Single module identifiability in linear dynamic networks

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Abstract— A recent development in data-driven modeling addresses the problem of identifying dynamic models of interconnected systems, represented as linear dynamic networks. For these networks the notion of network identifiability has been introduced recently, which reflects the property that different network models can be distinguished from each other. Network identifiability is extended to cover the uniqueness of a single module in the network model, and conditions for single module identifiability are derived and formulated in terms of path-based topological properties of the network models.

I. INTRODUCTION

Systems in engineering are becoming increasingly complex and interconnected. In many control, monitoring and optimization applications it is advantageous to model a system as a set of interconnected modules. *Linear dynamic networks* are formed by interconnecting modules according to a structured topology. Given the increasing availability of sensors, it is attractive to develop tools for data-driven modeling of linear dynamic networks. There are several interesting topics of research, including the development of methods to estimate the dynamics of one, several or all modules embedded in the network from a given data set, or estimating its topology [1], [2], [3], [4], [5], [6].

When identifying either a full network or a subnetwork, it is important that the candidate models can be distinguished from each other. For this purpose, the concept of network identifiability has been introduced in [7], as a follow up on system theoretic results of [1]. In this setting, network identifiability is dependent on the presence and location of external excitation signals, on structural information on the network topology and the disturbance correlation structure.

The analysis in [7] has been concentrated on identifiability of a full network on the basis of all node variables being measured. An alternative dual problem is formulated in [8] where identifiability of all or some of the modules is studied on the basis of a subset of node signals being measured in a noise-free network. Duality comes from the fact that in [7] all nodes are measured but not necessarily excited directly, while in [8] all nodes are excited directly but not necessarily measured. In [9] the notion of generic network identifiability has been explicitly defined, which differs from the definition in [7] by requiring distinguishability of models for almost all of their model coefficient values. With this definition, the conditions for identifiability can be recast into attractive path-based conditions on the network topology.

In this paper we will extend the identifiability analysis of [7] in two different ways. First we will cover identifiability of a single module in a noise disturbed network, i.e. under what conditions is a module of interest identifiable? And secondly we will show that, in line with the approach in [8], [9], the conditions for identifiability can be formulated as path-based conditions, if we consider the identifiability concept in a generic sense. This allows for a simple verification of identifiability based on the topology of the network models.

A network setup and the main results of [7] are provided in Section II. In Section III single module identifiability is defined and conditions are derived for this property. Then in Section IV these conditions are formulated as path-based conditions, for the situation of generic network identifiability. Finally a discussion on the definition of identifiability is included in Section V.

II. PRELIMINARIES

A. Network setup

Following the basic setup of [5], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1, \ldots, L$, and K *external variables* r_k, $k = 1, \ldots K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^{L} G_{jl}(q)w_l(t) + \sum_{k=1}^{K} R_{jk}(q)r_k(t) + v_j(t) \quad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl}, R_{jk} are proper rational transfer functions, and the single transfers G_{jl} are referred to as *modules*.
- r_k are *external variables* that can directly be manipulated by the user;
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density, such that there exists a p-dimensional white noise process $e := [e_1 \cdots e_p]^T$, $p \leq L$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$.

For $p = L$, H is square, stable, monic and minimum-phase. The situation $p < L$ is referred to as the *rank-reduced* noise case, of which a detailed description can be found in [7].

The L nodes combined lead to the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & & G_{1L} \\ G_{21} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & G_{L-1\,L} \\ G_{L1} & \cdots & G_{L\,L-1} & & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + Rr + He. \tag{2}$$

The network transfer function that maps the external signals r and e into the node signals w is denoted $T(q) = \begin{bmatrix} T_{wr}(q) & T_{we}(q) \end{bmatrix}$

$$\text{with} \quad T_{wr}(q) := (I - G(q))^{-1} R(q), \tag{3}$$

$$\text{and} \quad T_{we}(q) := (I - G(q))^{-1} H(q). \tag{4}$$

As a shorthand notation we use $U(q) := \begin{bmatrix} H(q) & R(q) \end{bmatrix}$. The considered identification problem is identification of all or part of the network dynamics (G, R, H, Λ) on the basis of measured variables w and r.

Remark 1: The dynamic network formulation above is related to what has been called the *Dynamic Structure Function (DSF)* as considered for disturbance-free systems in [10], [3], [11].

In order to arrive at a definition of network identifiability we need to specify a network model and network model set.

Definition 1 (network model): A network model of a network with L nodes, and K external excitation signals, with a noise process of rank $p \leq L$ is defined by the quadruple:

$$M = (G, R, H, \Lambda), \quad \text{with}$$

- $G \in \mathbb{R}^{L \times L}(z)$, proper, stable[1] and diagonal entries 0;
- $R \in \mathbb{R}^{L \times K}(z)$, proper;
- $H \in \mathbb{R}^{L \times p}(z)$, stable, with a stable left inverse, and the top $p \times p$ block monic.
- $\Lambda \in \mathbb{R}^{p \times p}$, $\Lambda > 0$;
- the network is well-posed[2] [6], with $(I - G)^{-1}$ proper and stable. $\square$

The noise model H is defined to be non-square in the case of a rank-reduced noise ($p < L$).

Definition 2 (network model set): A network model set for a network of L nodes, K external excitation signals, and a noise process of rank $p \leq L$, is defined as a set of parametrized matrix-valued functions:

$$\mathcal{M} := \left\{ M(\theta) = \big(G(q, \theta), R(q, \theta), H(q, \theta), \Lambda(\theta) \big), \theta \in \Theta \right\},$$

with all models $M(\theta)$ satisfying the properties as listed in Definition 1. $\square$

There is a *path* in the network through nodes $w_{n_1}, \ldots, w_{n_k}$ if $G_{n_1 n_2} G_{n_2 n_3} \cdots G_{n_{(k-1)} n_k} \neq 0$.

B. Identifiability

Identification is usually performed on the basis of second-order properties of w and r, e.g. power spectral density functions. Therefore in [7] network identifiability has been defined on the basis of those second-order properties.

Definition 3 (Global network identifiability [7]): The network model set $\mathcal{M}$ is globally network identifiable at $M_0 := M(\theta_0)$ if for all models $M(\theta_1) \in \mathcal{M}$,

$$\left. \begin{array}{l} T_{wr}(q, \theta_1) = T_{wr}(q, \theta_0) \\ \Phi_{\bar{v}}(\omega, \theta_1) = \Phi_{\bar{v}}(\omega, \theta_0) \end{array} \right\} \Rightarrow M(\theta_1) = M(\theta_0), \tag{5}$$

where $\Phi_{\bar{v}}$ is the spectrum of $\bar{v}(t) := T_{we}(q)e(t)$. $\mathcal{M}$ is globally network identifiable if (5) holds for all $M_0 \in \mathcal{M}$. $\square$

Under some conditions on feedthrough in modules, the implication can be re-written.

Proposition 1 (from [7]): Let $\mathcal{M}$ be a network model set that satisfies either

- all modules in $G(q, \theta)$ are strictly proper, or
- there are no algebraic loops[3] and $\Lambda(\theta)$ is diagonal for all $\theta \in \Theta$.

Then $\mathcal{M}$ is globally network identifiable at $M_0 := M(\theta_0)$ if for all models $M(\theta_1) \in \mathcal{M}$,

$$\{T(q, \theta_1) = T(q, \theta_0)\} \Rightarrow \tag{6}$$
$$\{(G(\theta_1), R(\theta_1), H(\theta_1)) = (G(\theta_0), R(\theta_0), H(\theta_0))\}.$$

Network model set $\mathcal{M}$ is globally network identifiable if (6) holds for all $M_0 \in \mathcal{M}$. $\square$

Depending on the presence of external excitations, networks with algebraic loops can also be identifiable when (6) holds; see [7] for more details on the treatment of that situation.

In order to formulate necessary and sufficient conditions we introduce some notation considering row j. Denote with subscript $j\star$ row j of a matrix. The number of parameterized transfer functions in $G_{j\star}(\theta)$ and $U_{j\star}(\theta)$ are α_j and β_j respectively. Let P_j be the permutation matrix that gathers all parameterized modules in the left part of $G_{j\star}(\theta)P_j$, and let Q_j be the permutation matrix that gathers all parameterized transfer functions in the right part of $U_{j\star}(\theta)Q_j$. Define transfer function matrix $\check{T}_j$ as

$$\check{T}_j(q, \theta) := \begin{bmatrix} I_{\alpha_j} & 0 \end{bmatrix} P_j^{-1} T(q, \theta) Q_j \begin{bmatrix} I_{K+p-\beta_j} \\ 0 \end{bmatrix}, \tag{7}$$

which is the transfer function from signals r and e that are input to non-parameterized transfer functions in $U_{j\star}(\theta)$, to node signals w that are input to parameterized modules in $G_{j\star}(\theta)$.

Theorem 1 (Part of Theorem 2 from [7]): Let $\mathcal{M}$ satisfy the properties of Proposition 1, and additionally satisfy:

a. Every parametrized transfer function in the model $\{M(z, \theta), \theta \in \Theta\}$ covers the set of all proper rational transfer functions;

b. All parametrized transfer functions in the model $M(z, \theta)$ are parametrized independently (i.e. there are no common parameters).

Then

1) $\mathcal{M}$ is globally network identifiable at $M(\theta_0)$ if and only if

   i) each row j of the transfer function matrix $\begin{bmatrix} G(\theta) & U(\theta) \end{bmatrix}$ has at most $K + p$ parameterized entries, and

   ii) for each j, $\check{T}_j(\theta_0)$ defined by (7) has full row rank.

2) $\mathcal{M}$ is globally network identifiable if and only if i) holds, and ii) holds for all $\check{T}_j(\theta), \theta \in \Theta$. $\square$

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[1] The assumption of having all modules stable is made in order to guarantee that T_{we} (4) is a stable spectral factor of the noise process that affects the node variables.

[2] This implies that all principal minors of $(I - G(\infty))^{-1}$ are nonzero.

[3] an algebraic loop is a path where $n_1 = n_k$ and $\lim_{z \to \infty} G_{n_1 n_2}(z) G_{n_2 n_3}(z) \cdots G_{n_{(k-1)} n_k}(z) \neq 0$

III. EXTENSION TO SINGLE-MODULE IDENTIFIABILITY

In this section first we formalize identifiability of a row of M, which considers all modules that map into a particular node signal, after which identifiability of a particular module is treated. To this end we formalize identifiability of particular properties of M as suggested in [7].

Definition 4: For network models that satisfy the conditions of Proposition 1, row j of network model set $\mathcal{M}$ is globally network identifiable at $M_0 := M(\theta_0)$ if for all models $M(\theta_1) \in \mathcal{M}$,

$$T(q,\theta_1) = T(q,\theta_0)\} \Rightarrow \begin{cases} G_{j\star}(q,\theta_1) = G_{j\star}(q,\theta_0) \\ R_{j\star}(q,\theta_1) = R_{j\star}(q,\theta_0) \\ H_{j\star}(q,\theta_1) = H_{j\star}(q,\theta_0) \end{cases} . \quad (8)$$

Row j of network model set $\mathcal{M}$ is globally network identifiable if (8) holds for all $M_0 \in \mathcal{M}$. $\square$

The conditions in Theorem 1 are formulated independently for each row, so it is straightforward to obtain conditions under which a specific row of $\mathcal{M}$ is identifiable.

Corollary 1: Let $\mathcal{M}$ be a network model set defined as in Theorem 1, and $\check{T}_j(\theta)$ defined by (7), then

1) Row j of network model set $\mathcal{M}$ is globally network identifiable at $M(\theta_0)$ if and only if
   i) row j of transfer function matrix $[G(\theta) \quad U(\theta)]$ has at most $K + p$ parameterized entries, and
   ii) $\check{T}_j(\theta_0)$ has full row rank.
2) Row j of network model set $\mathcal{M}$ is globally network identifiable if and only if i) holds and and ii) holds for all $\check{T}_j(\theta), \theta \in \Theta$ $\square$

When we are interested in one specific module, then the above definition is conservative. It is possible that a module is identifiable, even when other modules of that row are not, which is illustrated by the following example.
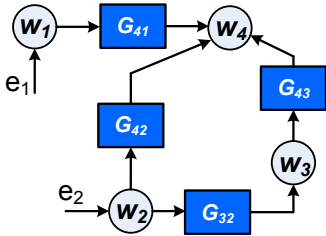

Fig. 1. Example network model where some modules are identifiable.

Example 1: Consider a set of network models of the topology shown in Figure 1, described by

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & G_{32} & 0 & 0 \\ G_{41} & G_{42} & G_{43} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (9)$$

where all modules G_{ji} are parameterized. The response of the node variables is given by $w = Te$ with

$$T = (I - G)^{-1} H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & G_{32} \\ G_{41} & G_{42} + G_{32}G_{43} \end{bmatrix}. \quad (10)$$

From T module G_{41} can directly be determined, but the other modules G_{42} and G_{43} on row 4 can not. This is because node 4 has three modules, and there are only two excitations. $\square$

Identifiability of a specific module is defined next.

Definition 5: For network models that satisfy the conditions of Proposition 1, module G_{ji} of network model set $\mathcal{M}$ is globally network identifiable at $M_0 := M(\theta_0)$ if for all models $M(\theta_1) \in \mathcal{M}$,

$$\{T(q,\theta_1) = T(q,\theta_0)\} \Rightarrow \{G_{ji}(q,\theta_1) = G_{ji}(q,\theta_0)\}. \quad (11)$$

Module G_{ji} of network model set $\mathcal{M}$ is globally network identifiable if (11) holds for all $M_0 \in \mathcal{M}$. $\square$

It is obvious that identifiability of every module holds for every model set that is globally network identifiable. However the interesting question is whether the conditions can be relaxed, such that identifiability of a module is guaranteed, even when other modules are not identifiable.

In order to find identifiability conditions for a single module G_{ji}, assume without loss of generality that this module corresponds to the top row of $\check{T}_j$. Then define $\check{T}_{j,(i,\star)}$ as the top row of $\check{T}_j$, and $\check{T}_{j,(-i,\star)}$ by

$$\check{T}_j(q,\theta) = \begin{bmatrix} \check{T}_{j,(i,\star)}(q,\theta) \\ \check{T}_{j,(-i,\star)}(q,\theta) \end{bmatrix} \quad (12)$$

So $\check{T}_{j,(-i,\star)}$ is $\check{T}_j$ with the row corresponding to node w_i removed. The following Theorem now specifies necessary and sufficient conditions for the identifiability condition (11).

Theorem 2: Let $\mathcal{M}$ be a network model set defined as in Theorem 1, then

1) Module G_{ji} of network model set $\mathcal{M}$ is globally network identifiable at $M(\theta_0)$ if and only if

$$\text{rank}(\check{T}_j(\theta_0)) > \text{rank}(\check{T}_{j,(-i,\star)}(\theta_0)). \quad (13)$$

2) Module G_{ji} of network model set $\mathcal{M}$ is globally network identifiable if and only if

$$\text{rank}(\check{T}_j(\theta)) > \text{rank}(\check{T}_{j,(-i,\star)}(\theta)) \quad (14)$$

   for all $\theta \in \Theta$. $\square$

The proof is collected in the appendix.

The essential part of the theorem is that if the row of $\check{T}_j$ corresponding to node w_i is a linear independent row, then the module is identifiable. Note that there is no explicit requirement on the number of parameterized modules. We do not require uniqueness of all modules, so we can have fewer equations than unknowns.

Example 2 (Example 1 continued): For node 4 there are three parameterized transfer functions, while there are only two external signals. To evaluate identifiability we use

$$\check{T}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & G_{32} \end{bmatrix}, \quad \check{T}_{4(-1,\star)} = \begin{bmatrix} 0 & 1 \\ 0 & G_{32} \end{bmatrix}. \quad (15)$$

In $\check{T}_4$ the first row is clearly linearly independent of the other rows, such that $\text{rank}(\check{T}_{4(-1,\star)}) = 1 < \text{rank}(\check{T}_4) = 2$, and the condition of Theorem 2 is satisfied for G_{41}. It can be shown that rows 2 and 3 of $\check{T}_4$ are linearly dependent, and so G_{42} and G_{43} are both not identifiable. $\square$

IV. PATH-BASED IDENTIFIABILITY CONDITIONS

In this section the rank conditions that appear in the network identifiability results are formulated as topology based conditions. The core idea is that the rank of T depends on the topology of the network. We base our reasoning on concepts presented in [8], [9], such as generic identifiability, which we adapt to our problem setting. Then the rank conditions for identifiability are adapted to this definition. The notion of *vertex disjoint paths* and its relation to the rank of the transfer matrix will be introduced, and used to formulate topological conditions under which the network is identifiable.

Definition 6 (Generic network identifiability):

- $\mathcal{M}$ is generically network identifiable if for all models $M(\theta_1) \in \mathcal{M}$ the implication (6) holds for *almost all* $M_0 \in \mathcal{M}$.
- Row j of network model set $\mathcal{M}$ is generically network identifiable if for all models $M(\theta_1) \in \mathcal{M}$ the implication (8) holds for *almost all* $M_0 \in \mathcal{M}$.
- Module G_{ji} of network model set $\mathcal{M}$ is generically globally network identifiable for all models $M(\theta_1) \in \mathcal{M}$ the implication (11) holds for *almost all* $M_0 \in \mathcal{M}$. □

The only difference between Definitions 3 and 6 is the exception of a set of zero measure, i.e. identifiability of almost all models is accounted for. The consequences of this change in the definitions are discussed in Section V.

The rank conditions of Theorems 1 and 2, Corollary 1 can directly be formulated for the generic network identifiability.

Corollary 2: The model set $\mathcal{M}$, row j of model set $\mathcal{M}$, or module G_{ji} of model set $\mathcal{M}$ is generically network identifiable by the conditions 2) of Theorem 1, Corollary 1, Theorem 2 respectively upon replacing the phrase "for all $\theta \in \Theta$" by "for *almost all* $\theta \in \Theta$". □

The proof is a trivial extension of the proof of Theorem 1 found in [7] and the proof of Theorem 2.

In [12], the name vertex is used for a node, and the rank of a transfer matrix is connected to the notion of a set of vertex disjoint paths. The notion of vertex disjoint paths can be used to formulate topological conditions under which a model set is generically network identifiable, following the approach in [8]. As defined in [12], two paths in a network between external signals or nodes are vertex disjoint if they have no common nodes, including their start and end nodes. For a set of l paths, these paths are vertex disjoint if every pair of paths is vertex disjoint.

The essential meaning is the following: If there exists a set of vertex disjoint paths from some excitations r_k, e_l to some nodes w_i, then every one of those nodes has 'its own' source of excitation. Note that when two paths are vertex disjoint, there may still exist modules that connect the nodes in the paths, and there may exist loops around the nodes.

The connection to rank of a transfer matrix is on the basis of state-space systems in the following way. A parameterized state-space system is defined with matrices A, B, C, and the open-loop transfer from input to output is defined as $T_{ss} := C(sI - A)^{-1}B$. Then the *generic rank* of the transfer matrix

T_{ss} is defined as the rank of T_{ss} for almost all parameters. This is formalized in the following theorem.

Theorem 3 (Theorem 2 from [12]): Let G_Σ be the graph corresponding to the state-space system

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{16}$$

The maximum number of vertex disjoint paths in G_Σ from signals in u to signals in y equals the generic rank of T_{ss}. □

This result can be employed for the dynamic networks by coupling its topological properties with a state-space system.

Proposition 2: Let $\mathcal{M}$ be a set of network models M with strictly proper modules in G. Let $\mathcal{U}$ be a set of external signals, i.e. a set of some r_k and e_l, and let $\mathcal{Y}$ be a set of nodes w_i. The maximum number of vertex disjoint paths in M from signals in $\mathcal{U}$ to nodes in $\mathcal{Y}$ is equal to the generic rank of the transfer $T_{\mathcal{YU}}(q, \theta)$ from signals in $\mathcal{U}$ to nodes in $\mathcal{Y}$. □

The proof is collected in the appendix.

The rank conditions on $\check{T}_j$ of Corollary 2 can now be formulated in terms of vertex disjoint paths. The generic row rank of $\check{T}_j$ can be checked by checking whether there are a sufficient number of vertex disjoint paths from selected external signals to node signals.

Proposition 3: Let $\mathcal{M}$ be a set of network models M with strictly proper modules in G. Let $\mathcal{Y}_j$ be the set of nodes w_k which are an input to a parameterized $G_{jk}(\theta)$, and let α_j be the the cardinality of $\mathcal{Y}_j$. Let $\mathcal{U}_j$ be the set of external signals r_k, e_l that are an input to *non*-parameterized R_{jk}, H_{jl}.

1) The model set $\mathcal{M}$ is generically network identifiable if and only if i) of Theorem 1 holds and for each j, there is a set of α_j vertex disjoint paths from excitations in $\mathcal{U}_j$ to nodes in $\mathcal{Y}_j$.
2) Row j of model set $\mathcal{M}$ is generically network identifiable if and only if i) of Corollary 1 holds and there is a set of α_j vertex disjoint paths from excitations in $\mathcal{U}_j$ to nodes in $\mathcal{Y}_j$.
3) For module G_{ji}, let $\bar{\mathcal{Y}}_j = \mathcal{Y}_j \setminus w_i$. Module G_{ji} of model set $\mathcal{M}$ is generically network identifiable if and only if there exists a set $\mathcal{P}$ of the maximum number of vertex disjoint paths from signals in $\mathcal{U}_j$ to nodes in $\bar{\mathcal{Y}}_j$, and there is an additional path from signals in $\mathcal{U}_j$ to w_i, such that this path and the paths in $\mathcal{P}$ are vertex disjoint. □

Proof: By combining Corollary 2 with Proposition 2. In order to satisfy condition 1) or 2) there is an implicit requirement on the number of available external signals, which is directly related to the maximum number of parameterized elements in conditions 1) and 2) of Corollary 2. For condition 3) there is no minimum number of external signals, but there is the implicit requirement that there is a 'surplus' excitation that can form a vertex disjoint path to the module of interest.

In order to check the conditions of Proposition 3, all that must be done is check which transfer functions are parameterized, and check whether the necessary paths are present in the network. This is illustrated in an example.

Example 3 (Example 2 continued): Topology based conditions for identifiability are checked for various modules

in the network in Figure 1. When checking identifiability of modules that map into node w_4 we see that $\mathcal{Y}_4 = \{w_1, w_2, w_3\}$, so $\alpha_4 = 3$. There are only 2 excitations, which can never form 3 vertex disjoint paths, so the row is not generically network identifiable.

For identifiability of module G_{42} there are two vertex disjoint paths from external signals to the other inputs w_1, w_3, and there is no surplus excitation available for w_2. However for identifiability of module G_{41} there is just one vertex disjoint path from external signals to the other inputs w_2, w_3, and there is the surplus excitation available for w_1, so G_{41} is generically network identifiable.

$\square$

V. DISCUSSION ON DEFINITION OF IDENTIFIABILITY

Path-based conditions are based on generic rank, and not 'standard' rank. The difference between the two definitions of identifiability is the exclusion of a zero-measure set of models, so network identifiability is stricter than generic network identifiability. When one model in $\mathcal{M}$ is not identifiable, then $\mathcal{M}$ is not network identifiable, but it can be generically network identifiable. Next an example is given of a single non-identifiable model in the model set, which is in particular relevant in case the objective is to identify the network topology.

Example 4: Suppose we have a parameterized set of models as depicted in Figure 2, with

$$G = \begin{bmatrix} 0 & G_{12} \\ G_{21} & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T^0 = \begin{bmatrix} \frac{1}{1 - G_{12}^0 G_{21}^0} \\ \frac{G_{21}^0}{1 - G_{12}^0 G_{21}^0} \end{bmatrix}.$$

Identifiability of G_{12} and G_{21} is determined from the rank of $\check{T}_1 = \frac{G_{21}}{1 - G_{12} G_{21}}$ and $\check{T}_2 = \frac{1}{1 - G_{12} G_{21}}$ respectively. For all θ where $G_{21}(\theta) = 0$ the $\check{T}_1$ loses rank and G_{12} is not identifiable. $\square$

In a situation where the topology is known, i.e. it is known that $G_{21} \neq 0$, then we want to classify the model set as identifiable, which can be done with generic network identifiability. However when the topology is not known a-priori, we would like to determine whether G_{12} and G_{21} are zero or non-zero. Then the possibility that $G_{21}^0 = 0$ must be taken into account such that the generic network identifiability concept is less attractive, and the global network identifiability concept is more powerful.

VI. CONCLUSIONS

The notion of network identifiability has been extended to cover the case of single-module identifiability. Necessary and sufficient conditions for single module identifiability have
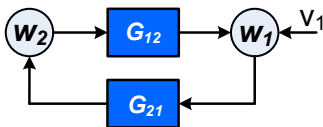

Fig. 2. A closed-loop network representing a set of models.

been obtained, and it has been shown that when considering the concept of generic identifiability, the necessary and sufficient conditions can be reformulated in terms of path-based conditions that can simply be verified on the basis of the network topology.

APPENDIX

A. Proof of Theorem 2

The left hand side of the implication (11) is written as

$$(I - G(\theta))T = U(\theta), \quad (17)$$

where we use shorthand notation $T = T(\theta_0)$, $G(\theta) = G(\theta_1)$ and $U(\theta) = U(\theta_1)$. By inserting the permutation matrices P and Q into row j of (17) similar to (7) we obtain

$$(I - G(\theta))_{j\star} P P^{-1} T Q = U_{j\star}(\theta) Q. \quad (18)$$

Let $\left[(I - G(\theta))_{j\star}^{(1)} \ (I - G)_{j\star}^{(2)} \right] = (I - G(\theta))_{j\star} P$ be the partitions containing parameterized and non-parameterized modules after permutation, and similarly let $\left[U_{j\star}^{(1)} \ U(\theta)_{j\star}^{(2)} \right] = U(\theta)_{j\star} Q$ be the partitions of non-parameterized and parameterized modules, then

$$(I - G(\theta))_{j\star}^{(1)} T_j^{(1)} + (I - G)_{j\star}^{(2)} T_j^{(2)} = \left[U_{j\star}^{(1)} \quad U(\theta)_{j\star}^{(2)} \right], \quad (19)$$

with $P^{-1} T Q = \begin{bmatrix} T_j^{(1)} \\ T_j^{(2)} \end{bmatrix}$. Note that $\check{T}_j = T_j^{(1)} \begin{bmatrix} I_{K+p-\beta} \\ 0 \end{bmatrix}$. The right-hand block in (19) corresponding to $U(\theta)_{j\star}^{(2)}$ does not add to the uniqueness of the module of interest since it is fully parameterized (conditions a,b of Theorem 1). Let ρ be the left $1 \times (K + p - \beta)$ block of $(I - G)_{j\star}^{(2)} T_j^{(2)}$, which does not contain parameters, so equivalently we can consider

$$(I - G(\theta))_{j\star}^{(1)} \check{T}_j + \rho = U_{j\star}^{(1)}. \quad (20)$$

Now since ρ and $U_{j\star}^{(1)}$ are independent of θ we have that $(I - G(\theta))_{ji}^{(1)}$ is uniquely specified if and only if $(I - G(\theta))_{ji}^{(1)}$ is uniquely specified in the left-nullspace of $\check{T}_j$.

Sufficiency: Define some transfer matrix $X(q)$ of dimension $(K + p - \beta) \times 1$ with the following properties:

- $\check{T}_{j(-i,\star)}(q, \theta_0) X(q) = 0$, and
- $\check{T}_{j(i,\star)}(q, \theta_0) X(q) \neq 0$,

where $\check{T}_{j(-i,\star)}$ and $\check{T}_{j(i,\star)}$ are defined in (12). This X exists because condition (13) requires that $\check{T}_{j(-i,\star)}(q, \theta_0)$ is not full column rank, and condition (13) implies that $\check{T}_{j(i,\star)}(q, \theta_0)$ is linearly independent from the rows of $\check{T}_{j(-i,\star)}(q, \theta_0)$. Now define an $(K+p-\beta) \times (K+p-\beta)$ full rank transfer matrix Z which has X as its first column. Then (20) is post-multiplied with Z to obtain an equivalent set of equations, leaving the set of solutions for G_{ji} invariant. The first column of $\check{T}_j Z$ is

$$\check{T}_j(q, \theta_0) X(q) = \begin{bmatrix} \check{T}_{j(i,\star)}(q, \theta_0) X(q) \\ 0 \end{bmatrix}, \quad (21)$$

such that, for this choice of Z, G_{ji} can be uniquely determined from

$$(I - G(\theta))_{j\star}^{(1)} \begin{bmatrix} \check{T}_{j(i,\star)} X \\ 0 \end{bmatrix} = (U_{j\star}^{(1)} - \rho) X. \quad (22)$$

If G_{ji} is unique for this particular choice of Z, it must be unique in the original problem also.

Necessity: The converse of condition (13) is that $\text{rank}(\check{T}_j(q, \theta_0)) = \text{rank}(\check{T}_{j(-i,\star)}(q, \theta_0))$. In this case the row of $\check{T}_j(q, \theta_0)$ corresponding to $G_{ji}(\theta)$ is linearly dependent on other rows of $\check{T}_j(q, \theta_0)$. When $\check{T}_{j(i,\star)}$ is linearly dependent on another row $\check{T}_{j(k,\star)}$, an equation equivalent to (20) can be created where the element G_{j1} and row $\check{T}_{j(i,\star)}$ are deleted, and where $(G_{ji}F + G_{jk})$ replaces G_{jk}, such that G_{ji} can not uniquely be distinguished.

Proof of situation (2): For all $\theta \in \Theta$: For every $\theta \in \Theta$ we can construct $T(\theta)$ with related $\check{T}_j(\theta)$. If condition (13) applies for every model as stated by condition (14), then the reasoning as presented before fully applies to every model. If for some $\theta \in \Theta$ the condition (13) is not met, there exists a model in the model set which is not identifiable, and hence the model set is not globally network identifiable in $\mathcal{M}$. $\square$

B. Proof of Proposition 2

A state-space system is defined, which we show has paths related to paths in M, such that the vertex disjoint paths of Theorem 3 also appear in M.

Theorem 3 is not suitable for state-space systems with D matrix. In situations where U contains direct terms, i.e. $\lim_{z \to \infty} U(z) \neq 0$, we add 1 delay to U. The network then is $q^{-1}w(t) = (I - G)^{-1}(q^{-1}U)u(t)$, which has the same topology, and the rank of T is unaffected.

Node w_j can be written as a state-space system with minimal and observable canonical form, i.e.

$$x_j^+ = A_j x_j + \sum_l B_{jl}^w w_l + \sum_k B_{jk}^u u_k, \quad w_j = C_j x_j, \qquad (23)$$

via relation $G_{j\star} = C_j(zI - A_j)^{-1}B_{j\star}^w$ and $q^{-1}U_{j\star} = C_j(zI - A_j)^{-1}B_{j\star}^u$. Note that $B_{jl}^w = 0$ if w_l is not an input to w_j. The full network can be written in state-space form by interconnecting all of the individual state-space systems (23)

$$x^+ = Ax + B^u u, \quad w = Cx, \quad \text{with} \qquad (24)$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}, \quad B^u = \begin{bmatrix} B_{11}^u & B_{12}^u & \cdots \\ B_{21}^u & B_{22}^u & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad B^w = \begin{bmatrix} B_{11}^w & B_{12}^w & \cdots \\ B_{21}^w & B_{22}^w & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$C = diag(\{C_i\}_{i=1\cdots L}), \quad A = diag(\{A_{ii}\}_{i=1\cdots L}) + B^w C.$$

The network topology is encoded by B^w. We consider the graph G_M with nodes w, u associated with the network model M, and the graph G_Σ, with nodes being the elements of the state vector x and u, while $w_j = x_{j\{1\}}$, where $x_{i\{1\}}$ denotes the first element of x_i. The proof now proceeds along the following steps.

Claim 1: The path w_i, w_j (or u_i, w_j) exists in G_M if and only if the path w_i (or u_i), $x_{j,\{s\}}, x_{j,\{s-1\}}, \cdots x_{j,\{1\}} = w_j$ exists in G_Σ for some value of s. The path w_i, w_j implies $G_{ji} \neq 0$, which implies $B_{ji}^w \neq 0$, such that the ji block of A denoted with $A_{\{ji\}} = B_{ji}^w C_i$ has a non-zero first column since $C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$, such that there is a path w_i (or u_i), $x_{j,\{s\}}$. The implications act also in the other direction. The path $x_{j,\{s\}}, x_{j,\{s-1\}}, \cdots x_{j,\{1\}}$ is a direct consequence of the chosen observable canonical form of the state-space model. The interpretation is that the path in G_M from w_i to its out-neighbor w_j in G_Σ becomes a path only through states in x_j.

Now we make a claim on paths from u to w. *Claim 2:* Path $u_i, w_{k_1}, \cdots, w_{k_n}, w_j$ exists in G_M if and only if path $u_i, x_{k_1\{1\}}, \cdots, x_{k_n\{1\}}, x_{j\{1\}}, w_j$ exists[4] in G_Σ, for any sequence $k_1, \cdots, k_n$. This is a direct consequence of Claim 1. The interpretation of the claim is that the additional states/nodes in G_Σ can not change the vertex joint/disjoint properties of paths between the nodes that are also in G_M.

Transfer $T_{\mathcal{YU}} = C_{\mathcal{Y}}(zI - A)^{-1}B_{\mathcal{U}}$ where $C_{\mathcal{Y}}$ and $B_{\mathcal{U}}$ are C and B with appropriate rows and columns removed. Then by Theorem 3 and Claim 2, the generic rank of $T_{\mathcal{YU}}$ equals the maximum number of vertex disjoint paths $\mathcal{U}$ to $\mathcal{Y}$ in G_M. $\square$

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[4] Technically the $x_{k\{1\}}$ is to be replaced with $x_{k\{s\}}, \cdots, x_{k\{1\}}$ for some s