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Identifiability of dynamic networks with part of the nodes noise-free

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Abstract: In dynamic network identification a major goal is to uniquely identify the topology and dynamic links between the measured node variables. It is common practice to assume that process noises affect every output in multivariable system identification, and every node in dynamic networks with a full rank noise process. For many practical situations this assumption might be overly strong. This leads to the question of how to handle situations where the process noise is not full rank, i.e. when the number of white noise processes driving the network is strictly smaller than the number of nodes. In this paper a first step towards answering this question is taken by addressing the case of a dynamic network where some nodes are noise-free, and others are disturbed with a (correlated) process noise. In this situation the predictor filters that generate the one-step-ahead prediction of the node signals are non-unique, and the appropriate identification criterion leads to a constrained optimization problem. It is assessed when it is possible to distinguish between models on the basis of this criterion, leading to new notions of network identifiability. It appears that a sufficient condition for network identifiability is that every node signal in the network is excited by an external excitation signal or a process noise signal that is uncorrelated with other node excitations.

Keywords: System identification, dynamic networks, network topology detection, reduced rank noise process, identifiability

1. INTRODUCTION

Interesting topics in the system identification field are the problems of topology detection and identification of dynamics in a dynamic network setting. In this setting the topology refers to the (boolean) interconnection structure in the network. Multiple authors have developed methods to perform identification in dynamic networks (Torres et al., 2015; Van den Hof et al., 2013; Dankers, 2014), and topology detection (Chiuso and Pillonetto, 2012; Yuan et al., 2011; Sanandaji et al., 2011; Materassi and Salapaka, 2012). In these publications the general setup is as follows: the dynamic network consists of dynamic links between (measured) node variables, (known) external excitation signals and unknown stochastic process noises. External excitation signals are not incorporated in all the methods. For all of the methods it is assumed that the process noise variables affecting different nodes are mutually uncorrelated, and that each node is affected by a process noise variable. This type of noise is referred to as a full-rank mutually uncorrelated process noise.

In many practical situations it can be unrealistic to assume full rank and uncorrelated process noise. An example of such a situation is ship modeling where multiple variables are affected by only one disturbance, the waves, see (Linder, 2014). In a situation like this, the spectral density matrix of the noise will typically be singular. Also in the classical closed-loop network, see Figure 1, with a noise-disturbed plant output and a controller output that is noise-free, the vector noise process will be singular. In (Weerts et al., 2015) the question has been addressed whether models (topology and dynamics) can be distinguished from each other for the case of networks with correlated full rank process noise. It has been shown in that paper that uniqueness of the detected topology is essentially an identifiability issue and that because of the noise correlations all node variables need to be treated simultaneously. Rather than decomposing the problem into several MISO problems, this requires the handling of a MIMO problem. The analysis in (Weerts et al., 2015) is however no longer valid in case the noise is of reduced rank.

In the current paper we treat the most simple situation of a reduced rank noise process, namely a network where part of the nodes are noise-free, and the remaining nodes are contaminated by a full rank process noise, while it is known up front which nodes are noise-free. The main question to address is: *In the considered situation of a dynamic network with (some) given noise-free nodes, under what conditions (on external excitation, network topology and model structures) can we distinguish between two network models on the basis of measured signals*?

In the identification literature little attention is paid to rank reduced noise processes, even though the classical closed-loop system (Figure 1) has this property. Closed-loop identification methods typically work around the issue by either replacing the external excitation signal r by a stochastic noise process, as e.g. in the join-IO method (Caines and Chan, 1975), or by only focussing on identifying the plant model (and not the controller), as e.g. in the direct method. In econometrics dynamic factor models have been developed to deal with the situation of rank reduced noise (Deistler et al., 2015).

In the full rank noise case it is rather obvious that unique estimates can be obtained from network transfers that describe the mappings from external excitation signals to measurable node signals. In the singular noise case this is not obvious. Therefore we will set up a framework for identification in the singular situation, that we expect to be valid not only for the situation of noise-free nodes, but also for the more general situation of singular process noise.

In our approach we first formally define the network (section 2), after which the identification setup is formulated (section 3). The predictor is derived, and it is shown that the predictor filters are not unique due to the presence of noise-free nodes. An appropriate -constrained- identification criterion is introduced to deal with the noise-free signals and the resulting nonuniqueness of the predictor filters. In section 4, network identifiability is addressed and analyzed, after which two illustrative examples are provided (section 5). The Proofs of the results in this paper are collected in a report version of the paper (Weerts et al., 2016).

2. DYNAMIC NETWORK SETTING

In this paper a dynamic network consisting of L scalar *internal variables* or *nodes* w_j, $j = 1 \cdots L$ is considered. Only the first p nodes are affected by noise. This leads to a network defined by the equation

$$\begin{bmatrix} w_a(t) \\ w_b(t) \end{bmatrix} = \underbrace{\begin{bmatrix} G^0_{aa}(q) & G^0_{ab}(q) \\ G^0_{ba}(q) & G^0_{bb}(q) \end{bmatrix}}_{G^0(q)} \begin{bmatrix} w_a(t) \\ w_b(t) \end{bmatrix} \cdots + \underbrace{\begin{bmatrix} R^0_a(q) \\ R^0_b(q) \end{bmatrix}}_{R^0(q)} r(t) + \underbrace{\begin{bmatrix} H^0_s(q) \\ 0 \end{bmatrix}}_{H^0(q)} e(t), \tag{1}$$

where:

- $G^0_{aa} \in \mathbb{R}^{p\times p}(z)$, $G^0_{ab} \in \mathbb{R}^{p\times(L-p)}(z)$, $G^0_{ba} \in \mathbb{R}^{(L-p)\times p}(z)$, $G^0_{bb} \in \mathbb{R}^{(L-p)\times(L-p)}(z)$ are proper rational transfer function matrices;
- nodes $w_a \in \mathbb{R}^p$, $w_b \in \mathbb{R}^{L-p}$;
- $\begin{bmatrix} H^0_s(q) \\ 0 \end{bmatrix} e(t)$ is the *process noise* affecting the nodes w_a, it is modeled as a realization of a stationary stochastic process with rational spectral density;
- $e(t) \in \mathbb{R}^p$, a stationary white noise process with diagonal covariance matrix $\Gamma > 0$;
- $R^0_a \in \mathbb{R}^{p\times K}(z)$, $R^0_b \in \mathbb{R}^{(L-p)\times K}(z)$, $K \in \mathbb{N}_0$;
- $r(t) \in \mathbb{R}^K$, it is the quasi-stationary *external excitation variable* that can directly or indirectly be manipulated by the user.

The diagonal of $G^0_{aa}(q)$ and $G^0_{bb}(q)$ is 0, i.e. nodes are not connected to themselves directly. There are no algebraic loops in the network, i.e. when individual elements of $G^0(q)$ are denoted by $G^0_{n_i n_j}(q)$ then for any sequence $n_1, \cdots, n_k$: $\lim_{z\to\infty} G^0_{n_1 n_2}(z) G^0_{n_2 n_3}(z) \cdots G^0_{n_k n_1}(z) = 0$. Nodes w_a are affected by noise, and nodes w_b are noise-free. $H^0_s(q)$ is square, monic, stable and stably invertible, $H^0_s \in \mathbb{R}^{p\times p}(z)$. Note that $\lim_{z\to\infty} H^0_s(z) = I$, such that the innovations process of the network is $e_0(t) := [I\ 0]^T e(t)$ where $e_0 \in \mathbb{R}^L$ which is in line with the definition of the innovations process in (Caines, 1987). For convenience we will denote the internal and external variables as

$$z(t) = \begin{bmatrix} w_a(t) \\ w_b(t) \\ r(t) \end{bmatrix}.$$

The topology of the network is defined as the set of indices that represent which interconnections in the network are nonzero.

Definition 1. Set $\mathcal{N}$ represents the boolean topology of (1), it is defined by

$$\mathcal{N} = \left\{ (n_i, n_j) \mid \exists\ z \text{ such that } G_{n_i,n_j}(z) \neq 0 \right\},$$

where n_i and n_j indicate the specific interconnection

$$n_i, n_j \in \{1, 2, \cdots, L\}. \qquad \square$$

For a dynamic network with noise-free nodes as in (1), the resulting identification problem then becomes to identify the topology and/or the network dynamics $\{G^0, H^0, R^0\}$ on the basis of measured node variables $\{w_j,\ j = 1, \cdots L\}$ and external variables $\{r_k,\ k = 1, \cdots K\}$. In this paper we will identify the topology through identification of the dynamic networks $\{G^0, H^0, R^0\}$.

We assume that it is known which nodes of a network are noise-free such that the network can be written in the form described above. In practice it is possible to estimate the covariance of noise in a network, hence it is possible to determine which nodes are noise-free. We can use that information to partition the nodes into the noisy w_a and noise-free w_b groups.

3. NETWORK PREDICTOR AND IDENTIFICATION CRITERION

In this section a prediction error identification setup is presented that is suited for dealing with the situation of noise-free nodes. This setup will be used to identify the network dynamics. All node signals in the network are treated symmetrically, i.e. no distinction is made between input and output node signals. First we define the one-step-ahead predictor as follows:

Definition 2. The one-step-ahead predictor is defined for $j = 1, \cdots, L$ as

$$\hat{w}_j(t|t-1) := \mathbb{E}\left\{ w_j(t) \mid w_j^{t-1},\ w_i^t\ \forall\ i \neq j,\ r^t \right\}$$

where $w_i^t := \{w_i(0), \cdots, w_i(t)\}$ and $r^t := \{r(0), \cdots, r(t)\}$. $\square$

Although algebraic loops are not allowed, transfer functions without delay are allowed, which leads to the predictor expression above. Due to the non-square noise filter H^0 the expressions for the predictor will be different from the classical full rank case. This is shown in the following result.

Proposition 3. The one-step-ahead predictor of (1) is determined by the set of equations

$$\begin{bmatrix}\hat{w}_a(t|t-1)\\ \hat{w}_b(t|t-1)\\ 0\end{bmatrix} = \begin{bmatrix}W_{aa}(q) & W_{ab}(q) & W_{ar}(q)\\ 0 & I & 0\\ G^0_{ba} & G^0_{bb}-I & R^0_b\end{bmatrix} z(t) \qquad (2)$$

with

$$\begin{aligned} W_{aa}(q) &= \left[I - H_s^{-1}(q)\left(I - G^0_{aa}(q)\right)\right],\\ W_{ab}(q) &= \left[H_s^{-1}(q)G^0_{ab}(q)\right],\\ W_{ar}(q) &= \left[H_s^{-1}(q)R^0_a(q)\right]. \end{aligned}$$

□

While the predictions $\hat{w}_a$ and $\hat{w}_b$ are unique, see also the orthogonal projection theorem (Caines, 1987), the predictor filters are *not* unique, due to the deterministic relationship between the signals w_a, w_b and r in z, induced by the noise-free nodes. By combining the third row in (2) with any one of the other two, different expressions for the predictor filters result. We can pre-multiply both sides of the equation with

$$S(q) := \begin{bmatrix}I & 0 & A(q)\\ 0 & I & B(q)\\ 0 & 0 & I\end{bmatrix},$$

where $A(q)$ and $B(q)$ must be proper rational matrices. The pre-multiplication does not change the left-hand side, but the filters on the right-hand side do change. Next to the basic expressions for W_{aa}, W_{ab} and W_{ar}, as present in Proposition 3, we can e.g. also construct a version where $W_{ab} = 0$ by taking $A(q) = -H_s^{-1}G^0_{ab}(G^0_{bb} - I)^{-1}$.

For purpose of identification a network model will be parametrized through a model structure $\mathcal{M}$ defined as

$$\mathcal{M} := \{G(q,\theta),\ H(q,\theta),\ R(q,\theta),\ \theta \in \Theta\},$$

with

$$\begin{aligned} G(q,\theta) &= \begin{bmatrix}G_{aa}(q,\theta) & G_{ab}(q,\theta)\\ G_{ba}(q,\theta) & G_{bb}(q,\theta)\end{bmatrix},\\ H(q,\theta) &= H_s(q,\theta),\\ R(q,\theta) &= \begin{bmatrix}R_a(q,\theta)\\ R_b(q,\theta)\end{bmatrix} \end{aligned}$$

where all matrices have the same dimensions as their nonparameterized counterparts.

We make the assumption that the model structure $\mathcal{M}$ only contains models that do not contain any algebraic loops, i.e. any sequence $n_1, \cdots, n_k$:

$$\lim_{z\to\infty} G_{n_1n_2}(z,\theta)G_{n_2n_3}(z,\theta)\cdots G_{n_kn_1}(z,\theta) = 0$$

for all θ. A parameterized predictor for $w_a(t)$ can then be constructed as:

$$\hat{w}_a(t|t-1,\theta) = \left[W_{aa}(q,\theta)\ W_{ab}(q,\theta)\ W_{ar}(q,\theta)\right] z(t) \quad (3)$$

with the three filters parametrized in any of the forms that conforms to the of equations (2) pre-multiplied by $S(q)$.

Since the expression for $\hat{w}_b(t|t-1)$ in (2) does not directly lead to a parametrized version of this predictor, we will pre-multiply (2) with $S(q)$, with $B(q) = I$. Then $\hat{w}_b(t|t-1)$ is parameterized by taking the summation of the two last equations in (2), leading to

$$\begin{aligned} &\hat{w}_b(t|t-1;\theta) = \qquad (4)\\ &\left[G_{ba}(q,\theta)\quad G_{bb}(q,\theta)\quad R_b(q,\theta)\right] z(t)\quad \text{for all } t. \end{aligned}$$

This choice guarantees that the parametrized transfer functions G_{ba}, G_{bb} and R_b appear in the prediction error, and thus can be identified. Note that with this parameterization $\hat{w}_b(t|t-1;\theta)$ is not necessarily equal to $w_b(t)$. The related prediction error is defined by

$$\begin{bmatrix}\varepsilon_a(t,\theta)\\ \varepsilon_b(t,\theta)\end{bmatrix} := \begin{bmatrix}w_a(t)\\ w_b(t)\end{bmatrix} - \begin{bmatrix}\hat{w}_a(t|t-1,\theta)\\ \hat{w}_b(t|t-1,\theta)\end{bmatrix}. \qquad (5)$$

When choosing an identification criterion we need to properly weigh the contributions of both prediction errors ε_a and ε_b. For minimum variance reasons the respective weights should be inversely proportional to the innovations variance related to each term. Since w_b is noise-free, the resulting weight for the term ε_b should be infinite. Accordingly we need to choose an identification criterion with a constraint, formulated as:

$$J(z,\mathcal{M}) = \left\{\begin{array}{l}\underset{M(\theta)}{\arg\min}\ \underset{\theta\in\Theta}{\bar{\mathbb{E}}}\ \varepsilon_a^T(t,\theta)\Lambda\varepsilon_a(t,\theta)\\ \text{subject to: } \varepsilon_b(t,\theta) = 0\ \text{ for all } t.\end{array}\right\}, \qquad (6)$$

with Λ a strictly positive definite matrix of appropriate dimensions. In this definition J is a set containing the models $M(\theta)$ which satisfy the constraint and minimize the cost function. The constraint optimization problem (6) is the natural identification method for dealing with noise-free nodes.

In this identification setup we have so far neglected the possible effect of non-zero initial conditions, that could render the formulated constraint infeasible. This issue can be resolved by parametrizing the unknown initial conditions too, and including them in the parameter vector θ, see e.g. Pintelon and Schoukens (2012).

In the next section it will be analyzed under which conditions network models can be distinguished in this identification setup.

4. NETWORK IDENTIFIABILITY

Models can typically be distinguished from each other through measurement data, if they have different properties in view of the induced prediction errors that are used as a basis for identification. In the classical situation of full rank process noise, this is typically reflected in different predictor filters. In the current situation however, the predictor filters are non-unique, and so they cannot directly serve as a basis for deciding whether two network models can be distinguished on the basis of data. Therefore we need a more detailed analysis and some new concepts for characterizing when network models can be distinguished. In the setup to be presented, the general question will be whether the identification criterion can distinguish between the models in $\mathcal{M}$. The setup that is being presented here is expected to be applicable also to the general situation of having singular noise processes (not necessarily having noise-free nodes). First we formalize when two models can be distinguished by the criterion.

Definition 4. (J-equivalent models (Van den Hof, 1994)). Two models $M_1, M_2 \in \mathcal{M}$ are J-equivalent within $\mathcal{M}$, denoted $M_1 \overset{J}{\sim} M_2$, if for all possible data sequences z [1],

$$M_1 \in J(z,\mathcal{M}) \Leftrightarrow M_2 \in J(z,\mathcal{M}). \qquad \square$$

[1] This refers to all possible quasi-stationary stochastic processes $z = [w_a^T\ w_b^T\ r^T]^T$ that can be generated by $w = G(q)w + H(q)e +$

In other words: two J-equivalent models always appear together in the solution set of an identification problem (if one of them appears, the other appears also and vice versa). As a result two J-equivalent models can not be distinguished from each other by the identification criterion J, no matter what the data set is. This notion of J-equivalent models leads to a natural definition of network identifiability.

Definition 5. (Global network identifiabilitity). A network model structure $\mathcal{M}$ is globally network identifiable in $M_0 \in \mathcal{M}$ with respect to the identification criterion J, if for any model $M_1 \in \mathcal{M}$ it holds that

$$\{M_1 \overset{J}{\sim} M_0\} \Longrightarrow M_1 = M_0.$$

A network model structure $\mathcal{M}$ is globally network identifiable, if it is globally network identifiable in every model $M_0 \in \mathcal{M}$ with respect to J. □

Equality of models is to be interpreted as the dynamics of the models being equal. The presented identifiability concept has some relation to the classical notion of *system identifiability* (Ljung, 1976) that operates under a criterion and on transfer function level as well. However unlike that classical notion, the above concept is not based on consistency of the estimates, and therefore not based on a particular data-generating system. It is closely related to the notion of discriminability of model sets (Van den Hof, 1989). It is also different from the classical notion of identifiability ((Ljung, 1999)) as it is not concerned with uniqueness of parameters, but rather with uniqueness of models in the form of their transfer function representation.

Proposition 6. A sufficient condition for a model set $\mathcal{M}$ to be globally network identifiable in M_0 with respect to J, is that there exists a data sequence z such that $J(z, \mathcal{M}) = \{M_0\}$ [2]. □

This result follows directly from the definition of network identifiability. There must exist some data sequence such that the solution is a singleton, hence this model M_0 is not J-equivalent to any other model in $\mathcal{M}$. The importance of this property is that it implies that topology and dynamics of M_0 can be distinguished from other topologies and dynamics. We are now going to formulate conditions under which the property of Proposition 6 holds. For this purpose we choose a particular data sequence, namely a data sequence z that is generated by M_0 through (1) in which r is persistently exciting of a sufficiently high order.

Proposition 7. Consider a data sequence z generated by M_0, where M_0 has the form (1), then $J(z, \mathcal{M}) = \{M_0\}$ if the following implication holds true:

$$\left.\begin{array}{l} \hat{w}_b(t|t-1;\theta_1) = w_b(t)\ \forall t \\ \bar{\mathbb{E}}[\hat{w}_a(t|t-1;\theta_1) - \hat{w}_a(t|t-1;\theta_0)]^T \Lambda \cdot \\ \cdot\, [\hat{w}_a(t|t-1;\theta_1) - \hat{w}_a(t|t-1;\theta_0)] = 0 \end{array}\right\} \Rightarrow M(\theta_1) = M(\theta_0) \tag{7}$$

□

The condition (7) can be interpreted as a condition that warrants that the data sequence used for identification is sufficiently informative for the model structure considered. Informativity of data is usually defined as a condition like (7) but then without the constraint on $\hat{w}_b$, see Ljung (1999). Because of the noise-free nodes, the additional constraint needs to be added.

Under the influence of the constrained identification criterion J (6), we can now rewrite the predictor filters into a form that becomes unique, and therefore can be used as an appropriate basis for analyzing global network identifiability. This is done by substituting the constraint $\hat{w}_b(t|t-1,\theta) = w_b(t)$ into the expression for the predictor for w_a, and removing w_b as a predictor input.

Proposition 8. Under the constraint $\hat{w}_b(t|t-1,\theta) = w_b(t)$, the predictors for the network can be rewritten as

$$\begin{bmatrix} \hat{w}_a(t|t-1,\theta) \\ \hat{w}_b(t|t-1,\theta) \end{bmatrix} = P(q,\theta) \begin{bmatrix} w_a(t) \\ r(t) \end{bmatrix}, \tag{8}$$

where

$$P = \begin{bmatrix} P_{aa}(q,\theta) & P_{ar}(q,\theta) \\ P_{ba}(q,\theta) & P_{br}(q,\theta) \end{bmatrix},$$

with (omitting arguments q, θ)

$$\begin{aligned} P_{aa} &= I - H_s^{-1}(I - G_{aa}) + H_s^{-1} G_{ab} P_{ba}, \\ P_{ar} &= H_s^{-1} R_a + H_s^{-1} G_{ab} P_{br}, \\ P_{ba} &= (I - G_{bb})^{-1} G_{ba}, \\ P_{br} &= (I - G_{bb})^{-1} R_b. \end{aligned}$$

□

Proof of the proposition is collected in a report version of the paper (Weerts et al., 2016). The predictor for the network has been restructured into a filtered version of w_a and r only. Because of the noise-free character of w_b, this signal can indeed be discarded as input to the predictor filters. Since the node signals w_a are driven by a full rank noise process, this makes this representation of the network predictor filters unique. In contrast with the earlier predictor expression (3), the input signals to the predictor now constitute a full rank process (provided that r is sufficiently exciting), and therefore informativity of the data to estimate $P(q,\theta)$ uniquely is guaranteed. As a consequence of this, we can now specify the conditions for global network identifiability under criterion J, in terms of the predictor filters $P(q,\theta)$.

Corollary 9. A network model structure $\mathcal{M}$ is globally network identifiable in $M_0 := M(\theta_0)$ with respect to the identification criterion J (6) if for all models $M(\theta_1) \in \mathcal{M}$ the following implication holds

$$P(q,\theta_1) = P(q,\theta_0) \Rightarrow M(\theta_1) = M(\theta_0). \qquad \square$$

In the particular situation that we consider here, i.e. a number of indicated nodes that are noise-free, the predictor filters in the above corollary are uniquely related to the transfer functions that map the external signals (r, e) in the network to the internal network nodes $w_a, w_b)$.

Proposition 10. Consider the network model structure $\mathcal{M}$, and define

$$T(q,\theta) := (I - G(q,\theta))^{-1} \begin{bmatrix} H_s(q,\theta) & R_a(q,\theta) \\ 0 & R_b(q,\theta) \end{bmatrix}.$$

$R(q)r$ with G, H, R being linear, time-invariant filters satisfying the usual conditions, e a white noise, and r quasi-stationary.

[2] In line with the classical definitions for full rank noise prediction error methods, a data sequence z that leads to a singleton $J(z, \mathcal{M})$ can be called informative for a globally network identifiable model set $\mathcal{M}$ with respect to criterion J.

being the parameterized transfer function from $\begin{pmatrix} e \\ r \end{pmatrix}$ to $\begin{pmatrix} w_a \\ w_b \end{pmatrix}$ [3]. Then for any $\theta_1, \theta_2 \in \Theta$

$$P(q,\theta_1) = P(q,\theta_2) \Leftrightarrow T(q,\theta_1) = T(q,\theta_2).$$

□

This proposition shows that in the condition of Corollary 9, the matrices $P(q,\theta)$ can equivalently be replaced by the matrices $T(q,\theta)$. On the basis of the above results we can now formulate the conditions for verifying global network identifiability of a model structure.

Theorem 11. A network model structure $\mathcal{M}(\theta)$ is globally network identifiable with respect to identification criterion J (6) if there exists a nonsingular and parameter-independent transfer function matrix $\tilde{P} \in \mathbb{R}^{(K+p)\times(K+p)}(z)$ such that

$$\begin{bmatrix} H_s(q,\theta) & R_a(q,\theta) \\ 0 & R_b(q,\theta) \end{bmatrix} \tilde{P}(q) = [D(q,\theta) \;\; F(q,\theta)]$$

with $D(\theta) \in \mathbb{R}^{L\times L}(z)$, diagonal and full rank for all $\theta \in \Theta$, and $F(\theta) \in \mathbb{R}^{L\times(p+K-L)}(z)$.
In the case that the off-diagonal terms of $G(q,\theta)$ are fully parametrized and all transfer functions in $\mathcal{M}(\theta)$ are parametrized independently [4] the condition is also necessary. □

Given the result of Proposition 10, the proof of this theorem is similar to the proof of Theorem 2 in Weerts et al. (2015). One of the important consequences of this theorem is formulated in the next corollary.

Corollary 12. A network model structure $\mathcal{M}(\theta)$ is globally network identifiable with respect to J if every node signal in the network is excited by either an external excitation signals r or a noise signal v, that is uncorrelated with the excitaton/noise signals on the other nodes. □

Uncorrelated excitation can come from noise or external variables, although on the noise-free nodes it must come from an external variable. The condition in Theorem 11 can be relaxed when considering a model structure with structure restrictions in G. A theorem for such structure restricted models can be formulated in a similar fashion by using Theorem 2 from Weerts et al. (2015).

5. ILLUSTRATIVE EXAMPLES

5.1 Closed-loop system

One of the very simple examples to which the results of this paper apply is the situation of a single-loop feedback system, with a disturbance signal on the process output, and a reference input at the process input (controller output), see Figure 1.

The process output y will take the role of node variable w_a, while the process input u will be represented by the (noise-free) w_b. When parametrizing process $G(q,\theta)$ and controller $C(q,\theta)$, as well as noise model $v(t) = H(q,\theta)e(t)$

[3] Strictly speaking it is the transfer function from $\begin{pmatrix} \varepsilon \\ r \end{pmatrix}$ to $\begin{pmatrix} w_a \\ w_b \end{pmatrix}$.

[4] i.e. parameters used in one transfer function entry are different from parameters used in any other transfer function entry.

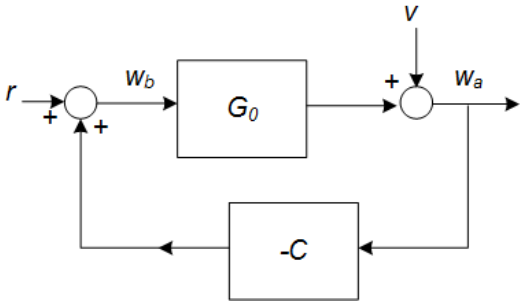

Fig. 1. Simple classical closed-loop configuration.

and the fixed reference filters $R_a(q) = 0$, $R_b(q) = 1$, it appears that the essential identifiability result of Theorem 11 is reflected by the matrix

$$\begin{bmatrix} H_s(q,\theta) & R_a(q,\theta) \\ 0 & R_b(q,\theta) \end{bmatrix} = \begin{bmatrix} H_s(q,\theta) & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix is square and equal to the diagonal matrix D in the theorem. Since it is square we have that matrix F will have dimension 2×0. The conditions of Theorem 11 are satisfied with $\tilde{P} = I$, and therefore the closed-loop system is globally network identifiable w.r.t. J. This implies that a consistent estimates of G_0 and C can be obtained, when identified simultaneously.

In this closed-loop example we treat all signals w symmetrically in a direct-method approach, meaning that all internal variables, w_a and w_b, are predicted. We identify both G and C, which is in contrast with the classical direct method that identifies only G_0. The joint-IO method for closed-loop identification has symmetric treatment of all signals as well. Our method however allows for the noise on the input node to be replaced by a known external excitation signal.

5.2 Network example

In this example we analyze the 5 node network of Figure 2 where the noises on nodes 1 and 2 are correlated. The

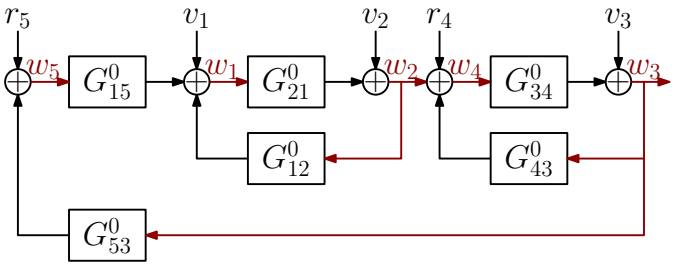

Fig. 2. 5 node network.

nodes are labeled such that the last two are noise-free. Process noise will be modeled by

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{11}(q,\theta) & H_{12}(q,\theta) & 0 \\ H_{21}(q,\theta) & H_{22}(q,\theta) & 0 \\ 0 & 0 & H_{33}(q,\theta) \end{bmatrix}}_{H_s(q,\theta)} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}$$

such that dynamic correlation can be represented. The condition of Theorem 11 is checked by attempting to diagonalize the matrix

$$\begin{bmatrix} H_{11}(q,\theta) & H_{12}(q,\theta) & 0 & 0 & 0 \\ H_{21}(q,\theta) & H_{22}(q,\theta) & 0 & 0 & 0 \\ 0 & 0 & H_{33}(q,\theta) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

by postmultiplication with some filter $\tilde{P}(q)$ which does not depend on θ. Due to the correlated noise it is not possible to diagonalize the matrix in this way. In case some external excitations are added to nodes 1 and 2 we can make the problem identifiable.

6. CONCLUSIONS

In this paper extensions of identification methods for networks where the process noise is of a reduced rank are considered. The predictor for these networks is by definition unique, while the predictor filters contain degrees of freedom. By using a constrained identification criterion it is shown that conditions under which unique estimates are obtained are similar to the case of full rank noise. The introduced concepts of uniqueness are *J-equivalence* which describes when two models can not be distinguished by an identification criterion, and *global network identifiability w.r.t. a criterion* which describes that there are no distinct J-equivalent models in a model set. These generalized notions allow the treatment of identification criteria that go beyond the classical prediction error schemes. We consider the network identification problem under fully symmetric treatment of variables.

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