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Brief Paper

An Indirect Method for Transfer Function Estimation from Closed Loop Data*

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Key Words—System identification, closed loop identification, prediction error methods, approximate models.

Abstract—An indirect method is introduced that is able to estimate consistently the transfer function of a linear plant on the basis of data obtained from closed loop experiments, even in the situation where the model of the noise disturbance on the data is not accurate. Moreover, the method allows approximate identification of the open loop plant with an explicit and tunable expression for the bias distribution of the resulting model.

1. *Introduction*

THE PROBLEM of parametric identification of a linear system on the basis of data obtained from closed loop experiments has received considerable attention in the literature. Several methods have been proposed and analyzed, either in the framework of (least-squares) prediction error methods (Söderström *et al.*, 1976; Gustavsson *et al.*, 1977; Anderson and Gevers, 1982), or in terms of instrumental variable methods (Söderström and Stoica, 1981). In the prediction error context, well-known approaches are the direct method, the indirect method and the joint input–output method. It has been established that—under weak conditions—the system's transfer function can be consistently identified, provided that the system is in the set of models that is considered. This rather restrictive condition refers to the input–output transfer function of the system, as to the noise-shaping filter of the noise contribution on the data. For instrumental variable methods, similar results have been derived, restricting only the input–output transfer function of the system to being present in the model set.

In many practical situations, our primary interest is not the consistent identification of the system, but the gathering of a good approximation of its input–output transfer function. In this paper, this problem will be discussed for a closed loop system configuration, in which an external (sufficiently exciting) reference signal or setpoint signal is present and measurable, and the controller is not assumed to be known. In the light of the remarks made above, we would like to come up with an identification method that is able to

(i) consistently identify the input–output transfer function regardless of whether the noise contribution on the data can be modelled exactly, and
(ii) formulate an explicit expression for the asymptotic bias distribution of the identified model when the input–output transfer function of the system cannot be modelled exactly.

Note that property (i) alone can also be reached through instrumental variable methods (Söderström and Stoica, 1981).

We will propose and analyze a two-stage identification method that satisfies the two requirements mentioned above, while still being composed of classical prediction error methods and using standard identification tools. Knowledge of the controller will not be required. Firstly, the sensitivity function of the closed loop system is identified through a high order linear model. This sensitivity function is used to simulate a noise-free input signal for an open loop identification of the plant to be identified. Using output error methods, in accordance with Ljung (1987), an explicit approximation criterion can be formulated, characterizing the bias of identified models in the case of undermodelling.

2. *Problem setting*

The identification framework we consider is adopted from Ljung (1987). We will consider a single-input single-output data-generating system that is defined as

$$S:\quad y(t) = G_0(q)u(t) + H_0(q)e(t) \tag{1}$$

with $y(t)$ the output signal, $u(t)$ the input signal, and $e(t)$ a zero mean unit variance white noise signal. $G_0(q)$ and $H_0(q)$ are proper rational functions in q, the forward shift operator, with $H_0(q)$ stable and stably invertible. The input signal is determined according to

$$u(t) = r(t) - C(q)y(t) \tag{2}$$

with C a linear controller and $r(t)$ a reference or setpoint signal. The closed loop system configuration that we consider is depicted in Fig. 1.

The parametrized set of models, considered to model the system S is denoted by

$$\mathcal{M}:\quad y(t) = G(q,\theta)u(t) + H(q,\theta)\varepsilon(t) \quad \theta \in \Theta \subset \mathbb{R}^d \tag{3}$$

with $G(q, \theta)$ and $H(q, \theta)$ proper rational transfer functions, depending on a real-valued parameter vector θ that is lying in a set Θ of admissible values, and ε the one step ahead prediction error (Ljung, 1987). The notation $S \in \mathcal{M}$ is used to indicate that there exists a $\theta_0 \in \Theta$ such that $G(z, \theta_0) = G_0(z)$ and $H(z, \theta_0) = H_0(z)$ for almost all $z \in \mathbf{C}$. The notation $G_0 \in \mathcal{G}$ accordingly refers to the situation that only $G(z, \theta_0) = G_0(z)$ for almost all $z \in \mathbf{C}$.

In the open loop case $C(q) \equiv 0$, it is well known (Ljung, 1987) that when $G_0 \in \mathcal{G}$, $S \notin \mathcal{M}$, it is possible—under weak conditions—to estimate G_0 consistently using prediction error methods, provided that $G(q, \theta)$ and $H(q, \theta)$ are independently parametrized within $\mathcal{M}$. To this end prediction error estimates are very often suggested with a fixed noise model: $H(q, \theta) = L(q)$, as e.g. the output error model structure, having $L(q) = 1$. In this situation the asymptotic parameter estimate is characterized by the explicit

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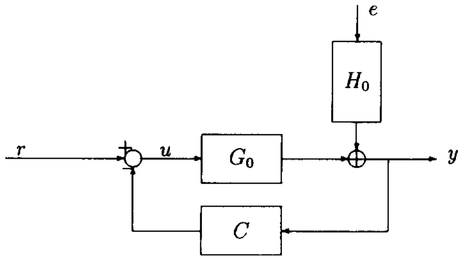

FIG. 1. The closed loop system configuration.

approximation criterion

$$\lim_{N\to\infty} \hat{\theta}_N = \arg\min_{\theta} \int_{-\pi}^{\pi} |G_0(e^{i\omega}) - G(e^{i\omega}, \theta)|^2 \times \frac{\Phi_u(\omega)}{|L(e^{i\omega})|^2} d\omega \quad \text{w.p. 1}, \quad (4)$$

with $\Phi_u(\omega)$ the spectral density of u.

In the closed loop situation, this consistency property of G_0 is lost, as well as the validity of the approximation criterion (4), due to the fact that the input signal u is not uncorrelated with the noise disturbance e. We will show that, by reorganizing the closed loop configuration (1), (2), we are able to create a situation where we can repeatedly apply the open loop results in order to reach our goals.

3. *A two-stage identification strategy*

Let us consider the sensitivity function of the closed loop system (1), (2),

$$T_0(q) = \frac{1}{1 + G_0(q)C(q)}. \quad (5)$$

Using T_0 we can rewrite equations (1), (2):

$$u(t) = T_0(q)r(t) - C(q)T_0(q)H_0(q)e(t) \quad (6)$$

$$y(t) = G_0(q)u(t) + H_0(q)e(t). \quad (7)$$

Since r and e are uncorrelated signals, and u and r are available from measurements, it follows from (6) that we can identify the sensitivity function T_0 in an open loop way. Using the open loop results as mentioned in the previous section, we can even identify $T_0(q)$ consistently, irrespective of the noise contribution $C(q)T_0(q)H_0(q)e(t)$ in (6), using any model structure

$$u(t) = T(q, \beta)r(t) + R(q, \gamma)\varepsilon_u(t) \quad \beta \in B \subset \mathbb{R}^{d_\beta}; \quad \gamma \in \Gamma \subset \mathbb{R}^{d_\gamma} \quad (8)$$

where $\varepsilon_u(t)$ is the one step ahead prediction error of $u(t)$, and T and R are parametrized independently.

The estimate $T(q, \hat{\beta}_N)$ of $T_0(q)$ is determined according to a least-squares criterion

$$\hat{\beta}_N = \arg_\beta \min_{\beta,\gamma} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_u(t)^2. \quad (9)$$

Consistency of $T(q, \hat{\beta}_N)$ can of course only be reached when $T_0 \in \mathcal{T} := \{T(q, \beta) \mid \beta \in B\}$.

By again manipulating equations (6), (7), we can write

$$u^r(t) := T_0(q)r(t) \quad (10)$$

$$y(t) = G_0(q)u^r(t) + T_0(q)H_0(q)e(t). \quad (11)$$

Since u^r and e are uncorrelated, it follows from (11) that when u^r would be available from measurements, G_0 could be estimated in an open loop way, using the common open loop techniques. Instead of knowing u^r, we have an estimate of this signal available through

$$\hat{u}^r_N(t) = T(q, \hat{\beta}_N)r(t). \quad (12)$$

Consider the model structure

$$y(t) = G(q, \theta)\hat{u}^r_N(t) + H(q, \eta)\varepsilon_y(t) \quad (13)$$

with $G(q, \theta)$, $H(q, \eta)$ parametrized independently, $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$, $\eta \in \Omega \subset \mathbb{R}^{d_\eta}$. It will be shown that the estimate $G(q, \hat{\theta}_N)$ of $G_0(q)$, determined by

$$\hat{\theta}_N = \arg_\theta \min_{\theta,\eta} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_y(t)^2 \quad (14)$$

under weak conditions, converges to $G_0(q)$ with probability 1. This result is formalized in the following theorem.

Theorem 3.1. Given the closed loop system determined by (1), (2), with $T_0(q)$ asymptotically stable, e and r uncorrelated quasi-stationary signals, and r persistently exciting of sufficient order.

Consider the two-stage identification procedure presented in this section with model structures and identification criteria (8), (9) for step 1, and (13), (14) for step 2.

If $T_0 \in \mathcal{T}$ and $G_0 \in \mathcal{G}$ then, under weak conditions, $G(q, \hat{\theta}_N) \to G_0(q)$ with probability 1 as $N \to \infty$.

Proof. The identification procedure in the first step, determined by (8), (9), is known to yield a consistent estimate of the transfer function T_0, provided that r is persistently exciting of sufficient order. This implies that

$$T(q, \hat{\beta}_N) \to T_0(q) \quad \text{with probability 1, as } N \to \infty. \quad (15)$$

For the second step (13), (14), we can write

$$\hat{\theta}_N = \arg_\theta \min_{\theta,\eta} \frac{1}{N} \sum_{t=1}^{N} V_N(\theta, \eta, t)$$

with

$$V_N(\theta, \eta, t) = \{H(q, \eta)^{-1}[y(t) - G(q, \theta)T(q, \hat{\beta}_N)r(t)]\}^2 \quad (16)$$

$$= \{H(q, \eta)^{-1}[G_0(q)T_0(q) - G(q, \theta)T(q, \hat{\beta}_N)]r(t) + H(q, \eta)^{-1}[H_0(q)T_0(q)]e(t)\}^2. \quad (17)$$

We know from Ljung (1987) that under weak conditions,† for $N \to \infty$,

$$\hat{\theta}_N \to \arg_\theta \min_{\theta,\eta} EV_N(\theta, \eta, t) \quad \text{with probability 1.}$$

Since r and e are uncorrelated, and $T(q, \hat{\beta}_N) \to T_0(q)$ with probability 1 as $N \to \infty$, the two terms that contribute to (17), being correlated with either r or e, are asymptotically uncorrelated, as $\hat{\beta}_N$ becomes independent of e. With the assumption that for very large N we may neglect the correlation, it follows that

$$\hat{\theta}_N \to \arg\theta \min_{\theta,\eta} E\{\{H(q, \eta)^{-1}[G_0(q) - G(q, \theta)]T_0(q)r(t)\}^2 + \{H(q, \eta)^{-1}[H_0(q)T_0(q)]e(t)\}^2\}. \quad (18)$$

This reasoning is similar to that in multi-step instrumental variable identification methods (see Stoica and Söderström, 1983). Exact uncorrelation also for finite values of N, can be obtained by using two separate and independent data sequences for the two steps in the algorithm.

If r is persistently exciting of sufficient order, $G_0 \in \mathcal{G}$, and G and H are parametrized independently, (18) implies that $G(q, \theta_N) \to G_0(q)$ with probability 1 as $N \to \infty$.

In the case that we accept undermodelling in the second step of the procedure, ($G_0 \notin \mathcal{G}$), the bias distribution of the asymptotic model can be characterized.

Proposition 3.2. Consider the situation of Theorem 3.1.

If $T_0 \in \mathcal{T}$, and if in step 2 of the identification procedure, determined by (13), (14), a fixed noise model is used, i.e. $H(q, \eta) = L(q)$, then, under weak conditions, $\hat{\theta}_N \to \theta^*$ with

† In this situation the 'weak conditions' include that the family of filters $\{T(q, \beta), \beta \in B\}$ is uniformly stable.

probability 1 as $N \to \infty$, with

$$\theta^* = \arg\min_{\theta} \int_{-\pi}^{\pi} |G_0(e^{i\omega}) - G(e^{i\omega}, \theta)|^2 \times \frac{|T_0(e^{i\omega})|^2 \Phi_r(\omega)}{|L(e^{i\omega})|^2} d\omega. \quad (19)$$

Proof. The proposition follows from transforming equation (18) to the frequency domain, employing Parsseval's relation.

In this situation of approximate modelling of G_0, the asymptotic estimate can be characterized by the explicit approximation criterion (19). It is remarkable, and at the same time quite appealing, that in this closed loop situation, the approximation of G_0 is obtained with an approximation criterion that has the sensitivity function T_0 of the closed loop system as a weighting function in the frequency domain expression (19). The fixed noise model $L(q)$ can be used as a design variable in order to 'shape' the bias distribution (20) to a desired form.

An even more general result is formulated in the following proposition, dealing also with the situation $T_0 \notin \mathcal{T}$.

Proposition 3.3. Consider the situation of Theorem 3.1.

If in both step 1 and 2 of the identification procedure fixed noise models are used, i.e. $R(q, \gamma) = K(q)$ and $H(q, \eta) = L(q)$, then, under weak conditions, $\hat{\theta}_N \to \theta^*$ with probability 1 as $N \to \infty$, with

$$\theta^* = \arg\min_{\theta} \int_{-\pi}^{\pi} |G_0(e^{i\omega})T_0(e^{i\omega}) - G(e^{i\omega}, \theta)T(e^{i\omega}, \beta^*)|^2 \frac{\Phi_r(\omega)}{|L(e^{i\omega})|^2} d\omega \quad (20)$$

and

$$\beta^* = \arg\min_{\beta} \int_{-\pi}^{\pi} |T_0(e^{i\omega}) - T(e^{i\omega}, \beta)|^2 \frac{\Phi_r(\omega)}{|K(e^{i\omega})|^2} d\omega. \quad (21)$$

Proof. The proof follows from similar reasoning as in the proof of Theorem 3.1, and proposition 3.2; however, now with the sensitivity function T_0 substituted by its limiting estimate $T(q, \beta^*)$.

Proposition 3.3 shows that even when in both steps of the procedure nonconsistent estimates are obtained, the bias distribution of $G(q, \theta^*)$ is characterized by a frequency domain expression which is dependent on the identification result from the first step [cf. (21)].

Remark 3.4. Note that in (20) the integrand expression can be rewritten, using the relation

$$\begin{aligned} G_0(e^{i\omega})T_0(e^{i\omega}) - G(e^{i\omega}, \theta)T(e^{i\omega}, \beta^*) &= [G_0(e^{i\omega}) - G(e^{i\omega}, \theta)]T_0(e^{i\omega}) \\ &+ G(e^{i\omega}, \theta)[T_0(e^{i\omega}) - T(e^{i\omega}, \beta^*)] \quad (22) \end{aligned}$$

which shows how an error made in the first step affects the estimation of G_0. If $T(q, \beta^*) = T_0(q)$ then (20) reduces to (19). If the error made in the first step is sufficiently small it will have a limited effect on the final estimate $G(q, \theta^*)$.

Note that the results presented in this section show that a consistent estimation of the sensitivity function T_0 is not even necessary to get a good approximate identification of the transfer function G_0. Equations (20) and (22) suggest that as long as the error in the estimated sensitivity function is sufficiently small, the input–output transfer function can be identified accurately. In this respect, one could also think of applying an FIR (finite impulse response) model structure (Ljung, 1987) in the first step, having a sufficient polynomial degree to describe the essential dynamics of the sensitivity function. This model structure will be applied in the simulation example described in the next section. Another possibility is to consider models consisting of a finite number of alternative orthogonal functions, such as Laguerre functions or generalized versions (Heuberger and Bosgra, 1990; Wahlberg, 1991).

Remark 3.5. In the procedure presented we have made use of three available signals r, u and y to perform two identification steps: first estimating the sensitivity of the closed loop system, and second estimating the open loop plant. If, instead of r, we had knowledge of the controller C, an alternative method could meet our requirements (i), (ii) as mentioned in the introduction, by solving a least-squares identification problem, using the model structure

$$y(t) = \frac{G(q, \theta)}{1 + G(q, \theta)C(q)} u(t) + \varepsilon(t).$$

This has been suggested in problem 14T.2 in Ljung (1987). However, on the one hand this would require the solution of a complicatedly parameterized identification problem for which no standard tools are available. On the other hand, the problem of estimating C, when it is not available *a priori*, again leads to a closed loop identification problem, for which a consistent estimate might be obtained, but for which an approximate model shows dependency on the plant dynamics and the noise contribution in the loop.

An alternative nonparametric solution to the problem, using *a priori* knowledge of the controller, is discussed in Schrama (1991).

Remark 3.6. The feedback structure (2) is conformable to the one used in related identification papers. Similar results can be derived for the alternative one-degree-of-freedom controller, $u = C(r - y)$.

4. *Simulation example*

In order to illustrate the results presented in this paper, we consider a linear system operating in closed loop according to Fig. 1, with

$$G_0 = \frac{1}{1 - 1.6q^{-1} + 0.89q^{-2}} \quad (23)$$

$$C = q^{-1} - 0.8q^{-2} \quad (24)$$

$$H_0 = \frac{1 - 1.56q^{-1} + 1.045q^{-2} - 0.3338q^{-3}}{1 - 2.35q^{-1} + 2.09q^{-2} - 0.6675q^{-3}}. \quad (25)$$

The noise signal e and the reference signal r are independent unit variance zero mean random signals. The controller is designed in such a way that the closed loop transfer function G_0T_0 has a denominator polynomial $(z - 0.3)^2$. The two-step identification strategy is applied to a data set generated by this closed loop system, using data sequences of length $N = 2048$.

In the first step, the sensitivity function is estimated by applying an FIR output error model structure, estimating 15 Markov parameters:

$$T(q, \beta) = \sum_{k=0}^{14} \beta(k)q^{-k}; \quad R(q, \gamma) = 1. \quad (26)$$

Note that the real sensitivity function T_0 is a rational transfer function of order 4. The magnitude Bode plot of the estimated sensitivity function is depicted in Fig. 2, together with the exact one.

The estimate $T(q, \hat{\beta}_N)$ is used to reconstruct a noise-free input signal $\hat{u}_N^r$ according to (12). Figure 3 shows this reconstructed input signal, compared with the real input $u(t)$ and the optimally reconstructed input signal $u^r(t) = T_0(q)r(t)$. Note that, despite the severe noise contribution on the signal u caused by the feedback loop, the reconstruction of u^r by $\hat{u}_N^r$ is extremely accurate.

In the second step an output error model structure is applied such that $G_0 \in \mathcal{G}$, by taking

$$G(q, \theta) = \frac{b_0 + b_1q^{-1} + b_2q^{-2}}{1 + a_1q^{-1} + a_2q^{-2}} \quad \text{and} \quad H(q, \eta) = 1. \quad (27)$$

Figure 4 shows the result of estimating G_0. The magnitude Bode plot is compared with the second order model obtained

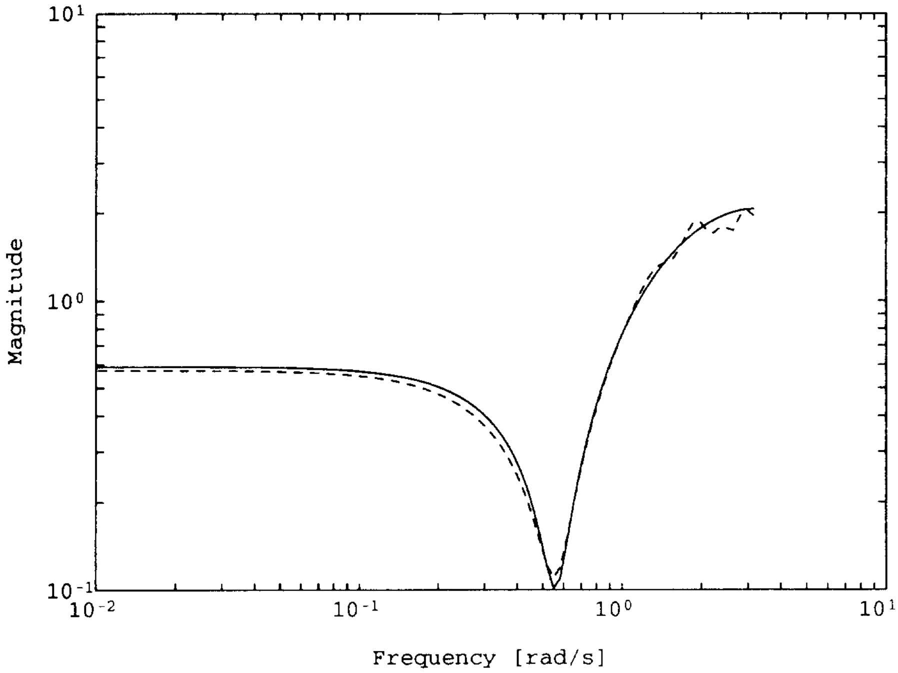

FIG. 2. Bode amplitude plot of exact sensitivity function T_0 (solid line) and estimated sensitivity function $T(q, \hat{\beta}_N)$ (dashed line).

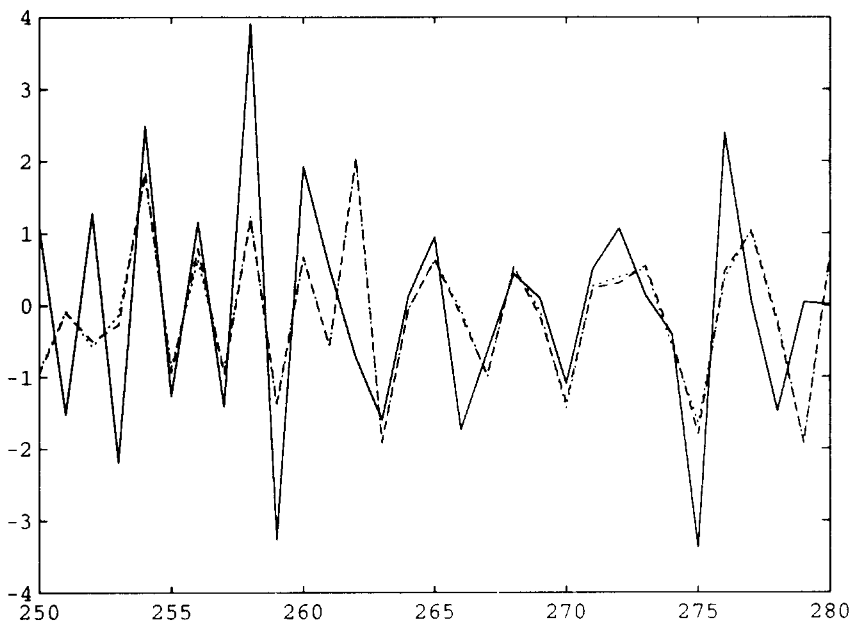

FIG. 3. Simulated input signal u (solid line), non-measurable input signal u^r caused by r (dashed line) and reconstructed input signal $\hat{u}^r_N$ (dotted line).

from a direct (one-step) output error method, using only the measurements of u and y. The results clearly show the degraded performance of the direct identification strategy, while the indirect method gives accurate results. This is also clearly illustrated in the Nyquist plot of the same transfer functions, as depicted in Fig. 5.

5. *Conclusions*

An indirect method has been analyzed for identification of transfer functions based on data obtained from closed loop experiments. It is assumed that a persistently exciting external reference signal is available; the controller is not assumed to be known. Using classical prediction error methods, the two-stage procedure is shown to yield consistent estimates of the open loop plant, irrespective of the noise dynamics. Similar to the open loop case, an explicit and tunable frequency domain expression is given for the bias distribution of the asymptotic model.

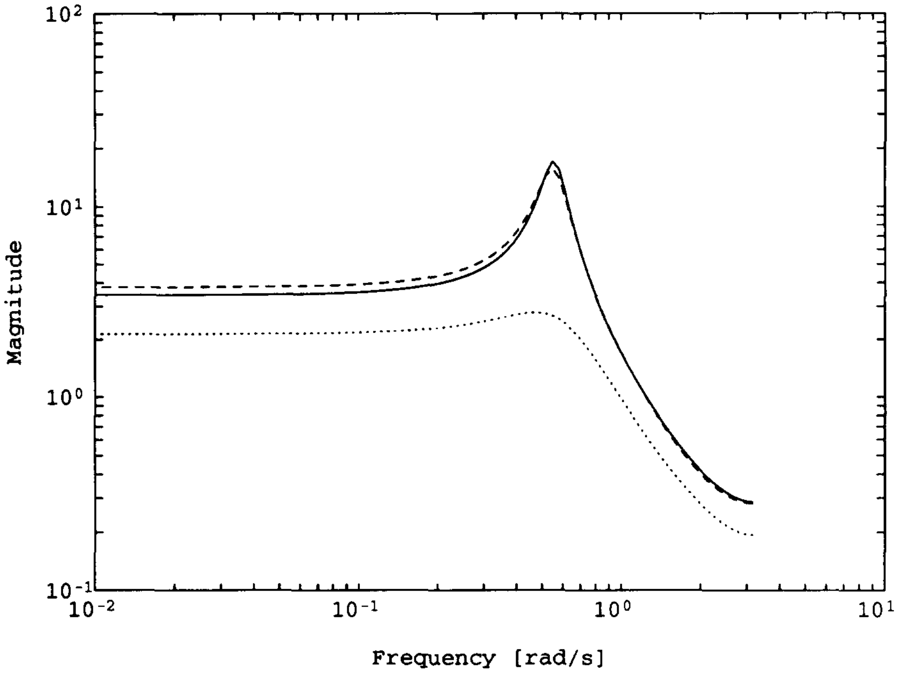

FIG. 4. Bode amplitude plot of transfer function G_0 (solid line), output error estimate $G(q, \hat{\theta}_N)$ obtained from the indirect method (dashed line) and output error estimate obtained from the direct method (dotted line). Order of the models is 2.

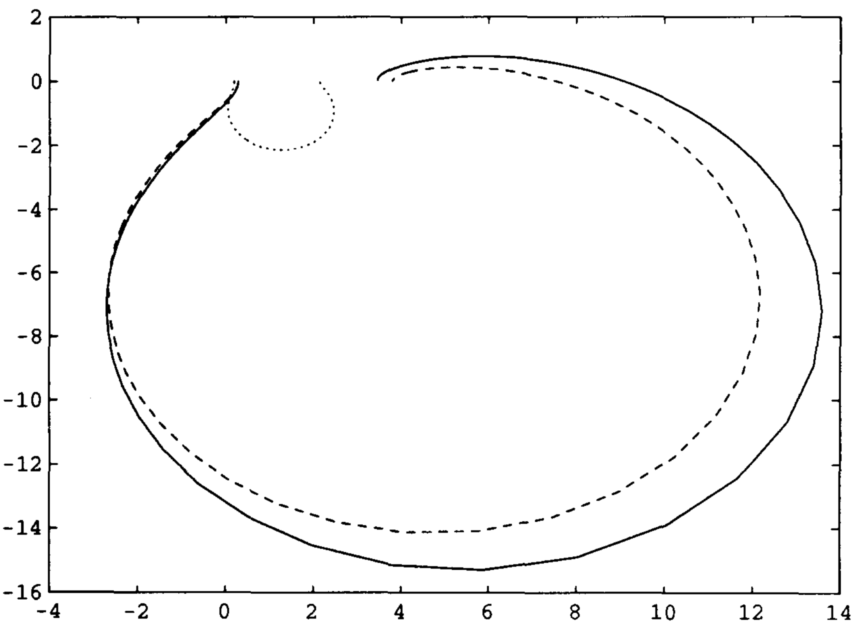

FIG. 5. Nyquist curve of transfer function G_0 (solid line), output error estimate $G(q, \hat{\theta}_N)$ obtained from the indirect method (dashed line) and output error estimate obtained from the direct method (dotted line). Order of the models is 2.

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