_2=UT_1$.

Using (iii) this equation can be written as

$\Gamma_2\Lambda(z)+\bar{T}_2(z) = U[\Gamma_1\Lambda(z)+\bar{T}_1(z)]$, or equivalently (A15.7)

$\Gamma_2+\bar{T}_2(z)\Lambda^{-1}(z) = U[\Gamma_1+\bar{T}_1(z)\Lambda^{-1}(z)]$ (A15.8)

with $\bar{T}_2(z)\Lambda^{-1}(z)$ and $\bar{T}_1(z)\Lambda^{-1}(z)$ strictly antiproper, i.e. $\lim_{z\to 0} \bar{T}_2(z)\Lambda^{-1}(z) =$

$\lim_{z\to 0} \bar{T}_1(z)\Lambda^{-1}(z) = 0$.

Since rank Γ_1 = p, there exists a matrix $V\in\mathbb{R}^{(p+m+p)xp}$ such that $\Gamma_1 V=I_p$, leading to: $[\Gamma_2+\bar{T}_2(z)\Lambda^{-1}(z)]V = U[I+\bar{T}_1(z)\Lambda^{-1}(z)V]$ (A15.9)

from which follows that U is antiproper, i.e. $\lim_{z\to 0} U(z,z^{-1})$ exists. (A15.10)

Combining (A15.10) with (A15.6) shows that $\{(i)\wedge(ii)\wedge(iii) \Rightarrow U=I\}$.

For applying condition (iv), let T_1, T_2 be written as:

$T_j(z)=\lambda_j(z)\Gamma_{lr}(T_j)+\bar{T}_j(z)$ j=1,2 (A15.11)

with $\lambda_j(z) = z^d$, $d\in\mathbb{Z}$, and $\nu_i^{(\ell)}(\bar{T}_j)>d$, 1=1,..,p; j=1,2; (A15.12)

Now $T_2=UT_1$ can be written as:

$$[\lambda_2(z)\Gamma_{lr}(T_2)+\bar{T}_2(z)] = U[\lambda_1(z)\Gamma_{lr}(T_1)+\bar{T}_1(z)] \tag{A15.13}$$

and with condition (iv), $\lambda_2(z)=\lambda_1(z)=\lambda(z)$, leading to:

$$[\Gamma_{lr}(T_2)+\lambda(z)^{-1}\bar{T}_2(z)] = U[\Gamma_{lr}(T_1)+\lambda(z)^{-1}\bar{T}_1(z)] \tag{A15.14}$$

with $\lambda(z)^{-1}\bar{T}_2(z)$ and $\lambda(z)^{-1}\bar{T}_1(z)$ strictly antiproper.

With a similar reasoning as used for arriving at (A15.10), it follows that $U(z,z^{-1})$ is antiproper. Combining this with (A15.6) it is shown that $\{(i)\wedge(ii)\wedge(iv) \Rightarrow U=I\}$. □

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NOTATION

The symbols that are used in the text are listed with a short explanation and with reference to the page number on which they first appear.

General notations

$\mathbb{Z}$	Set of integers {..,-1,0,1,2,..}	18
$\mathbb{Z}_+$	Set of nonnegative integers; $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$	18
$\mathbb{Z}_-$	Set of negative integers; $\mathbb{Z}_- = \mathbb{Z} \cap (-\infty,-1]$	18
$\mathbb{R}$	Set of real numbers	17
$\mathbb{R}^q$	Set of q-dimensional real vectors	18
$\mathbb{R} \cup \{\infty\}$	Set of real numbers extended with infinity	32
$\mathbb{N}$	Set of natural numbers; $\mathbb{N} = \mathbb{Z}_+ \cap [1,\infty)$	24
$\underline{p}$	Set $\mathbb{N} \cap [1,p]$ with $p \in \mathbb{N}$	70
$\mathbb{C}$	Set of complex numbers	
$A \backslash \{a_0\}$	General set A with exclusion of the element $a_0 \in A$	22
$\overline{A}$	Closure of A	24
$\{a_1,a_2,.\}$	Set consisting of the elements a_1, a_2 etc.	
$(a_1,a_2,.)$	Mapping with (a subset of) $\mathbb{N}$ as domain	
$\cup$	Union of sets	33
$\cap$	Intersection of sets	35
$\subset$	Subset (not necessarily a strict subset) of	17
$\oplus$	Direct sum of two sets; $A = A_1 \oplus A_2$ means that for every $a \in A$ there exists a unique decomposition $a=a_1+a_2$, such that $a_1 \in A_1$ and $a_2 \in A_2$	25
$\wedge$	Logical and	111
$col(w_1,w_2) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$		77
I_p	(p×p) identity matrix	74
Im	Image (range) of a mapping	43
$\arg\min f(\theta)$	with f a mapping f: $F \to G$ $:= \{\theta' \in F \mid f(\theta') \leq f(\theta) \text{ for all } \theta \in F\}$	32
$\arg\max f(\theta)$	with f a mapping f: $F \to G$ $:= \{\theta' \in F \mid f(\theta') \geq f(\theta) \text{ for all } \theta \in F\}$	38
dim	dimension	83
$\delta(\tau)$	Dirac function δ: $\mathbb{Z} \to \mathbb{N}$, with $\delta(\tau)=1$ for $\tau=0$ and $\delta(\tau)=0$ for $\tau \neq 0$	110

Signals

E	Residual signal set ($=\mathbb{R}^r$)	27
U	Input signal set ($=\mathbb{R}^m$)	19
Y	Output signal set ($=\mathbb{R}^p$)	19
V	Signal set of measurable signals: $V = Y \times U$ ($=\mathbb{R}^{p+m}$)	27
W	Signal set of a dynamical system	17
T	General time set $T \subset \mathbb{R}$	17
w	w: $\mathbb{Z} \to W$ General signal or system variable $w \in W^{\mathbb{Z}}$	17
y,u,e,v	Signals $y \in Y^{\mathbb{Z}}$, $u \in U^{\mathbb{Z}}$, $e \in E^{\mathbb{Z}}$, $v \in V^{\mathbb{Z}}$	27
x	state $x \in (\mathbb{R}^n)^{\mathbb{Z}}$	48
$w\|_T$	Signal w restricted to $T \subset \mathbb{Z}$	20
w^+	$= w\|_{\mathbb{Z}_+}$ with $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$	18
w^-	$= w\|_{\mathbb{Z}_-}$ with $\mathbb{Z}_- = \mathbb{Z} \cap (-\infty,-1]$	18
0	0: $\mathbb{Z} \to W$, with $0(t)=0$ for $t \in \mathbb{Z}$	18
T_N	Specific time set $T_N = \mathbb{Z} \cap [0,N-1]$, with $N \in \mathbb{N}$	24
w^N	$= w\|_{T_N}$	24
$w_1 \overset{\wedge}{}_{t_1} w_2$	concatenation of signals: a map $\mathbb{Z} \to W$ with $(w_1 \overset{\wedge}{}_{t_1} w_2)(t) = w_1(t)$ for $t<t_1$, and $= w_2(t)$ for $t \geq t_1$; it can also be used for signals defined on some $\mathbb{Z}_T \subset \mathbb{Z}$	21
σ	shift operator σ: $W^{\mathbb{Z}} \to W^{\mathbb{Z}}$ with $(\sigma w)(t) = w(t+1)$	18
σ^{-1}	shift operator σ^{-1}: $W^{\mathbb{Z}} \to W^{\mathbb{Z}}$ with $(\sigma^{-1}w)(t) = w(t-1)$	18
W^T	set of all maps w: $T \to W$	17
$(l_2)^q$	q-dimensional l_2 sequence (Hilbert) space	40
$(l_\infty)^q$	q-dimensional l_∞ sequence space	34

Models, model sets and parametrizations

S	General dynamical system	17
S_c	Controllable part of a dynamical system	23
M	General dynamical input-output system, or (i/o/r)-model	27
M^{io}	Input/output part of an (i/o/r)-model	29
$\hat{M}$	General optimal (i/o/r)-model	31

Σ	Set of all linear, time invariant, finite dimensional systems on $\mathbb{Z}$	18
$\Sigma_{p,m}$	Set of all input-output systems $S \subset \Sigma$ with signal set $W = Y \times U = \mathbb{R}^p \times \mathbb{R}^m$	20
$\Sigma_{p,m}(n)$	Subset of $\Sigma_{p,m}$ of all controllable and causal models with order n	133
$\tilde{\Sigma}$	Set of all (i/o/r)-models	27
$\tilde{\Sigma}_{p,m,r}$	Subset of $\tilde{\Sigma}$ with $W = Y \times U \times E$ and $Y = \mathbb{R}^p$, $U = \mathbb{R}^m$ and $E = \mathbb{R}^r$	27
$\hat{\Sigma}$	Set of all (i/o/pr)-models	28
$\hat{\Sigma}_{p,m}$	Subset of $\hat{\Sigma}$ with $W = Y \times U \times E$ and $Y = \mathbb{R}^p$, $U = \mathbb{R}^m$ and $E = \mathbb{R}^p$	28
$\mathcal{M}$	General set of (i/o/r)-models	27
$\mathcal{M}_{p,m,r}$	Subset of $\mathcal{M}$ with $W = Y \times U \times E$ and $Y = \mathbb{R}^p$, $U = \mathbb{R}^m$ and $E = \mathbb{R}^p$	27
$2^{\mathcal{M}}$	Set of all subsets of $\mathcal{M}$	31
$\tilde{M}$	Parametrization	43
$\tilde{M}_s$	Parametrization in (A,B,C,D) state space form	48
$\tilde{M}_p$	Parametrization in polynomial matrix form	45
Θ	General parameter set $\Theta \subset \mathbb{R}^d$, with $d \in \mathbb{N}$	43
Θ_s	Parameter set for a parametrization in (A,B,C,D) state space form	48
Θ_p	Parameter set for a parametrization in polynomial matrix form	45
$(\gamma_i)_{i=1,..,p}$	Observability indices of a state representation with p outputs	87
$\{\rho_i\}_{i=1,..,p}$	Left dynamical indices of a rational matrix with full row rank p	87
$\{\kappa_i\}_{i=1,..,p}$	Left structure indices of an i-o dynamical system with p outputs	88
$(\zeta_i)_{i=1,..,p}$	Pseudo-structure indices of an i-o dynamical system	136
$\mathcal{M}_{p,m,\gamma_1,..,\gamma_p}$	Set of i-o dynamical systems with $Y = \mathbb{R}^p$, $U = \mathbb{R}^m$, with a state space representation having observability indices $\gamma_1, \cdot, \gamma_p$.	91
$\tau(\mathcal{M})$	Minimal number of parameters required for parametrizing $\mathcal{M}$	90
c(M)	Complexity of a dynamical system	92
n(M)	Model order	83
$\delta_M(H)$	McMillan degree of a rational matrix	83
H	General transfer function	20
H_{ey}	Transfer function from output to residual in an (i/o/pr)-model	63
H_{eu}	Transfer function from input to residual in an (i/o/pr)-model	63
H_{ye}	Transfer function from residual to output in an (i/o/r)-model	63
H_{yu}	Transfer function from input to output in an i-o dynamical system, or in an (i/o/r)-model	63
$H_{e_2e_1}$	Transfer function between residuals of two (i/o/r)-models	113

A,B,C,D	Matrices constituting a state space representation	48
B_u,B_e	Input matrices in a state space representation	48
D_u,D_e	Input-output matrices in a state space representation	48
$\mathscr{M}^{pe}_{p,m}(L)$	Specific set of output error models with m inputs and p outputs	104
$\mathscr{M}^{pe}_{p,m}(L)$	Specific set of prediction error models with m inputs and p outputs	104

Equivalence relations and identification criteria

$\sim$	General equivalence relation; behaviour-equivalence	29
$\sim$	Equivalence of polynomial matrices	127
$\overset{t}{\underset{io}{\sim}}$	Input/output controllable behaviour- or transfer-equivalence	30
$\overset{t}{\sim}$	Controllable behaviour- or transfer-equivalence	30
$\underset{io}{\sim}$	Input/output behaviour-equivalence	30
$\overset{c}{\underset{LS\infty}{\sim}}$	Scaling-equivalence with respect to $\ell^+_{LS\infty}$	120
$\overset{J^N}{\underset{\mathscr{M}}{\sim}}$	J^N-equivalence of models within $\mathscr{M}$	34
J^N	Identification criterion over T_N	31
J^+	Identification criterion over $\mathbb{Z}_+$	40
$J^+_{LS\infty}$	Least squares identification criterion over $\mathbb{Z}_+$	98
ℓ^N	Residual function	39
$\ell^+_{LS\infty}$	Least squares residual function over $\mathbb{Z}_+$	98
f	Criterion function	38

Behaviours and signal sets

$\mathscr{B}$	Behaviour	17
$\mathscr{B}(S)$	Behaviour of dynamical system S on $\mathbb{Z}$	17
$\mathscr{B}(S)\|_T$	Behaviour of S restricted to $T \subset \mathbb{Z}$	22
$\mathscr{B}^+(S)$	Behaviour of S restricted to $\mathbb{Z}_+$	18
$\mathscr{B}^-(S)$	Behaviour of S restricted to $\mathbb{Z}_-$	18
$\mathscr{B}^N(S)$	Behaviour of S restricted to T_N	24

$\mathcal{B}_c(S)$	Controllable part of $\mathcal{B}(S)$	23
$\mathcal{B}_0(S)$	Initial zero part of $\mathcal{B}(S)$	24
$\mathcal{B}^N_{obs,y}(S)$	Observable behaviour of S on T_N with respect to y	25
$\mathcal{B}^N_{init,y}(S)$	Initial condition behaviour of S on T_N with respect to y	25
$\mathcal{B}_{w_1}(S)$	Behaviour of variable w_1 within S	19
$\mathcal{B}^{io}(M)$	Input-output behaviour of an (i/o/r)-model	29
$\mathcal{Y}(u,S)$	Set of admissible output signals for a given input $u \in U^{\mathbb{Z}}$, for an i-o dynamical system S	26
$\mathcal{Y}^N(u^N,S)$	Restriction of $\mathcal{Y}(u,S)$ to T_N and u^N	26
$\mathcal{Y}^N_{obs}(u^N,S)$	Set of admissible observable output signals of S for a given input u^N	26
$\mathcal{Y}^N_{init}(S)$	Initial condition behaviour of y on T_N within S	26
$\mathcal{E}(v,M)$	Set of admissible residuals of an (i/o/r)-model M for a given $v \in V^{\mathbb{Z}}$	29
$\mathcal{E}^N(v^N,M)$	Restriction of $\mathcal{E}(v,M)$ to T_N and v^N	29
$\mathcal{E}^N_{obs}(v,M)$	Observable part of $\mathcal{E}^N(v^N,M)$	29
$\mathcal{E}^N_{init}(M)$	Initial conditions behaviour of residuals within an (i/o/r)-model M	29
$V^{T_N}_{J,\mathcal{M}}$	Set of selecting data sequences $v^N \in V^{T_N}$ such that $J^N(v^N,\mathcal{M}) \neq \emptyset$	34
$\mathcal{B}^{io,+}(M)$	Restriction of $\mathcal{B}^{io}(M)$ to $T=\mathbb{Z}_+$	54
$\mathcal{R}(M_1,M_2)$	Set of admissible residual trajectories of two (i/o/r)-models	109

Matrices

$\mathbb{R}[z,z^{-1}]$	Ring of polynomials (with finite degree) in the indeterminates z,z^{-1}	18
$\mathbb{R}^{r_1 \times r_2}[z,z^{-1}]$	$(r_1 \times r_2)$-matrices over the ring $\mathbb{R}[z,z^{-1}]$	18
$\mathbb{R}(z)$	Field of rational functions in the indeterminate z	20
$\mathbb{R}^{r_1 \times r_2}(z)$	$(r_1 \times r_2)$-matrices over the field $\mathbb{R}(z)$	
$\Gamma_{hc}(T)$	Leading column coefficient matrix of $T \in \mathbb{R}^{r_1 \times r_2}[z,z^{-1}]$	71
$\Gamma_{lc}(T)$	Trailing column coefficient matrix of $T \in \mathbb{R}^{r_1 \times r_2}[z,z^{-1}]$	71
$\Gamma_{hr}(T)$	Leading row coefficient matrix of $T \in \mathbb{R}^{r_1 \times r_2}[z,z^{-1}]$	71
$\Gamma_{lr}(T)$	Trailing row coefficient matrix of $T \in \mathbb{R}^{r_1 \times r_2}[z,z^{-1}]$	71
$\text{rank}_{\mathbb{C}}(T)$	Rank of a matrix Tover $\mathbb{C}$; $\text{rank}_{\mathbb{C}}(T) \in \mathbb{C}$	47

$\det_{\mathbb{C}}(T)$	Determinant of a matrix T over $\mathbb{C}$; $\det_{\mathbb{C}}(T)\in\mathbb{C}$	
$\operatorname{rank}_{\mathbb{R}(z)}(T)$	Rank of a matrix over T over $\mathbb{R}(z)$; $\operatorname{rank}_{\mathbb{R}(z)}(T)\in\mathbb{R}(z)$	46
$\det_{\mathbb{R}(z)}(T)$	Determinant of a matrix T over $\mathbb{R}(z)$; $\det_{\mathbb{R}(z)}(T)\in\mathbb{R}(z)$	20
T_f	General polynomial matrix in $\mathbb{R}[z]$	72
T_b	General polynomial matrix in $\mathbb{R}[z^{-1}]$	72
T_{i*}	i^{th} row of matrix T	70
T_{*j}	j^{th} column of matrix T	70
$\pi_i^{(u)}(T)$	Highest power of z over all (i×i)-minors of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	84
$\pi_i^{(\ell)}(T)$	Lowest power of z over all (i×i)-minors of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	84
$\delta_{ij}^{(u)}(T)$	maximum power of z in $T_{ij}\in\mathbb{R}[z,z^{-1}]$	70
$\delta_{ij}^{(l)}(T)$	minimum power of z in $T_{ij}\in\mathbb{R}[z,z^{-1}]$	70
$\delta_{ij}(T)$	degree of $T_{ij}\in\mathbb{R}[z,z^{-1}]$	70
$\delta^{(u)}(T)$	maximum power of z in $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\delta^{(\ell)}(T)$	minimum power of z in $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\delta(T)$	degree of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\nu_i^{(u)}(T)$	maximum power of z in the i^{th} row of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\nu_i^{(\ell)}(T)$	minimum power of z in the i^{th} row of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\nu_i(T)$	i^{th} row degree of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\mu_j^{(u)}(T)$	maximum power of z in the j^{th} column of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\mu_j^{(\ell)}(T)$	minimum power of z in the j^{th} column of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\mu_j(T)$	j^{th} column degree of $T\in\mathbb{R}^{r_1 x r_2}[z,z^{-1}]$	70
$\operatorname{diag}(a_1,.,a_p)$	Diagonal matrix with diagonal element $a_1,.,a_p$	124
tr	Trace of a matrix	112

SUBJECT INDEX

SAMENVATTING

Dit proefschrift behandelt het probleem van de systeemidentifikatie, het modelleren van dynamische systemen op basis van meetgegevens. Het identifikatieprobleem wordt in dit werk beschouwd als een probleem van deterministisch benaderend modelleren. In vergelijking met klassieke methoden van systeemidentifikatie behelst dit twee belangrijke aspekten:

- er wordt bij voorbaat geaccepteerd dat de verkregen modellen slechts een benadering kunnen vormen van het dynamische systeem dat de meetgegevens gegenereerd heeft; en
- in de methodologie van het konstrueren van modellen op basis van meetdata wordt geen gebruik gemaakt van enige statistische veronderstellingen omtrent de meetgegevens .

Als een konsekwentie van deze uitgangspunten wordt er van identifikatiemethoden geëist dat zij modellen opleveren die bepaald worden door de betreffende meetdata en door gespecificeerde gebruikerskeuzen, en niet door impliciete (statistische) en moeilijk verifieerbare veronderstellingen omtrent de data en het onderliggende proces.

Gebaseerd op systeemtheoretische concepten die recentelijk in de literatuur zijn geintroduceerd, wordt in dit proefschrift een raamwerk gepresenteerd waarin het identifikatieprobleem, als boven beschouwd, duidelijk wordt geformuleerd. In dit raamwerk worden de verschillende onderdelen van een identifikatiemethode: de modelverzameling $\mathcal{M}$, de parametrisering $\tilde{\mathrm{M}}$ en het identifikatiekriterium J op een fundamentele en natuurlijke wijze gedefinieerd. De modelverzameling $\mathcal{M}$ bevat alle modellen die in beschouwing worden genomen bij het zoeken naar "optimale" modellen voor de gegeven tijdreeks; het identifikatiekriterium selekteert, gegeven de tijdreeks, optimale modellen uit deze verzameling; de parametrisering representeert de modellen in de modelverzameling door middel van (reële) parameters om de bovengenoemde selektieprocedure te vereenvoudigen.
In dit proefschrift staat de vraag ter diskussie welke eisen er gesteld dienen te worden aan deze drie basisconcepten opdat de uiteindelijk verkregen modellen duidelijk gedefinieerde benaderingen vormen van de beschikbare tijdreeks.

Modellen voor identifikatie worden in dit werk gedefinieerd door middel van hun "gedrag"; dit is de ruimte van toegestane signaaltrajektorieën. Het begrip modelgedrag blijkt een overkoepelend begrip te zijn voor alle bestaande definities van

lineaire, tijdinvariante en eindig–dimensionale systemen, zoals overdrachtsfunkties, toestandsbeschrijvingen en beschrijvingen met behulp van differentievergelijkingen. Dit concept geeft de mogelijkheid om een duidelijk onderscheid te maken tussen modelverzamelingen en parametriseringen, en vervolgens ook tussen het identifikatie– en het parametriseringsprobleem.

In het gepresenteerde raamwerk voor systeemidentifikatie worden modellen gehanteerd die drie typen van externe signalen bevatten: ingangssignalen, uitgangssignalen en residusignalen. De residusignalen zijn kunstmatig aan de (residu–gebaseerde) modellen toegevoegd voor de weergave van modelfouten, en om te fungeren als een basis voor de bepaling van een maat van afwijking tussen een model en een gegeven tijdreeks.

De gebruikelijke identifikatiemethoden zijn gekarakteriseerd binnen dit raamwerk, waarbij speciale aandacht is besteed aan de verschillende typen van residusignalen (voorspellingsfout–, uitgangsfout– en vergelijkingsfout–residusignalen). De specifieke gebruikerskeuzen die aan deze methoden ten grondslag liggen worden onderzocht. Er wordt aangetoond dat voor de populaire klasse van vergelijkingsfout-identifikatiemethoden, de resultaten van de identifikatie in zekere zin arbitrair kunnen worden, beinvloed veeleer door toevallige situaties dan door duidelijk gedefinieerde gebruikerskeuzen. Dit fenomeen wordt veroorzaakt door het feit dat de aspekten van identifikatie en parametrisering tot dusverre niet goed op elkaar zijn afgestemd.

Een inleiding tot het identifikatieprobleem, zoals beschouwd in dit proefschrift, wordt gegeven in hoofdstuk 1.
In hoofdstuk 2 wordt een raamwerk gepresenteerd voor het formuleren van het systeemidentifikatieprobleem op basis van residusignalen. Algemeen gehanteerde begrippen als modelverzameling, parametrisering, identifikatiekriterium en identificeerbaarheid worden heroverwogen en duidelijk gedefinieerd. Discrimineerbaarheid van modelverzamelingen wordt geintroduceerd.
In hoofdstuk 3 worden de gebruikelijke systeemidentifikatiemethoden gekarakteriseerd binnen het gepresenteerde raamwerk. Het aspekt modelcomplexiteit komt aan de orde, hetgeen leidt tot een theorie omtrent de orde en struktuurindices van multivariabele systemen. Door gebruik te maken van polynoommatrices in twee onbepaalden, leidt deze theorie tot een generalisatie van de bestaande separate theorieën voor modellen gepresenteerd in voorwaartse dan wel achterwaartse differentievergelijkingen.

Hoofdstuk 4 presenteert een uitwerking van het probleem van de konstruktie van identificeerbare parametriseringen voor een identifikatiekriterium gebaseerd op kleinste kwadraten. Er wordt aangetoond dat, om te komen tot duidelijk gedefinieerde modellen, de identificeerbaarheid van parametriseringen beschouwd dient te worden in relatie tot het toegepaste identifikatiekriterium.
In de hoofdstukken 5 en 6 zijn een aantal aanvullende opmerkingen omtrent de behandelde identifikatiemethoden opgenomen, en worden de konklusies van dit werk samengevat.

LEVENSBERICHT

Paul Van den Hof werd op 7 maart 1957 geboren in Maastricht. Zijn middelbare schoolopleiding volgde hij aan het Sint–Maartenscollege en het Henric van Veldekecollege, beide te Maastricht, waar hij in 1975 het diploma gymnasium–β behaalde. Vanaf september 1975 studeerde hij elektrotechniek aan de Technische Universiteit Eindhoven. In december 1982 behaalde hij daar zijn ingenieursdiploma (met lof). Zijn afstudeerprojekt verrichtte hij in de vakgroep Meten en Regelen van de Faculteit der Elektrotechniek; voor dit afstudeerwerk werd hem de KIVI Regeltechniekprijs 1982 toegekend. Van januari 1983 tot juli 1984 was hij, als gewetensbezwaarde militaire dienst, door het Ministerie van Sociale Zaken tewerkgesteld bij de vakgroep Meten en Regelen van de faculteit Elektrotechniek van de Technische Universiteit Eindhoven. In dezelfde vakgroep was hij werkzaam als wetenschappelijk assistent van september 1984 tot augustus 1986. Vanaf 1 september 1986 is hij werkzaam als universitair docent aan de Faculteit der Werktuigbouwkunde en Maritieme Techniek van de Technische Universiteit Delft, bij de sektie Technische Systemen van de vakgroep Meet– en Regeltechniek.

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STELLINGEN

behorende bij het proefschrift

"On Residual–Based Parametrization
and Identification of Multivariable Systems"

van

Paul Van den Hof

Eindhoven
3 maart 1989

[1]

Beschouw een rij van N+1 Markovparameters, $\{M(k)\}$, k=0,.,N, met $M(k) \in \mathbb{R}^{pxm}$, en laat een partiële realisatie van een dergelijke rij gedefinieerd zijn door een viertal matrices (A,B,C,D) met $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $C \in \mathbb{R}^{pxn}$, $D \in \mathbb{R}^{pxm}$, die voldoen aan

$$M(k) = CA^{k-1}B \quad \text{voor} \quad 1 \leq k \leq N, \quad \text{en}$$

$$M(k) = D \quad \text{voor} \quad k=0,$$

waarbij n de dimensie van de partiële realisatie is.

(i) De minimale dimensie n_{min} van een partiële realisatie van $\{M(k)\}$, k=0,.,N wordt generiek gegeven door:

$$n_{min} = \max \{ p \cdot \text{entier}[(N+1)\tfrac{m}{p+m}], m \cdot \text{entier}[(N+1)\tfrac{p}{p+m}] \}.$$

(ii) Een minimale realisatie van $\{M(k)\}$ k=0,.,N is generiek uniek dan en slechts dan als N geschreven kan worden als $N = a\,\frac{m+p}{\ell} - 1$, met ℓ de grootste gemene deler van p en m, en $a \in \mathbb{N}$.

Damen, A.A.H., R.P. Guidorzi, A.K. Hajdasinski en P.M.J. Van den Hof (1985). On multivariable partial realization. *Int. J. Control, 41*, 589–613.

Van den Hof, P.M.J., A.A.H. Damen en A.K. Hajdasinski (1984). Multivariable partial realization and the aspect of uniqueness. In: J. Gertler en L. Keviczky (Eds.), *A Bridge between Control Science and Technology.* IFAC Proc. Series 1985, No.1, pp. 175–180. Proc. 9th IFAC World Congress, Budapest, 1984.

[2]

Het gebruik van een pagina matrix, als alternatief voor een Hankel matrix, bij methoden voor benaderende realisatie op basis van Markovparameters, kan leiden tot modellen die de gegeven Markovparameters beter benaderen.

Damen, A.A.H., P.M.J. Van den Hof en A.K. Hajdasinski (1982). Approximate realization based upon an alternative to the Hankel matrix: the Page matrix. *Syst. Control Lett., 2*, 202–208.

Van den Hof, P.M.J. (1984). Approximate realization of noisy linear multivariable systems. *Journal A, 25*, 21–26.

[3]

Het bijvoeglijk naamwoord "suboptimaal" duidt op het ontbreken van optimaliteitseigenschappen en heeft geen andere betekenis dan "niet optimaal". De gewekte suggestie dat er van enige optimaliteitseigenschappen sprake is, is derhalve onjuist.

[4]

De veelvuldig gehanteerde uitspraak dat bij parametriseringen in een observeerbaarheids kanonieke vorm *wel* een a priori bepaling van struktuurindices vereist is en bij parametriseringen in overlappende vorm niet, is in het algemeen onjuist indien de geparametriseerde modelverzamelingen gebruikt worden voor het toepassen van "equation error" identifikatiemethoden.

Van den Hof, P.M.J. en P.H.M. Janssen (1987). Some asymptotic properties of multivariable models identified by equation error techniques. *IEEE Trans. Autom. Control, AC–32*, 89–92.

Van den Hof, P.M.J. (1989). Criterion based equivalence for equation error models. *IEEE Trans. Autom. Control, AC–34*, No.2, February 1989.

[5]

Bij de aanpak van het identifikatieprobleem op basis van het Frisch schema, als beschreven in Kalman (1982), worden principieel andere uitgangspunten gehanteerd dan die welke ten grondslag liggen aan het werk gepresenteerd in dit proefschrift. Waar in het Frisch schema het centrale aspekt is gelegen in het exact modelleren onder verstoringskondities die ruimer zijn dan gebruikelijk, wordt er in dit proefschrift gekozen voor het uitgangspunt dat verkregen modellen in het algemeen benaderingen zullen zijn van onderliggende processen.

Kalman, R.E. (1982). System identification from noisy data. In: A.R. Bednarek and L. Cesari (Eds.). *Dynamical Systems II.* Academic Press, New York, pp. 135–164.

[6]

De verschuivingsoperator σ: $(\mathbb{R}^q)^{\mathbb{Z}} \rightarrow (\mathbb{R}^q)^{\mathbb{Z}}$, gedefinieerd door $(\sigma w)(t)=w(t+1)$, wordt in het algemeen aangeduid als voorwaartse of als achterwaartse verschuivingsoperator, afhankelijk van de mathematische dan wel technische achtergrond van auteurs. Teneinde verwarring te voorkomen verdient het aanbeveling om gebruik van deze benamingen altijd te laten vergezellen door een formele definitie. Gebruik van de notatie "z" voor de inverse operator σ^{-1} dient vermeden te worden in verband met een ongewenste verwarring met de z–transformatie.

[7]

Voor het vergroten van hun geloofwaardigheid verdient het aanbeveling dat politici, veel meer dan nu het geval is, aan kiezers duidelijk maken dat de standpunten die zij verkondigen het gevolg zijn van een keuzeproces waarin menselijke eigenschappen als onzekerheid en twijfel een rol spelen, en die derhalve niet noodzakelijkerwijze de enige mogelijke en juiste standpunten zijn.

[8]

De "Interimregeling ziektekosten ambtenaren 1982", (IZK 1982), benadeelt relatievormen waarin beide partners in deeltijd werken ten opzichte van situaties waarin slechts een van beide partners een werkkring heeft. Zowel uit het oogpunt van vrouwenemancipatie als van een gelijke verdeling van werk is zij derhalve niet meer van deze tijd.

[9]

Het identificeren van processen met behulp van de klassieke (statistische) methode van kleinste kwadraten kan eenvoudig leiden tot modellen die sterk worden bepaald door onbewuste keuzen van de gebruiker, en die slechts een marginaal verband vertonen met de onderliggende processen. Identifikatie op basis van deze methode kan derhalve beter worden aangeduid met "identifiktie".