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Model Set Determination and its Application to the Control of Compact Disc Players*

T. Zhou[1], H. G. M. Dotsch[2], P. M. J. Van den Hof[2]

[1]Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama, Japan; [2]Mechanical Engineering Systems and Control Group, Delft University of Technology, Delft, The Netherlands

This paper deals with the problem of obtaining a plant nominal model and its error uncertainty bound when a batch of plant models is provided. While the selection of model set structure is essential and probably depends on the model intended application, in this paper it is assumed that a nominal model is perturbed by its error through a homographic transformation. A necessary and sufficient condition is obtained for a nominal model to be suboptimal. Moreover, an algorithm is proposed to obtain a nominal model which is suboptimal and has a low complexity. Furthermore, the extraction of structured nominal model errors is discussed. The efficiency of the proposed model set determination algorithm is verified through the application to the simultaneous spiral control of two compact disc players.

Keywords: Compact disc player; Error bound; Homographic transformation; Nominal model; Robust control

1. Introduction

Robustness is one of the major properties required for control systems. In control engineering, one challenging task is to design a controller which performs satisfactorily when the plant works under different conditions, or which satisfactorily controls different plants that have different but similar dynamics. The former is due to the fact that the dynamics of a plant usually changes with time, environment, etc. [13,23] while the latter is usually met in mass production [16].

With a slight abuse of terminology, in this paper we refer to a plant working under different conditions as different plants.

While it is desirable to design a controller which simultaneously stabilises *just* these different plants and make the closed loop systems achieve satisfactory performance, the model set consisting of the models of these plants *only* is highly structured, and controller design is still not tractable [7,15]. A pragmatic approach to cope with this controller design problem is to find a model set which contains all of these plant models and can be handled by the available robust control theories [1,23].

In the last decade, H_∞ control theory has been well developed, in which nominal model error is regarded as norm bounded but unstructured, and Riccati equation or linear matrix inequality based solutions have been established [15].

When the structure of nominal model error is discarded, several ways are suggested for nominal model error description. For example, additive error, multiplicative error, relative error, coprime factorisation error, linear fractional transformation error, etc., have been widely applied in robust control theories [15]. The investigation on the suitability of

Correspondence and offprint requests to: T. Zhou, Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, 4259, Nagatsuta-cho, Midori-ku, Yokohama, Japan. Email: zhou@cs.dis.titech.ac.jp

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model set structure is essential in both identification and robust control, and needs further efforts.

In this research, we investigate the problem of model set determination when a batch of plant models is provided. Former results are extended to the case in which a nominal model is perturbed by its error through a homographic transformation. This model set structure can be regarded as one of the most general model set descriptions, because additive, multiplicative, relative, etc., model set structure can be regarded as its special form. While homographic transformation has been widely applied in classical circuit theory [19], its application in control and identification is relatively rare. Limited to our knowledge, the only exceptions seem to be Abrishamchian et al. [2] and Kimura [12]. In [12], Kimura applied homographic transformation to solving the H_∞ optimal control problem, while in [2] Abrischamchian et al. utilised homographic transformation in the description of plant nominal model errors. In this paper, it is proved that the formulated model set determination problem can be converted into a model-matching problem, and furthermore, to a frequency-weighted $\mathcal{L}_\infty$ norm model reduction problem. A necessary and sufficient condition is obtained for a nominal model to be suboptimal. Moreover, an algorithm based on Hankel norm model reduction is proposed for the determination of a nominal model which is suboptimal and has a low complexity.

To reduce the conservatism in controller design, sometimes structured error uncertainty bound is preferable [4,15,21]. A necessary condition is obtained for the existence of structured nominal model errors, and an algorithm is proposed for their extraction.

During the last two decades, optical disc systems have been extensively developed. One successful example is compact disc players. Due to the lack of mechanical guidance for optical information pickup, position controls are necessary [5,8]. Moreover, because of the characteristic variations of the mechanical elements, electromagnetic elements, optical elements, etc., every compact disc player has similar but different dynamics. To shorten production period and hence reduce production costs, one approach is to design a robust controller for a batch of compact disc players. In this paper, the proposed model set determination method is applied to the simultaneous spiral control of two compact disc players, with a combination of mixed sensitivity reduction methodology developed in robust controller design theory [7].

The rest of this paper is organised as follows. In the next section, the model set determination problem is formulated and its control engineering significance is briefly discussed. A necessary and sufficient condition for a nominal model to be suboptimal is obtained in Section 3. Based on this condition, a model set determination algorithm is proposed in Section 4. In Section 5, the problem of structured nominal model error extraction is discussed. The application of the proposed model set determination algorithm to the simultaneous design of two compact disc player spiral control systems is investigated in Section 6. This paper is concluded by Section 7, in which some further research issues are suggested. Finally, three appendices are included which give respectively a proof of Theorem 3, the frequency-weighting functions applied in model set determination, robust controller design and controller reduction, and the models of the compact disc radial position control systems.

The notation used in this paper is standard. R is used to represent the set of real numbers, while H_∞ is the set of stable proper transfer function matrices. For a complex matrix M, M^T and $\tilde{M}$ denote respectively its transpose and complex conjugate transpose, while $\bar{\sigma}(M)$ is its maximal singular value. A $m \times m$ dimensional unit matrix is denoted by I_m, and m is often omitted when the dimension is not very important. $\tilde{G}(s)$ stands for the conjugate system of $G(s)$ which is defined as $\tilde{G}(s) = -B^T(sI + A^T)C^T + D^T$ when $G(s) = C(sI - A)^{-1}B + D$. Finally, $\|G(s)\|_\infty$ and $\|G(s)\|_H$ are applied to represent the $\mathcal{L}_\infty$ norm and Hankel norm of transfer function matrix $G(s)$, respectively.

2. Model Set Determination Problem

The object of this research is to obtain a model set which includes a batch of prescribed models and is compatible with the available robust control theories.

When a plant nominal model is perturbed by unstructured errors through a homographic transformation, the model set determination problem can be formulated as follows.

Problem. Assume that plant models $G_1(s), \ldots, G_n(s)$, weighting functions $w_1(s), \ldots, w_n(s)$, transfer function matrices $N_1(s)$, $D_1(s)$, and positive number γ are given. Moreover, assume that $G_i(s)$, $i = 1, \ldots, m$, $N_1(s)$, $D_1(s)$ are square and stable, while $w_i(s)$, $i = 1, \ldots, n$, are both stable and invertibly stable. Find stable transfer function matrices $N_0(s)$, $D_0(s)$, such that

1. $N_0(s)$, $D_0(s)$ are right coprime.
2. $\tilde{D_0}(s)D_0(s) = I$.

3. There exists at least one stable $\Delta_i(s)$, such that $G_i(s) = [N_0(s) + N_1(s)\Delta_i(s)][D_0(s) + D_1(s)\Delta_i(s)]^{-1}, \quad i = 1,\ldots,n.$
4. $J = \|[w_1(s)\Delta_1^T(s) \ \ldots \ w_n(s)\Delta_n^T(s)]^T\|_\infty < \gamma.$

In this paper, a nominal model $N_0(s)D_0^{-1}(s)$ satisfying these four conditions is called suboptimal, and it achieves the optimal one with the dimunution of γ.

Several remarks on the control engineering significance of the above problem are now in order.

Remark 1. When $N_1(s) - G_i(s)D_1(s)$, $i = 1, \ldots,n$, is invertibly stable, it will become clear in the subsequent discussions that there exists a minimum phase transfer function $w(s)$, such that all the plant models $G_i(s)$, $i = 1, \ldots,n$, are included in transfer function matrix set $\mathcal{G}$ defined as

$$\mathcal{G} = \{G(s)|G(s) = [N_0(s) + w(s)N_1(s)\Delta(s)][D_0(s) + w(s)D_1(s)\Delta(s)]^{-1}, \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1\} \quad (1)$$

Remark 2. When $N_1(s) = I$, $D_1(s) = 0$, the above transfer function matrix set $\mathcal{G}$ can be expressed as

$$\mathcal{G} = \{G(s)|G(s) = [N_0(s) + w(s)\Delta(s)]D_0^{-1}(s), \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1\} \quad (2)$$

That is, in this case, the nominal model is perturbed by additive errors.

Remark 3. When $N_1(s) = 0$, $D_1(s) = I$, transfer function matrix $\mathcal{G}$ has an expression

$$\mathcal{G} = \{G(s)|G(s) = N_0(s)[I + w(s)D_0^{-1}(s)\Delta(s)]^{-1}D_0^{-1}(s), \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1\} \quad (3)$$

which means that the nominal model has a relative error.

Remark 4. It is a direct result of the small gain theorem that the transfer function matrix set $\mathcal{G}$ defined in Eq. (1) is robustly stabilised by a controller $C(s)$ if and only if

$$\|w(s)D_0^{-1}(s)\{D_1(s) + C(s)[I + N_0(s)D_0^{-1}(s)C(s)]^{-1}[N_1(s) - N_0(s)D_0^{-1}(s)D_1(s)]\}\|_\infty < 1 \quad (4)$$

Based on this condition, a controller can be designed on the basis of $\mathcal{H}_\infty$ optimal control theory (for robust stability and nominal performance) or structured singular value control theory (for robust stability and robust performance).

Remark 5. An essential problem in this model set determination problem is about the selection of transfer function matrices $N_1(s)$ and $D_1(s)$. It is $N_1(s)$ and $D_1(s)$ that determine the structure of the model set. It will become clear in the following discussions that to make the resulting model set compatible with the available robust control theories, it is desirable that $N_1(s)$, $D_1(s)$, $(N_1(s) - G_i(s)D_1(s))^{-1}$, $i = 1,\ldots,n$, are stable. However, the determination of $N_1(s)$ and $D_1(s)$ needs further investigation.

It is worth mentioning that from the viewpoint of controller design, it is more suitable to minimise the cost function $\max_i\|w_i(s)\Delta_i(s)\|_\infty$ in the above model set determination problem. The minimisation of this cost function, however, is difficult when the complexity of the nominal model is restricted.

On the other hand, the requirement that $\tilde{D_0}(s)D_0(s) = I$ does not sacrifice any generality of the formulated model set determination problem. The reasons are as follows. Assume that transfer function matrices $\bar{N}_0(s)$, $\bar{D}_0(s)$ are right coprime and satisfy conditions (3) and (4). When $\bar{D}_0(s)$ has no purely imaginary zeros, there exists a stable and invertibly stable transfer function matrix $M(s)$ such that $\tilde{M}(s)M(s) = \tilde{\bar{D}_0}(s)\bar{D}_0(s)$. Define $D_0(s) = \bar{D}_0(s) \ M^{-1}(s)$, $N_0(s) = \bar{N}_0(s)M^{-1}(s)$. Then, it is obvious that $N_0(s)$, $D_0(s)$ are right coprime and $\tilde{D_0}(s)D_0(s) = I$. The benefit obtained from the requirement of $\tilde{D_0}(s)D_0(s) = I$ is that it makes the formulated problem solvable. This will become clear in the subsequent discussions.

When $G_i(s)$, $i = 1,2,\ldots,n$, are unstable, the formulation of the above model set determination problem is more involved. A possible approach is first to perform coprime factorisations of $G_i(s)$, $i = 1,2,\ldots,n$, and then respectively approximate their coprime factors.

3. Conditions for the Suboptimality of a Nominal Model

To solve the model set determination problem formulated in the previous section, some properties of the cost function J are investigated first.

Theorem 1. Assume that transfer function matrix $N_1(s) - G_i(s)D_1(s)$ has no purely imaginary zeros. Then, there exists a transfer function matrix $X_i(s)$, such that

$$(N_1(s) - G_i(s)D_1(s))(N_1(s) - G_i(s)D_1(s))^{\sim} = X_i(s)\,\tilde{X_i}(s), X_i(s), X_i^{-1}(s) \in \mathcal{H}_\infty, \quad i = 1,\ldots,n.$$

Moreover,

$$J = \|T_1(s) - T_2(s)N_0(s)D_0^{-1}(s)\|_\infty$$

in which $T_1(s)$ and $T_2(s)$ are respectively defined as

$$T_1(s) = \begin{bmatrix} w_1(s)X_1^{-1}(s)G_1(s) \\ \vdots \\ w_n(s)X_n^{-1}(s)G_n(s) \end{bmatrix},$$

$$T_2(s) = \begin{bmatrix} w_1(s)X_1^{-1}(s) \\ \vdots \\ w_n(s)X_n^{-1}(s) \end{bmatrix}$$

Proof. The existence of transfer function matrix $X_i(s)$ is a direct result of spectral factorisation theory [9].

From

$$G_i(s) = [N_0(s) + N_1(s)\Delta_i(s)][D_0(s) + D_1(s)\Delta_i(s)]^{-1} \tag{5}$$

we have

$$[N_1(s) - G_i(s)D_1(s)]\Delta_i(s) = G_i(s)D_0(s) - N_0(s) \tag{6}$$

On the other hand, from the definition of $X_i(s)$, the next relation can be established:

$$[X_i^{-1}(s)(N_1(s) - G_i(s)D_1(s))]^\sim[X_i^{-1}(s)(N_1(s) - G_i(s)D_1(s))] = [N_1(s) - G_i(s)D_1(s)]^\sim[X_i(s)X_i^\sim(s)]^{-1}X$$
$$[N_1(s) - G_i(s)D_1(s)] = I \tag{7}$$

noting that $G_i(s)$, $N_1(s)$, $D_1(s)$ are assumed to be square and $N_1(s) - G_i(s)D_1(s)$ has no purely imaginary zeros.

Based on this relation, the next equation can be derived immediately:

$$\Delta_i^\sim(s)\Delta_i(s) = [X_i^{-1}(s)(N_1(s) - G_i(s)D_1(s))\Delta_i(s)]^\sim[X_i^{-1}i(s)(N_1(s) - G_i(s)D_1(s))\Delta_i(s)] \tag{8}$$

Hence,

$$\sum_{i=1}^{n} w_i^\sim(s)w_i\Delta_i^\sim(s)\Delta_i(s)$$
$$= \sum_{i=1}^{n} w_i^\sim(s)w_i(s)[X_i^{-1}(s)(N_1(s) - G_i(s)X D_1(s))\Delta_i(s)]^\sim[X_i^{-1}(s)(N_1(s) - G_i(s)D_1(s))\Delta_i(s)]$$
$$= \begin{bmatrix} w_1(s)X_1^{-1}(s)[G_1(s)D_0(s) - N_0(s)] \\ \vdots \\ w_n(s)X_n^{-1}(s)[G_n(s)D_0(s) - N_0(s)] \end{bmatrix} \sim X$$
$$\begin{bmatrix} w_1(s)X_1^{-1}(s)[G_1(s)D_0(s) - N_0(s)] \\ \vdots \\ w_n(s)X_n^{-1}(s)[G_n(s)D_0(s) - N_0(s)] \end{bmatrix} \tag{9}$$

Therefore,

$$J = \left\| \begin{bmatrix} w_1(s)X_1^{-1}(s)[G_1(s)D_0(s) - N_0(s)] \\ \vdots \\ w_n(s)X_n^{-1}(s)[G_n(s)D_0(s) - N_0(s)] \end{bmatrix} \right\|_\infty$$
$$= \|T_1(s)D_0(s) - T_2(s)N_0(s)\|_\infty \tag{10}$$
$$= \|T_1(s) - T_2(s)N_0(s)D_0^{-1}(s)\|_\infty$$

This completes the proof. □

It is clear from Eq. (6) that a sufficient condition for $\Delta_i(s)$ to be stable is that $N_1(s) - G_i(s)D_1(s)$ is invertibly stable, $i = 1,2,\ldots,n$. Note that in robust controller design theories it is preferable that nominal model perturbations are stable [7].

For brevity, define $G_0(s) = N_0(s)D_0^{-1}(s)$. Then

$$J = \|T_1(s) - T_2(s)G_0(s)\|_\infty \tag{11}$$

On the other hand, let $T_{20}(s)$ be the square stable transfer function matrix satisfying

$$T_{20}^\sim(s)T_{20}(s) = T_2^\sim(s)T_2(s), \quad T_{20}^{-1}(s) \in \mathcal{H}_\infty \tag{12}$$

and define transfer function matrix $T_{2I}(s)$ as

$$T_{2I}(s) = T_2(s)T_{20}^{-1}(s) \tag{13}$$

The existence of $T_{20}(s)$ is guaranteed by the assumptions that $w_i(s)$, $X_i(s)$, $i = 1,2,\ldots,n$, are both stable and invertibly stable. The matrix valued transfer functions $T_{20}(s)$ and $T_{2I}(s)$ are usually called the outer and inner factors of $T_2(s)$, respectively [9].

From the definition of transfer function matrix $T_{2I}(s)$, it is obvious that

$$T_{2I}^\sim(s)T_{2I}(s) = I, \quad T_2(s) = T_{2I}(s)T_{20}(s) \tag{14}$$

Moreover, there exists a transfer function matrix $T_{2\perp}(s)$, which belongs to $\mathcal{H}_\infty$ and satisfies [9]:

$$[T_{2I}(s)\ T_{2\perp}(s)]^\sim[T_{2I}(s)\ T_{2\perp}(s)]$$
$$= [T_{2I}(s)\ T_{2\perp}(s)][T_{2I}(s)\ T_{2\perp}(s)]^\sim = I \tag{15}$$

Hence,

$$J = \|T_1(s) - T_2(s)G_0(s)\|_\infty$$

$$= \left\| T_1(s) - [T_{2I}(s)\ T_{2\perp}(s)] \begin{bmatrix} T_{20}(s) \\ 0 \end{bmatrix} G_0(s) \right\|_\infty$$

$$= \left\| [T_{2I}(s)\ T_{2\perp}(s)]^\sim \left\{ T_1(s) - [T_{2I}(s)\ T_{2\perp}(s)] \begin{bmatrix} T_{20}(s) \\ 0 \end{bmatrix} G_0(s) \right\} \right\|_\infty$$

$$= \left\| \begin{matrix} T_{20}(s)[T_{20}^{-1}(s)T_{2I}^\sim(s)T_1(s) - G_0(s)] \\ T_{2\perp}^\sim(s)T_1(s) \end{matrix} \right\|_\infty \tag{16}$$

According to Eq. (16) and Theorem 1 of Chapter 8 in Francis [9], the following theorem is established, which gives a necessary and sufficient condition for a nominal model to be suboptimal.

Theorem 2. The cost function J is smaller than γ, if and only if

$$\|T_{2\perp}^\sim(s)T_1(s)\|_\infty < \gamma,$$

$$\|T_{20}(s)[T_{20}^{-1}(s)T_{2I}^\sim(s)T_1(s) - G_0(s)]R^{-1}(s)\|_\infty < 1$$

Here, the transfer function matrix $R(s)$ is both stable and invertibly stable and satisfies

$$R^\sim(s)R(s) = \gamma^2 I - T_1^\sim(s)T_{2\perp}(s)T_{2\perp}^\sim(s)T_1(s)$$

Note that matrix valued transfer function $T_{2\perp}^\sim(s)T_1(s)$ is independent of the plant nominal model $G_0(s)$. The conclusions of Theorem 2 also imply that $\|T_{2\perp}^\sim(s)T_1(s)\|_\infty$ is a lower bound of the cost function J.

When $D_0^\sim(s)D_0(s) \neq I$, cost function J can no longer be expressed as that of Eq. (16). For example, when $N_0^\sim(s)N_0(s) + D_0^\sim(s)D_0(s) = I$ is required, that is, transfer function matrices $N_0(s)$ and $D_0(s)$ are required to be the normalised right coprime factorisation of a suboptimal plant nominal model, by the same token as that in the derivation of equations (10) and (16), we have

$$J = \left\| \begin{bmatrix} T_{2I}^\sim(s)T_1(s) - T_{20}(s)G_0(s) \\ T_{2\perp}^\sim(s)T_1(s) \end{bmatrix} D_0(s) \right\|_\infty \tag{17}$$

Hence, a necessary and sufficient condition for the cost function J to be smaller than γ is

$$\begin{aligned} &[T_{2I}^\sim(s)T_1(s) - T_{20}(s)G_0(s)]^\sim[T_{2I}^\sim(s)T_1(s) \\ &-T_{20}(s)G_0(s)] + [T_{2\perp}^\sim(s)T_1(s)]^\sim[T_{2\perp}^\sim(s)T_1(s)] \\ &< \gamma^2[D_0^\sim(s)]^{-1}D_0^{-1}(s) \end{aligned} \tag{18}$$

On the other hand, from the assumption that matrix valued transfer functions $N_0(s)$ and $D_0(s)$ are normalised right coprime factors of $G_0(s)$, it is obvious that

$$[D_0^\sim(s)]^{-1}D_0^{-1}(s) = I + G_0^\sim(s)G_0(s) \tag{19}$$

Substituting Eq. (19) into inequality (18), we can declare that $G_0(s) = N_0(s)D_0^{-1}(s)$ is a suboptimal nominal model if and only if

$$\begin{aligned} &G_0^\sim(s)[Y_1^\sim(s)Y_1(s) - \gamma^2 I]G_0(s) + G_0^\sim(s)Y_1^\sim(s) \\ &Y_2(s) + Y_2^\sim(s)Y_1(s)G_0(s) < \gamma^2 I - Y_2^\sim(s)Y_2(s) \\ &\qquad - Y_3^\sim(s)Y_3(s) \end{aligned} \tag{20}$$

in which

$$\begin{aligned} &Y_1(s) = -T_{20}(s), \quad Y_2(s) = T_{2I}^\sim(s)T_1(s), \\ &Y_3(s) = T^\sim_{2\perp}(s)T_1(s) \end{aligned} \tag{21}$$

The problem of finding a $G_0(s)$ satisfying Eq. (20) is tractable only if the inertia of $Y_1^\sim(s)Y_1(s) - \gamma^2 I$ remains unchanged for all $s = jw$, $w \in [0, +\infty)$ [11]. This condition, however, is very restrictive.

When plant nominal model error is represented in additive form and the weighting functions $w_i(s)$, $i = 1,2,\ldots,n$ are selected to be identical, an optimal plant nominal model can be obtained through simply averaging $G_i(s)$, $i = 1,2\ldots,n$, provided that there are no restrictions on the complexity of the plant nominal model. This is an immediate result of Theorem 2, and is included here as a corollary.

Corollary 1. Assume that $N_1(s) = I$ and $D_1(s) = 0$. Moreover, assume that $w_1(s) = \ldots = w_n(s) = w(s)$. Then, $G_0(s) = \frac{1}{n}\sum_{i=1}^{n} G_i(s)$ is one of the optimal nominal models.

Proof. Using the same notation as that in the establishment of Theorems 1 and 2, it is obvious that if $N_1(s) = I$ and $D_1(s) = 0$, then

$$X_i(s) = I,\ i = 1,2,\ldots,n \tag{22}$$

Moreover, from the assumption that the weighting functions are identical, we have

$$T_1(s) = w(s)\begin{bmatrix} G_1(s) \\ \vdots \\ G_n(s) \end{bmatrix}, \qquad T_2(s) = w(s)\begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \tag{23}$$

Hence, the inner and outer factors of $T_2(s)$ are

$$T_{2I}(s) = \frac{1}{\sqrt{n}}\begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}, \qquad T_{20}(s) = \sqrt{n}\, w(s) \tag{24}$$

Therefore,

$$T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s) = \frac{1}{n}\sum_{i=1}^{n} G_i(s) \tag{25}$$

The proof of Corollary 1 can now be completed by applying the conclusions of Theorem 2 and Eq. (16). □

4. Model Set Determination Algorithm

From Theorem 2, it is obvious that $G_0(s) = T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s)$ is one of the suboptimal nominal models. From the viewpoint of controller design, however, a simple nominal model is preferable. Hence, it is more suitable to select $G_0(s)$ through frequency-weighted $\mathcal{L}^\infty$ norm model reduction of $T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s)$. On the other hand, it is well known that model reduction based on the criterion of frequency-weighted $\mathcal{L}^\infty$ norm is not tractable, while Hankel norm model reduction has been well developed [10,20]. Based on these arguments, a pragmatic algorithm is proposed for model set determination.

1. Compute the $\mathcal{L}^\infty$ norm of $\tilde{T}_{2\perp}(s)T_1(s)$. If it is smaller than γ, go to the next step; otherwise, γ is not achievable and it must be increased.
2. Fix the Smith–McMillan degree of $G_0(s)$ to be k. Perform frequency-weighted Hankel norm model reduction on $T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s)$, that is, $\|T_{20}(s)[T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s) - G_0(s)]R^{-1}(s)\|_H$, and obtain a stable transfer function matrix, say, $\bar{G}_0(s)$, with Smith–McMillan degree k. (To reduce plant nominal model error, convex optimisation can be applied to the determination of the numerator coefficient matrices of $\bar{G}_0(s)$, based on the minimisation of cost function $\|T_{20}(s)[T_{20}^{-1}(s)\tilde{T}_{2I}(s)T_1(s) - G_0(s)]R^{-1}(s)\|_\infty$.)
3. Compute the $\mathcal{L}^\infty$ norm of $T_{20}(s)[T_{2I}^{-1}(s)T_1(s) - \bar{G}_0(s)]R^{-1}(s)$. If it is smaller than 1, let $G_0(s) = \bar{G}_0(s)$; otherwise, let $k + 1 \rightarrow k$, repeat step (2).
4. Let $N_0(s) = G_0(s)$, $D_0(s) = I$.
5. Define $\Delta_i(s) = [N_1(s) - G_i(s)D_1(s)]^{-1}[G_i(s)D_0(s) - N_0(s)]$, $i = 1,\ldots,n$. Find a minimum phase transfer function $w(s)$, such that $|w(j\omega)| \geq \max_{1\leq i\leq n}\bar{\sigma}(\Delta_i(j\omega))$, $\omega \in [0, +\infty)$.

It is worthwhile to point out that when $N_0(s)$ and $D_0(s)$ are obtained, $w(s)$ can be obtained by means of frequency domain curve fitting and inner outer factorisation.

In the above model set determination algorithm, it is also possible to permit $\bar{G}_0(s)$ to be unstable. In this case, transfer function matrices $N_0(s)$ and $D_0(s)$ will be obtained as follows.

Assume that $C(sI - A)^{-1}B + D$ is the minimal realisation of transfer function matrix $G_0(s)$. Then, for arbitrary constant matrix K with compatible dimension, transfer function matrices

$$N_0(s) = D + (C - DK)(sI - A + BK)^{-1}B \tag{26}$$

$$D_0(s) = I - K(sI - A + BK)^{-1}B \tag{27}$$

are one of the right coprime factorisations of $G_0(s)$, provided that $A - BK$ is stable [14]. Let P be a positive definite matrix which satisfies the following Lyapunov equation:

$$AP + PA^T - BB^T = 0 \tag{28}$$

Moreover, define $K = B^TP^{-1}$. Then, it is an immediate result of transfer function matrix multiplications that $\tilde{D}_0(s)D_0(s) = I$. The stability of matrix $A - BK$ is guaranteed by the existence of a positive definite solution of Eq. (28).

When Eq. (28) has no positive definite solutions, assume that matrix A has no purely imaginary eigenvalues and K is one of its stabilising matrices. Then, there exists a stable transfer function matrix $M(s)$ such that $M^{-1}(s) \in \mathcal{H}_\infty$ and $\tilde{M}(s)M(s) = \tilde{D}_0(s)D_0(s)$. As a consequence $\bar{N}_o(s) = N_0(s)M^{-1}(s)$ and $\bar{D}_0(s) = D_0(s)M^{-1}(s)$ are the desirable transfer function matrices. This approach, however, usually increases the complexity of the transfer function matrices to be obtained.

The results on model set determination can be extended to the case in which there exist modelling errors in the prescribed plant models. This is more general in control engineering. For example, when plant model $G_i(s), i = 1,2,\ldots,n$, belongs to

$$\mathcal{G}_i = \{G(s) \mid G(s) = G_{io}(s) + w_{ei}(s)\Delta(s), \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1\} \tag{29}$$

and $N_1(s) = I$, $D_1(s) = 0$, the desired model set $\mathcal{G}$ can be determined as follows. Firstly, find a transfer function matrix $G_0(s)$ and a minimum phase transfer function $w(s)$, such that every transfer function matrix $G_{i0}(s)$, $i = 1,2,\ldots,n$, is included in model set

$$\bar{\mathcal{G}} = \{G(s) \mid G(s) = G_0(s) + w(s)\Delta(s), \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1\} \tag{30}$$

Secondly, find a minimum phase transfer function $\bar{w}(s)$, such that

$$|\bar{w}(j\omega)| \geq |w(j\omega)| + \max_{1\leq i\leq n} |w_{ei}(j\omega)|, 0 \leq \omega \leq \infty \tag{31}$$

Then, it is obvious that every plant model $G_i(s)$, $i = 1,2,\ldots,n$, belongs to model set $\mathcal{G}$ defined as

$$\mathcal{G} = \{G(s) \mid G(s) = G_0(s) + \overline{w}(s)\Delta(s),\ \Delta(s) \in \mathcal{H}_\infty,\ \|\Delta(s)\|_\infty \le 1\} \quad (32)$$

5. Structured Error Extraction

In the previous sections, we discussed the problem of determining a nominal model and its error bound from a batch of plant models. A model set has been obtained which includes all the provided plant models and is compatible with the well-developed H^∞ optimal control theory. In this transfer function matrix set, nominal model error is regarded as unstructured. Sometimes, however, it is preferable to represent nominal model error as structured, in order to reduce the conservatism in controller design [4,15,21]. Ideally, it is desirable to simultaneously obtain a nominal model, its structured and unstructured error bounds from the provided plant models. Unfortunately, this problem is not tractable at the moment, except that the denominator of the plant nominal model is prescribed [21]. To improve the performance of control systems, a two-step approach is suggested here. Firstly, a nominal model is obtained, regarding its error as unstructured. Secondly, the structured information of the nominal model errors is extracted.

Another reason for structured error extraction is as follows.

Generally, the nominal models that respectively minimise cost functions $\max_i\|w_i(s)\Delta_i(s)\|_\infty$ and $\|[w_1(s)\Delta_1^T(s)\ \ldots\ w_n(s)\Delta_n^T(s)]^T\|_\infty$ are different. While the former cost function is more natural in controller design, the latter one is mathematically easier to cope with. In our model set determination the latter cost function is chosen for its mathematical tractability. As a consequence, structured error is introduced into the model set determination due to the selection criterion. To make the unstructured error bound of the model set as small as possible, one approach is to suitably adjust the weighting functions $w_1(s),\ldots, w_n(s)$; another approach is to extract the structured error from the nominal model errors. While the former depends on the provided plant models, the latter is investigated in this section.

To simplify discussions, we assume, without loss of generality, that the $m \times p$ nominal model errors $\Delta_i(s)$, $i = 1,\ldots,n$, satisfy $m \ge p$. If $m < p$, the problem can be solved by just transposing $\Delta_i(s)$, $i = 1,\ldots,n$.

At first, we have the following results.

Theorem 3. Let $\Delta(s) = [\Delta_1^T(s)\ \ldots\ \Delta_n^T(s)]^T$. Assume that $\omega_0 = \arg\max_\omega \overline{\sigma}(w(j\omega)\Delta(j\omega))$. Moreover, assume that

$$w(j\omega_0)\Delta(j\omega_0) = [u_1^*\ \ldots u_{nm}^*]x \begin{bmatrix} diag\{\sigma_i|_{i=1}^p\} \\ 0 \end{bmatrix} [v_1\ \ldots\ v_p]$$

in which, $\sigma_1 \ge \ldots \ge \sigma_p \ge 0$, and matrices $[u_1^*\ \ldots u_{nm}^*]$ and $[v_1\ \ldots\ v_p]$ are unitary. Then, there exist a $\Delta_0(s) \in \mathcal{H}_\infty$ and a $\delta_i \in \mathcal{R}$, $i = 1,\ldots,n$, such that

$$\Delta_i(s) = \delta_i\Delta_0(s) + \hat{\Delta}_i(s), \quad \sum_{i=1}^n \delta_i^2 = 1$$

$$\|w(s)[\hat{\Delta}_1^T(s)\ \ldots\ \hat{\Delta}_n^T(s)]^T\|_\infty < \|w(s)\Delta(s)\|_\infty$$

only if vector u_1 can be expressed as

$$[\alpha_1\ k_1\alpha_1\ \ldots\ k_{m-1}\alpha_1\ \alpha_2\ k_1\alpha_2\ \ldots\ k_{m-1}\alpha_2\ \alpha_3\ \ldots\ k_{m-1}\alpha_n], \qquad \alpha_i \in \mathcal{R}$$

A proof of this theorem is given in Appendix A.

Based on the conclusions of Theorem 3, the following algorithm is proposed for structured error extraction:

1. Verify whether the conditions of Theorem 3 are satisfied. If the answer is negative, stop the computation; Otherwise, go to the next step.
2. Define δ_i, $i = 1, \ldots n$ and $\Delta_0(s)$ as
$$\delta_i = \frac{\alpha_i}{\sqrt{\sum_{i=1}^n \alpha_i^2}}, \quad \Delta_0(s) = \sum_{i=1}^n \delta_i\Delta_i(s)$$
3. Perform convex optimisation to find a $\overline{\delta}_i$, $i = 1,\ldots,n$, such that $\|w(s)[\Delta_i(s) - \overline{\delta}_i\Delta_0(s)]\|_\infty$ is minimised. Assume that the desirable $\overline{\delta}_i$ is $\overline{\delta}_i^0$.
4. Find a minimum phase transfer function $\hat{w}(s)$ satisfying
$|\hat{w}(j\omega)| \ge \max_{1\le i\le n}\overline{\sigma}(\Delta_i(j\omega) - \overline{\delta}_i^0\Delta_0(j\omega))$
for all $\omega \in [0, +\infty)$.
5. Define $\hat{\Delta}_0(s) = (\max_{1\le i\le n}|\overline{\delta}_i^0|)\ \Delta_0(s)$.

From the above algorithm, it is obvious that all the nominal model errors, $\Delta_i(s)$, $i = 1,\ldots,n$, are contained in transfer function matrix set Δ defined as

$$\left(\Delta = \left\{ \Delta(s) \,\middle|\, \Delta(s) = [I_m\ I_m] \begin{bmatrix} \delta I_m & 0 \\ 0 & \hat{\Delta}(s) \end{bmatrix} \begin{bmatrix} \hat{\Delta}_0(s) \\ \hat{w}(s)I_p \end{bmatrix} \right\}, \right.$$

$$\left. \delta \in \mathcal{R},\ |\delta| \le 1,\ \hat{\Delta}(s) \in \mathcal{H}_\infty,\ \|\hat{\Delta}(s)\|_\infty \le 1 \right) \quad (33)$$

These conclusions and algorithm can be simply extended to the case in which the parametric perturbation δ is permitted to be complex.

While it is possible to extract the structured error in $\hat{\Delta}(s)$ by means of the proposed algorithm, it is worthwhile to point out that with the increment of the number of parametric error blocks, robust controller design will become difficult [15].

Another application of this structured error extraction algorithm is to verify whether the frequency weighting functions applied in the model set determination have been well chosen or not. If $\|\hat{\Delta}_0(s)\|_\infty$ is small, then the frequency weighting functions have been appropriately chosen. Otherwise, some adjustments of the frequency weighting functions are preferable.

6. Simultaneous Spiral Control of Compact Disc Players

In Zhou and Van den Hof [22], the efficiency of the proposed model set determination algorithm and structured error extraction algorithm has been verified by a simulation example, through comparing the nominal model error bounds with those of the intuitively determined nominal models. In this section, we investigate the application of the proposed model set determination algorithm to the simultaneous spiral control of two compact disc players.

A compact disc player is an optical decoding device that reproduces high-quality audio signals from digitally coded signals recorded on a reflective disc, which is one of the successfully developed optical disc systems. It consists of a turntable DC motor for the rotation of the compact disc, and a balanced radial arm for track following. Both focus control and radial position control are required for optical information decoding. But the interaction between these two control loops is not significant and their controller design can be performed independently [5,8]. In this paper, we only discuss its radial position control.

In Fig. 1, a schematic structure of a compact disc control system is shown.

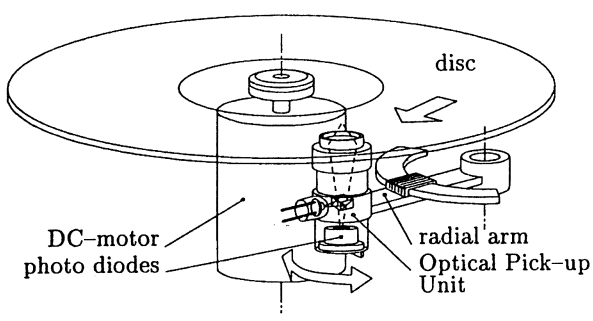

Fig. 1. Structure of a compact disc player.

The radial actuator is a permanent magnet/coil system. An optical element is mounted at the end of the radial arm, in which four photodiodes are utilised to detect the laser spot position error.

Due to the linear DC motor, the performance of the open loop radial position control system is very poor. Hence, a controller is necessary for stabilising the radial position control system and improving its performance [5,8].

When several conflicting factors are taken into account, the bandwidth of the radial position control system is required to be not smaller than 500 Hz. These conflicting factors include suppression of mechanical shocks, disturbance attenuation at the rotational frequency (4 ~ 8 Hz), ability to cope with significant disc eccentricity (not greater than 300 μm), playability of discs with faults, power consumption, etc. Moreover, noise attenuation below frequency 1100 Hz is required. Especially, the influence of noise below frequency 30 Hz on the closed loop system has to be significantly reduced. Furthermore, the radial track following error should be reduced to be smaller than 0.15 μm [5,8].

To shorten the production period, it is desirable to design a controller which performs satisfactorily for several compact disc players. On the other hand, it is almost impossible to make the dynamics of the open loop radial position control systems of compact disc players identical, because of the dynamics variations of the mechanical elements, electromagnetic elements, optical elements, etc. Hence, to achieve the object of simultaneous spiral control, one approach is to introduce robust controller design in which model set determination becomes unavoidable.

Based on closed loop identification theory [17,18] a 22nd order model $G_1(s)$ for compact disc player I, and a 26th order model $G_2(s)$ for compact disc player II, have been obtained. The transfer functions $G_1(s)$ and $G_2(s)$ are provided in Appendix B, while their frequency responses are shown in Fig. 2(a). These models represent accurately the frequency responses of the two compact disc players below 10^4 Hz, respectively.

Since the plant models are not minimum phase ones, we assume that $D_1 = 0$ and $N_1 = 1$ in the model set determination. On the basis of the closed loop system performance requirements, a 9th order weighting function $w(s)$ is selected for the determination of the nominal model in which $w_1(s) = w_2(s) = w(s)$. This weighting function is determined as follows. As stated in Section 3, $\frac{1}{2}(G_1(s) + G_2(s))$ is

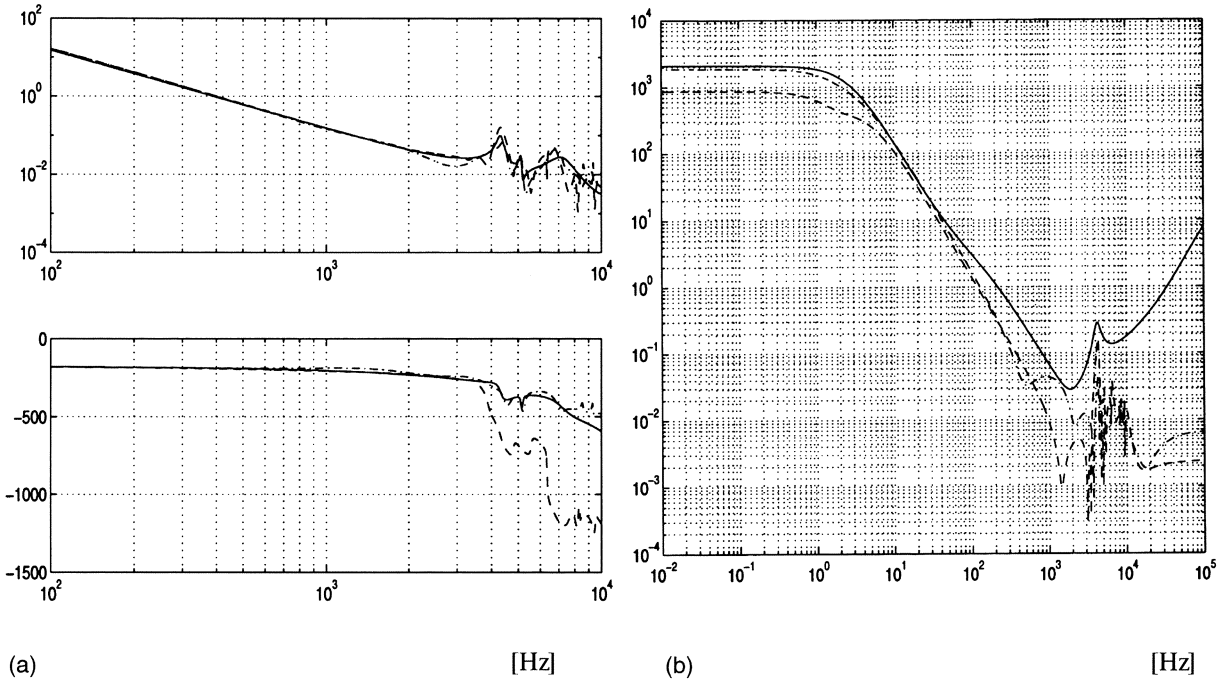

Fig. 2. Frequency response of CD player models and nominal model errors. ---: CD player I; --: CD player II; -: nominal model and error bound. (a) Models and nominal model. (b) Error magnitude and error bound.

one of the optimal nominal models when the nominal model errors are represented in additive form and $w_1(s) = w_2(s)$. On the other hand, it is preferable to reduce the nominal model error around the cross-over frequency of the control system. Taking into account the requirements on the closed loop control system, $w(s)$ is first chosen as

$$\overline{w}(s) = \frac{(1000 \times 2\pi)^2}{s^2 + 1000 \times 2\pi s + (1000 \times 2\pi)^2} \times (G_1(s) + G_2(s))^{-1}$$

in order to reduce nominal model errors in the neighbour of frequency 1000 Hz. To make the weighting function $w(s)$ have a minimal realisation and be a minimal phase transfer function, balanced model reduction without frequency weighting and inner outer factorisation are applied to $\overline{w}(s)$ [9]. That is, $w(s)$ is the outer factor of $\overline{w}(s)$ with minimal realisation. Note that in model set determination, only the magnitude of $w(j\omega)$, $\omega \in \mathcal{R}$, is important.

When $w(s)$ has been chosen, using the same notation of Section 3, direct matrix manipulations show that

$$T_{20}(s) = \sqrt{2}w(s), \tag{34a}$$

$$T_{20}^{-1}(s)T_{2l}^{\sim}(s)T_1(s) = \frac{1}{2}[G_1(s) + G_2(s)], \tag{34b}$$

$$T_{2\perp}^{\sim}(s)T_1(s) = \frac{w(s)}{\sqrt{2}}[G_2(s) - G_1(s)] \tag{34c}$$

Utilising the M-file *normhinf.m* of the commercial software Matlab, we have

$$\|T_{2\perp}^{\sim}(s)T_1(s)\|_\infty = 3.375 \times 10^3 \tag{35}$$

This implies that the cost function J cannot be reduced smaller than 3.375×10^3, no matter what plant nominal model is selected.

Taking into account the problems that may possibly arise in numerical computations and the requirements on the closed loop control system, γ is chosen to be 3.4×10^3.

The degree of plant nominal model $G_0(s)$ is increased from 1, in line with the algorithm proposed in Section 4. Table 1 shows the relation between the complexity of the plant nominal model and J^* which is defined as $J^* = \|T_{20}(s)[T_{20}^{-1}(s)T_{2l}^{\sim}(s)T_1(s) - \overline{G}_0(s)[R^{-1}(s)]\|_\infty$. Here, frequency-weighted Hankel norm model reduction and convex optimisation are applied in the search for the preferable $\overline{G}_0(s)$ when its Smith–McMillan degree is prescribed, using the algorithm suggested by Zhou [20].

For reference, J^* has been computed out until the degree of the plant nominal model is increased to 12.

From Table 1, it is obvious that there is an 8th

Table 1. Nominal model complexity and J^*

Nominal model order	1	2	3	4	5	6
J^*	9490	222.4	222.0	61.36	28.27	2.832
Nominal model order	7	8	9	10	11	12
J^*	1.039	0.9744	0.9585	0.8647	0.4136	0.3767

order nominal model which can make the cost function J smaller than 3.4×10^3. This nominal model is as follows:

$$G_0(s) = \frac{5.46 \times 10^{-3}s^8 - 2.35 \times 10^2 s^7 + 3.58 \times 10^7 s^6 - 1.01 \times 10^{12} s^5 + 6.10 \times 10^{16} s^4 - 1.35 \times 10^{21} s^3 + 3.99 \times 10^{25} s^2 - 5.75 \times 10^{29} s + 9.45 \times 10^{33}}{s^8 + 9.97 \times 10^3 s^7 + 3.86 \times 10^9 s^6 + 2.00 \times 10^{13} s^5 + 4.45 \times 10^{18} s^4 + 9.89 \times 10^{21} s^3 + 1.58 \times 10^{27} s^2 + 6.31 \times 10^{28} s + 4.73 \times 10^{29}} \quad (36)$$

It is worth pointing out that while it is possible to reduce J^* to zero through increasing the complexity of $G_0(s)$, a nominal model with degree greater than 8 does not significantly reduce the cost function J. This may possibly be obvious by noting that 3.4×10^3 is very near to 3.375×10^3, a lower bound of J. On the other hand, a simple nominal model is generally more appreciative in robust controller synthesis. Based on these arguments, $G_0(s)$ of Eq. (36) is selected for robust controller synthesis.

At every frequency, the error of the above nominal model is bounded by an 8th order minimal phase transfer function $w_{au}(s)$.[1] This transfer function is obtained through frequency domain curve fitting and inner outer factorisation. The latter is applied to guarantee that $w_{au}(s)$ has a minimal phase.

The frequency response of the obtained nominal model is presented in Fig. 2(a), while in Fig. 2(b) the frequency characteristics of the error magnitude of the obtained nominal model and its error bound are shown. It is obvious that the models of these two compact disc players are included in the model set

$$\mathcal{G} = \{ G(s) \mid G(s) = G_0(s) + w_{au}(s)\Delta(s), \quad \Delta(s) \in \mathcal{H}_\infty, \|\Delta(s)\|_\infty \leq 1 \} \quad (37)$$

[1] $|w_{au}(j\omega)$ is selected greater than $\max\{|G_1(j\omega) - G_0(j\omega)|, |G_2(j\omega) - G_0(j\omega)|\}$ *when* $\omega \geq 10^4 \times 2\pi$, in order to prevent the actual control system from being destabilised by the plant unmodelled dynamics.

The mixed sensitivity reduction approach is applied in robust controller design [7,9]. Briefly, this controller design method is to find an internally stabilising controller $C(s)$, such that

$$\left\| \begin{matrix} w_{au}(s)C(s)[I + G_0(s)C(s)]^{-1} \\ w_s(s)[I + G_0(s)C(s)]^{-1} \end{matrix} \right\|_\infty < \alpha \quad (38)$$

for a prescribed positive number α. The optimal controller is obtained through decreasing α until the desirable controller does not exist. Here, transfer functions $[I + G_0(s)C(s)]^{-1}$ and $C(s)[I + G_0(s)C(s)]^{-1}$ are usually referred to as the sensitivity function and controller sensitivity function, respectively.

According to robust control theories, a sufficient condition for the existence of a controller $C(s)$ that simultaneously stabilises these two compact disc players is that $\|w_{au}(s)C(s)[I + G_0(s)C(s)]^{-1}\|_\infty < 1$, which is guaranteed by assigning α to be 1. On the other hand, the requirements on low-frequency noise attenuation of the designed controller are mainly reflected by the sensitivity function $[I + G_0(s)(s)]^{-1}$, while the bandwidth of the closed loop system can be achieved by appropriately shaping the complementary sensitivity function $G_0(s)C(s)[I + G_0(s)C(s)]^{-1}$ [7,9]. Note that $G_0(s)C(s)[I + G_0(s)C(s)]^{-1} = I - [I + G_0(s)\ C(s)]^{-1}$; it is also possible to satisfy the requirements on the closed loop system's bandwidth through suitably selecting a weighting function for the sensitivity function $[I + G_0(s)C(s)]^{-1}$.

When the requirements on the closed loop systems' performance are taken into account, an 8th order weighting function $w_s(s)$ is selected for the sensitivity function $[I + G_0(s)C(s)]^{-1}$, in order to have a significant noise attenuation in the low-frequency range ($\leq$30 Hz) and a bandwidth of approximately 1000 Hz.

Based on the selected nominal model $G_0(s)$, the weighting functions $w_{au}(s)$ and $w_s(s)$, a controller

$C(s)$ is obtained for $\alpha = 1$, by means of the M-file *linf.m* of the commercial software Matlab.[2] This controller is of degree 38. To make it realisable, an 8th order controller $\overline{C}(s)$ is obtained through frequency-weighted Hankel norm model reduction and convex optimisation [20]:

$$\overline{C}(s) = \frac{1.62 \times 10^{-2}s^8 + 4.55 \times 10^4 s^7 + 1.98 \times 10^{10}s^6 + 1.49 \times 10^{14}s^5 + 1.67 \times 10^{19}s^4 + 6.57 \times 10^{21}s^3 + 2.80 \times 10^{24}s^2 + 3.39 \times 10^{26}s + 2.56 \times 10^{27}}{s^8 + 8.51 \times 10^4 s^7 + 3.87 \times 10^9 s^6 + 1.28 \times 10^{14}s^5 + 1.90 \times 10^{18}s^4 + 2.40 \times 10^{22}s^3 + 2.30 \times 10^{24}s^2 + 1.53 \times 10^{26}s + 9.56 \times 10^{23}} \tag{39}$$

In controller reduction, a 15th order frequency-weighting function $w_c(s)$ is selected in order to make the deterioration of the closed loop system's performance as small as possible [3]. This frequency-weighting function is obtained as follows. To minimise the performance deterioration of the control system due to controller reduction, at first a frequency-weighting function

$$\overline{w}_c(s) = \frac{s^2 + 6.35 \times 10^2 s + 3.95 \times 10^3}{s^2 + 6.28 \times 10^3 s + 3.95 \times 10^3} C^{-1}(s) \tag{40}$$

is selected for controller reduction. Because minimal phase weighting functions are preferable in model reduction [20], balanced model reduction without frequency weighting and inner outer factorisation are applied in the determination of weighting function $\overline{w}_c(s)$. Therefore, $w_c(s)$ is in fact the outer factor of $\overline{w}_c(s)$ after its uncontrollable and/or unobservable states have been deleted.

All the frequency-weighting functions used in the model set determination, robust controller design and controller reduction are given in Appendix C, while their magnitude frequency responses are presented in Fig. 3.

The frequency responses of the original 38th order controller and the reduced 8th order controller are presented in Fig. 4, while the magnitude frequency responses of the sensitivity functions and the controller sensitivity functions are shown in Fig. 5.

From Figs 4 and 5 it is clear that although the frequency response of the reduced controller does not match very well with that of the original high-order controller at some frequencies, significant performance deterioration has been successfully avoided in controller simplification. Moreover, the designed sensitivity functions have a magnitude not greater than 2 for all frequencies, not greater than 10^{-2} for frequencies below 30 Hz, and not greater than 10^{-3} for frequencies below 10 Hz.

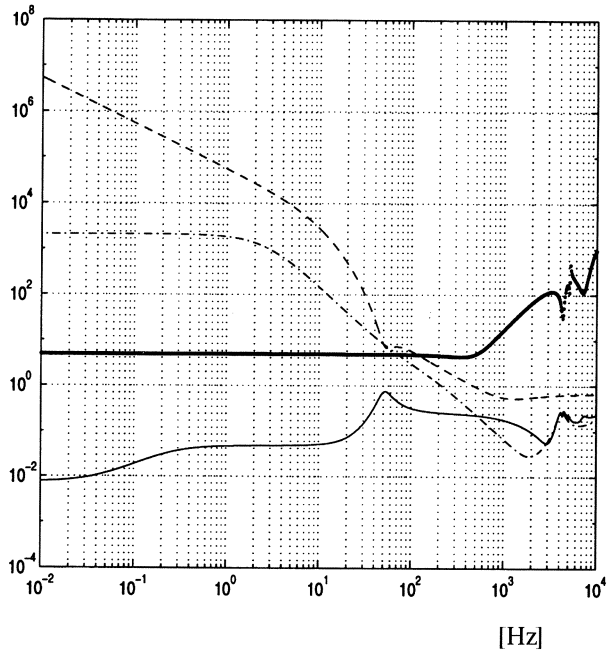

Fig. 3. Magnitude frequency response of the weighting functions. ••: $w(s)$; -·-: $w_{au}(s)$; --: $w_s(s)$; -: $w_c(s)$.

The reduced controller is realised by a digital signal processor with a sampling period of 4×10^{-5} s. In controller discretisation, the first order holder approach is applied.

Here, we would like to mention that the designed controller does not have a high gain at high frequencies. This is obvious through comparing the coefficients of the numerator and denominator of the controller $\overline{C}(s)$. In fact, the magnitude of the controller $\overline{C}(s)$ begins to roll down from approximately 7.0×10^3 Hz. In Figs 4 and 5, the frequency characteristics of the controllers, sensitivity functions and controller sensitivity functions are presented only below frequency 10^4 Hz. This is mainly because we are most interested in the frequency range $0 \sim 10^4$ Hz. Another reason is due to the realisation of the controller. While a 4×10^{-5} s sampling period is short enough for properly implementing the designed controller, the frequency characteristics of the controller are significantly violated at frequencies greater than 6.25×10^3 Hz.

[2]The M-file *hinf.m* has also been applied to the robust controller design, but no desirable controller has been obtained. This may possibly be due to the numerical problems in solving the Riccati equations.

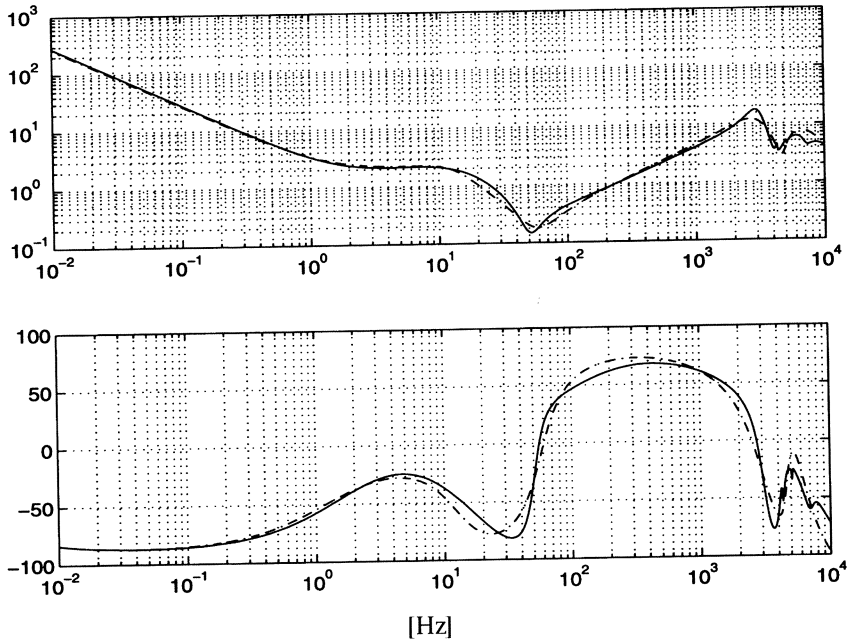

Fig. 4. Controller frequency response. -: before reduction; ---: after reduction.

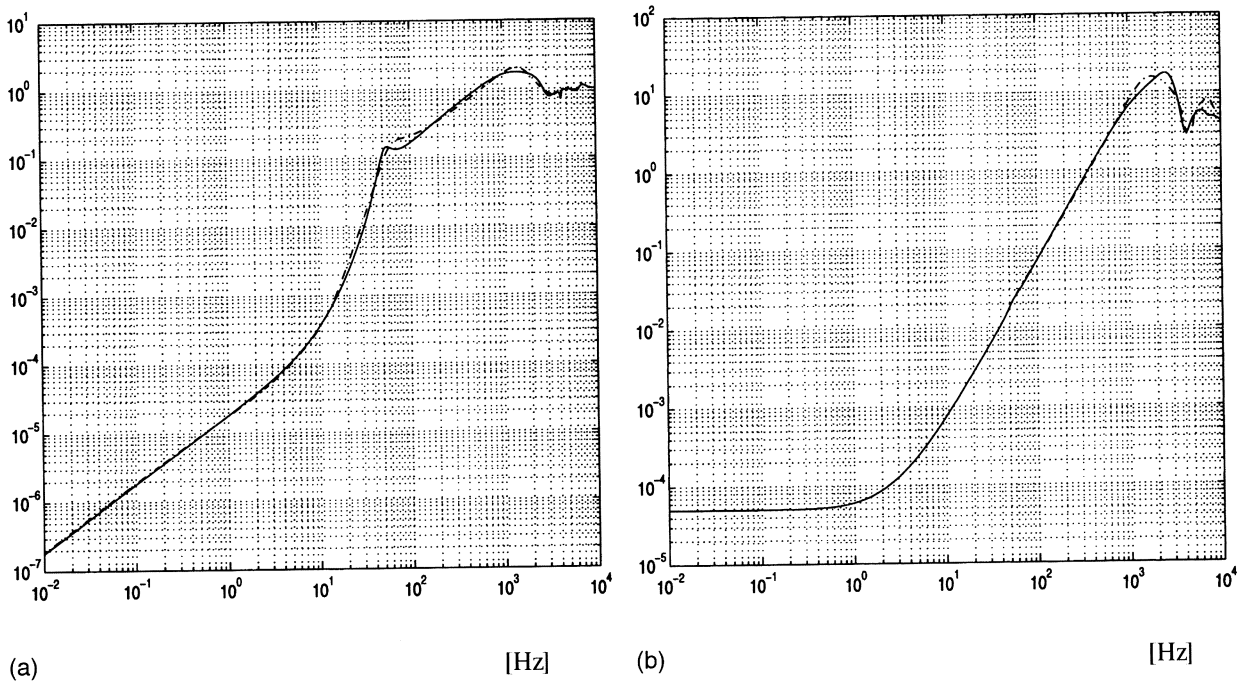

Fig. 5. Designed frequency domain performance. -: original controller; ---: reduced controller. (a) Sensitivity function. (b) Controller sensitivity function.

Figure 6 presents the measured magnitude frequency responses of the sensitivity functions and the complementary sensitivity functions of the two closed loop compact disc radial position control systems, while in Fig. 7 their measured time domain track-following errors are shown.

It is worth pointing out that it is preferable to measure the magnitudes of the sensitivity functions for all the frequencies not greater than 10^4 Hz. The measurement of the sensitivity functions below 100 Hz, however, is very difficult. This is due to the fact that there are many severe disturbances

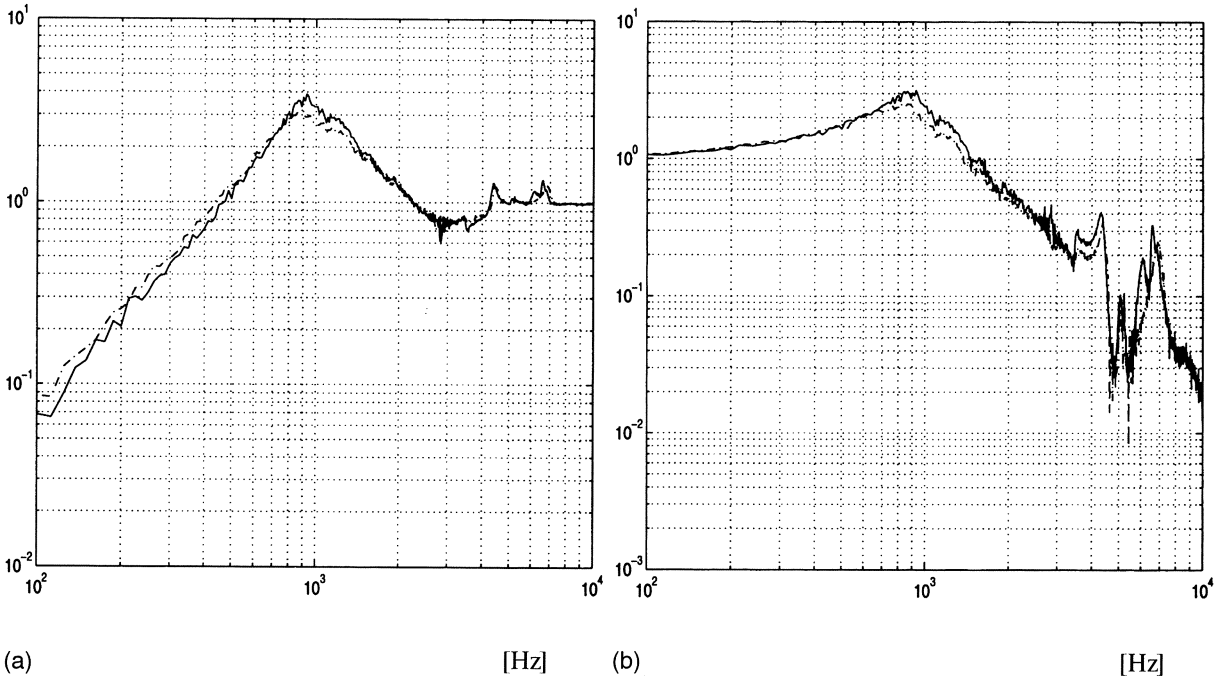

Fig. 6. Measured frequency domain performance. ---: CD player I; -: CD player II. (a) Sensitivity function. (b) Complementary sensitivity function.

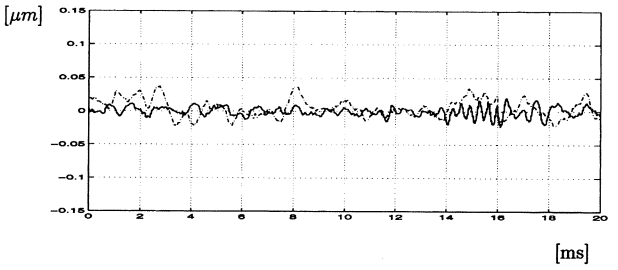

Fig. 7. Measured radial track following error. ---: CD player I; -: CD player II.

below 100 Hz in compact disc systems, and it is very hard to increase the signal–noise ratio in the measurement of this frequency range. These disturbances possibly result from the focus control loop, rotation control loop, etc.

From the measured closed loop systems' frequency domain and time domain performance, it is obvious that the bandwidth of every closed loop system is approximately 10^3 Hz, while the magnitudes of the sensitivity functions are smaller than 4 at every frequency. Especially, the sensitivity function magnitudes are smaller than 1 for all frequencies below 400 Hz, and have been reduced to be smaller than 0.1 at 100 Hz. Moreover, the radial track-following errors have been reduced to be smaller than 0.05 μm. Hence, the required performance on the radial position systems has been achieved by the designed controller.

There are some differences between the designed and measured sensitivity functions. This may be mainly due to the following two reasons. First, note that there exist differences between the nominal model $G_0(s)$ and the actual dynamics of the compact disc systems. Secondly, the influence of the time delay caused by the digital signal processor calculation is not negligible.[3]

On the other hand, from the time domain measurements, it is clear that there exist some periodic disturbances in the compact disc systems. These disturbances are mainly due to the deficiencies of the mechanical structure and the eccentric rotation. To reduce these periodic track-following errors, adaptive repetitive control method has been proved to be useful [6].

7. Concluding Remarks

In this paper, we have discussed the problem of model set determination when a batch of plant mod-

[3]At frequency 900 Hz, the 4×10^{-5} s sampling period results in a phase lag of approximately 13°. A controller has been designed and implemented which has almost the same magnitude frequency response around 900 Hz as that of controller $\bar{C}(s)$, but its phase lead around that frequency is approximately 10° greater than that of $\bar{C}(s)$. The measured sensitivity functions have magnitudes about 2.7 in the region of 900 Hz, which matches better the simulation results shown in Fig. 5(a).

els is provided. Previous results have been extended to the case in which the plant model is described by a homographic transformation. It has been proved that this model set determination problem can be reduced to a model-matching problem, and furthermore, to a frequency-weighted $\mathcal{L}_\infty$ norm model reduction problem. A necessary and sufficient condition has been obtained for a nominal model to be suboptimal. An algorithm is proposed to obtain a suboptimal nominal model which has a low Smith–McMillan degree, together with its error bound. This algorithm is based on frequency-weighted Hankel norm model reduction and convex optimisation. In addition, the problem of extracting structured information from nomimal model errors has also been investigated. A necessary condition is obtained for the existence of structured errors. Based on this condition, a computationally attractive algorithm is suggested.

The proposed model set determination algorithm has been applied to the simultaneous spiral control of two compact disc players. Acceptable performance has been obtained by the designed controller.

However, some important issues concerned with this model set determination problem still remain unsolved. One of them is about the selection of transfer function matrices $N_1(s)$ and $D_1(s)$. These transfer function matrices determine the model set structure and therefore play essential roles in both model set determination and robust controller design.

Another important issue is to extend the established results to the case in which a nominal model is perturbed by its error through a linear fractional transformation. Currently, linear fractional transformation is the most general model set description applied in robust control theories, and all of the other model set descriptions can be regarded as its special case [15].

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Appendix A: A Proof of Theorem 3

Proof. For brevity, define $J = \|w(s)[\hat{\Delta}_1^T(s) \ \ldots \ \hat{\Delta}_n^T(s)]^T\|_\infty$. Since $\delta_1^2 + \ldots + \delta_n^2 = 1$, we have

$$\nabla^T\nabla = I_m \tag{A.1}$$

in which, $\nabla = [\delta_1 I_m \ \delta_2 I_m \ \ldots \ \delta_n I_m]^T$. As a consequence, there exists an $nm \times (n-1)m$ matrix T, such that

$$[\nabla\ T]^T[\nabla\ T] = [\nabla\ T][\nabla\ T]^T = I_{nm} \tag{A.2}$$

On the other hand, $\hat{\Delta}_i(s) = \Delta_i(s) - \delta_i\Delta_0(s)$. Therefore,

$$\begin{aligned} J &= \left\| w(s)\left\{\begin{bmatrix}\Delta_1(s)\\ \vdots \\ \Delta_n(s)\end{bmatrix} - \begin{bmatrix}\delta_1 I_m\\ \vdots \\ \delta_n I_m\end{bmatrix}\Delta_0(s)\right\}\right\|_\infty \\ &= \|w(s)[\nabla\ T]^T[\Delta(s) - \nabla\Delta_o(s)]\|_\infty \\ &= \left\|\begin{bmatrix} w(s)[\nabla^T\Delta(s) - \Delta_0(s)] \\ w(s)T^T\Delta(s)\end{bmatrix}\right\|_\infty \end{aligned} \tag{A.3}$$

Then, according to Theorem 1 of Chapter 8 in Francis [9], J is smaller than a positive number, say, γ, if and only if

$$\|w(s)T^T\Delta(s)\|_\infty < \gamma \tag{A.4}$$

$$\|w(s)[\nabla^T\Delta(s) - \Delta_0(s)]\hat{R}^{-1}(s)\|_\infty < 1 \tag{A.5}$$

in which both $\hat{R}(s)$ and $\hat{R}^{-1}(s)$ belong to $\mathcal{H}_\infty$, and $\hat{R}^\sim(s)\hat{R}(s) = \gamma^2 I_p - (w(s)T^T\Delta(s))^\sim(w(s)T^T\Delta(s))$.

Now, assume that there exist $\delta_i \in \mathcal{R}$, $i = 1,\ldots,n$, and a $\Delta_0(s) \in \mathcal{H}_\infty$, such that $\Delta_i(s) = \delta_i\Delta_0(s) + \hat{\Delta}_i(s)$, $i = 1,\ldots,n$; $\delta_1^2 + \ldots + \delta_n^2 = 1$, and $J < \gamma < \|w(s)\Delta(s)\|_\infty$. Then, $\|w(s)T^T\Delta(s)\|_\infty < \gamma$, which implies that

$$w(j\omega_0)w^*(j\omega_0)T^T\ \Delta(j\omega_0)\Delta^*(j\omega_0)T < \gamma^2 I_{(n-1)m} \tag{A.6}$$

From the definition of matrix T, it is obvious that

$$T^T T = I_{(n-1)m} \tag{A.7}$$

Hence,

$$T^T\{\gamma^2 I_{nm} - [w(j\omega_0)\Delta(j\omega_0)]\times [w(j\omega_0)\Delta(j\omega_0)]^*\}T > 0 \tag{A.8}$$

Define matrix U as $U = [u_1^T \ \ldots \ u_{nm}^T]^T$. Then,

$$T^T\{\gamma^2 I_{nm} - [w(j\omega_0)\Delta(j\omega_0)][w(j\omega_o)\Delta(j\omega_0)]^*\}T = (UT)^*\begin{bmatrix} diag\{(\gamma^2 - \sigma_i^2)|_{i=1}^{p}\} & 0 \\ 0 & \gamma^2 I_{nm-p}\end{bmatrix} UT \tag{A.9}$$

Since $\gamma < \|w(s)\Delta(s)\|_\infty = \sigma_1$, to guarantee that inequality (A.8) is satisfied, it is necessary that

$$u_1 T = 0, \qquad or\ T^T u_1^T = 0 \tag{A.10}$$

From Eq. (A.2), it is obvious that

$$T^T\nabla = 0, \quad rank(T) = (n-1)m, \quad rank(\nabla) = m \tag{A.11}$$

Hence, there exist real numbers β_i and γ_i, $i = 1, \ldots, n$, such that

$$Re[u_1]^T = \nabla\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{bmatrix}, \tag{A.12a}$$

$$Im[u_1]^T = \nabla\begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_m\end{bmatrix} \tag{A.12b}$$

and there is at least one i, $1 \le i \le n$, such that $\beta_i + j\gamma_i \neq 0$.

We assume, without loss of generality, that $\beta_1 + j\gamma_1 \neq 0$. Define

$$\hat{\alpha}_i = (\beta_1 + j\gamma_1)\delta_i, \qquad i = 1,\ldots,n \tag{A.13}$$

$$k_i = \frac{\beta_{i+1} + j\gamma_{i+1}}{\beta_1 + j\gamma_1}, \qquad i = 1,\ldots,m-1 \tag{A.14}$$

Then, it is clear that

$$u_1 = Re[u_1] + jIm[u_1] = [\hat{\alpha}_1\ k_1\hat{\alpha}_1 \ \ldots\ k_{m-1}\hat{\alpha}_1\ \hat{\alpha}_2\ k_1\hat{\alpha}_2 \ldots k_{m-1}\hat{\alpha}_2\ \hat{\alpha}_3 \ldots k_{m-1}\hat{\alpha}_n] \tag{A.15}$$

Multiply vector u_1 by $\dfrac{\beta_1 - j\gamma_1}{|\beta_1 + j\gamma_1|}$ and define

$$\alpha_i = \frac{\beta_1 - j\gamma_1}{|\beta_1 + j\gamma_1|}\hat{\alpha}_i, \qquad i = 1,\ldots,n \tag{A.16}$$

Then, $\alpha_i \in \mathcal{R}$, $i = 1,\ldots,n$ and $\hat{u}_1 = \dfrac{\beta_1 - j\gamma_1}{|\beta_1 + j\gamma_1|}u_1$ has the required form. Moreover, it is obvious that

$$w(j\omega_0)\Delta(j\omega_0) = [\hat{u}_1^* \; u_2^* \; \cdot\cdot u_{nm}^*]X \begin{bmatrix} diag\{\sigma_i|_{i=1}^p\} \\ 0 \end{bmatrix} [v_1 \; v_2 \; \cdots \; v_p] \tag{A.17}$$

and matrix $[\hat{u}_1^* \; u_2^* \; \cdots \; u_{nm}^*]$ is unitary. This completes the proof. □

Appendix B: Models for the Compact Disc Systems

1. Model for compact disc player I:

$$G_1(s) = \frac{\begin{gathered} 1.03s^{22} - 1.11 \times 10^6 s^{21} + 1.76 \times 10^{12} s^{20} + \\ 1.74 \times 10^{16} s^{19} + 3.24 \times 10^{22} s^{18} + 5.44 \times 10^{26} s^{17} + 2.42 \times 10^{32} s^{16} + \\ 4.30 \times 10^{36} s^{15} + 9.59 \times 10^{41} s^{14} + 1.62 \times 10^{46} s^{13} + 2.22 \times 10^{51} s^{12} + \\ 3.34 \times 10^{55} s^{11} + 3.12 \times 10^{60} s^{10} + 4.02 \times 10^{64} s^9 + 2.69 \times 10^{69} s^8 + \\ 2.92 \times 10^{73} s^7 + 1.39 \times 10^{78} s^6 + 1.31 \times 10^{82} s^5 + 4.17 \times 10^{86} s^4 + \\ 3.53 \times 10^{90} s^3 + 6.43 \times 10^{94} s^2 + 4.17 \times 10^{98} s + 2.57 \times 10^{102} \end{gathered}}{\begin{gathered} s^{22} + 1.55 \times 10^7 s^{21} + 1.34 \times 10^{14} s^{20} + 2.40 \times 10^{19} s^{19} + 2.44 \times 10^{24} s^{18} + \\ 3.12 \times 10^{29} s^{17} + 1.78 \times 10^{34} s^{16} + 1.67 \times 10^{39} s^{15} + 6.81 \times 10^{43} s^{14} + \\ 4.76 \times 10^{48} s^{13} + 1.50 \times 10^{53} s^{12} + 7.77 \times 10^{57} s^{11} + 1.95 \times 10^{62} s^{10} + \\ 7.37 \times 10^{66} s^9 + 1.47 \times 10^{71} s^8 + 3.89 \times 10^{75} s^7 + 5.92 \times 10^{79} s^6 + \\ 1.01 \times 10^{84} s^5 + 1.05 \times 10^{88} s^4 + 8.85 \times 10^{91} s^3 + 3.88 \times 10^{95} s^2 + \\ 1.57 \times 10^{97} s + 1.18 \times 10^{98} \end{gathered}} \tag{B.1}$$

2. Model for compact disc player II:

$$G_2(s) = \frac{\begin{gathered} 2.49s^{26} + 7.24 \times 10^6 s^{25} - 1.58 \times 10^{11} s^{24} + \\ 1.45 \times 10^{17} s^{23} - 2.84 \times 10^{21} s^{22} + 1.25 \times 10^{27} s^{21} - 2.05 \times 10^{31} s^{20} + \\ 6.06 \times 10^{36} s^{19} - 8.66 \times 10^{40} s^{18} + 1.83 \times 10^{46} s^{17} - 2.49 \times 10^{50} s^{16} + \\ 3.63 \times 10^{55} s^{15} - 5.25 \times 10^{59} s^{14} + 4.90 \times 10^{64} s^{13} - 8.18 \times 10^{68} s^{12} + \\ 4.83 \times 10^{73} s^{11} - 8.90 \times 10^{77} s^{10} + 3.92 \times 10^{82} s^9 - 5.97 \times 10^{86} s^8 + \\ 2.79 \times 10^{91} s^7 - 1.79 \times 10^{95} s^6 + 1.52 \times 10^{100} s^5 + 3.20 \times 10^{103} s^4 + \\ 4.98 \times 10^{108} s^3 + 3.62 \times 10^{112} s^2 + 6.90 \times 10^{116} s + 6.74 \times 10^{120} \end{gathered}}{\begin{gathered} s^{26} + 2.31 \times 10^9 s^{25} + 1.02 \times 10^{14} s^{24} + 4.89 \times 10^{19} s^{23} + 1.92 \times 10^{24} s^{22} + \\ 4.50 \times 10^{29} s^{21} + 1.57 \times 10^{34} s^{20} + 2.37 \times 10^{39} s^{19} + 7.35 \times 10^{43} s^{18} + \\ 7.98 \times 10^{48} s^{17} + 2.17 \times 10^{53} s^{16} + 1.79 \times 10^{58} s^{15} + 4.24 \times 10^{62} s^{14} + \\ 2.72 \times 10^{67} s^{13} + 5.53 \times 10^{71} s^{12} + 2.81 \times 10^{76} s^{11} + 4.80 \times 10^{80} s^{10} + \\ 1.92 \times 10^{85} s^9 + 2.68 \times 10^{89} s^8 + 8.25 \times 10^{93} s^7 + 9.01 \times 10^{97} s^6 + \\ 2.01 \times 10^{102} s^5 + 1.59 \times 10^{106} s^4 + 2.09 \times 10^{110} s^3 + 1.04 \times 10^{114} s^2 + \\ 4.34 \times 10^{115} s + 3.24 \times 10^{116} \end{gathered}} \tag{B.2}$$

Appendix C: Frequency-Weighting Functions

1. Frequency-weighting function for model set determination:

$$w(s) = \frac{\begin{array}{c}89.8s^9 + 6.54 \times 10^7 s^8 + 1.19 \times 10^{12} s^7 + 2.53 \times 10^{17} s^6 + 2.49 \times 10^{21} s^5 + \\ 2.94 \times 10^{26} s^4 + 1.73 \times 10^{30} s^3 + 1.07 \times 10^{35} s^2 + 3.48 \times 10^{38} s + 1.05 \times 10^{42}\end{array}}{\begin{array}{c}s^9 + 1.60 \times 10^5 s^8 + 1.33 \times 10^{10} s^7 + 1.03 \times 10^{15} s^6 + 4.26 \times 10^{19} s^5 + \\ 1.82 \times 10^{24} s^4 + 4.71 \times 10^{28} s^3 + 1.16 \times 10^{33} s^2 + 1.66 \times 10^{37} s + 2.15 \times 10^{41}\end{array}} \tag{C.1}$$

2. Frequency-weighting function for controller sensitivity function:

$$w_{au}(s) = \frac{\begin{array}{c}43.5s^8 + 2.50 \times 10^7 s^7 + 3.38 \times 10^{12} s^6 + 1.31 \times 10^{17} s^5 + \\ 3.51 \times 10^{21} s^4 + 4.62 \times 10^{25} s^3 + 4.26 \times 10^{29} s^2 + 1.78 \times 10^{33} s + 5.20 \times 10^{35}\end{array}}{\begin{array}{c}s^8 + 3.01 \times 10^6 s^7 + 3.02 \times 10^{12} s^6 + 1.02 \times 10^{18} s^5 + 8.41 \times 10^{21} s^4 + \\ 6.95 \times 10^{26} s^3 + 8.85 \times 10^{29} s^2 + 2.95 \times 10^{31} s + 2.49 \times 10^{32}\end{array}} \tag{C.2}$$

3. Frequency-weighting function for sensitivity function:

$$w_s(s) = \frac{\begin{array}{c}0.667s^8 + 7.33 \times 10^3 s^7 + 4.01 \times 10^7 s^6 + 6.10 \times 10^{10} s^5 + \\ 4.31 \times 10^{13} s^4 + 1.81 \times 10^{16} s^3 + 5.25 \times 10^{18} s^2 + 1.02 \times 10^{21} s + 9.08 \times 10^{22}\end{array}}{\begin{array}{c}s^8 + 1.04 \times 10^4 s^7 + 1.19 \times 10^7 s^6 + 5.68 \times 10^9 s^5 + 1.39 \times 10^{12} s^4 + \\ 1.80 \times 10^{14} s^3 + 1.14 \times 10^{16} s^2 + 2.63 \times 10^{17} s + 1.69 \times 10^{15}\end{array}} \tag{C.3}$$

4. Frequency-weighting function for controller reduction:

$$w_c(s) = \frac{\begin{array}{c}58.6s^{15} + 3.46 \times 10^7 s^{14} + 5.28 \times 10^{12} s^{13} + 3.74 \times 10^{17} s^{12} + \\ 2.73 \times 10^{22} s^{11} + 1.12 \times 10^{27} s^{10} + 5.12 \times 10^{31} s^9 + 1.38 \times 10^{36} s^8 + \\ 4.34 \leqq 10^{40} s^7 + 7.30 \times 10^{44} s^6 + 1.64 \times 10^{49} s^5 + 1.37 \times 10^{53} s^4 + \\ 2.10 \times 10^{57} s^3 + 4.23 \times 10^{59} s^2 + 4.13 \times 10^{61} s + 1.04 \times 10^{61}\end{array}}{\begin{array}{c}s^{15} + 3.02 \times 10^6 s^{14} + 3.08 \times 10^{12} s^{13} + 1.08 \times 10^{18} s^{12} + 3.69 \times 10^{22} s^{11} + \\ 5.00 \times 10^{27} s^{10} + 1.05 \times 10^{32} s^9 + 7.93 \times 10^{36} s^8 + 1.17 \times 10^{41} s^7 + \\ 5.24 \times 10^{45} s^6 + 5.51 \times 10^{49} s^5 + 1.24 \times 10^{54} s^4 + 8.60 \times 10^{57} s^3 + \\ 1.05 \times 10^{60} s^2 + 8.65 \times 10^{62} s + 1.38 \times 10^{63}\end{array}} \tag{C.4}$$