A Unified Approach to Stability Robustness for Uncertainty Descriptions based on Fractional Model Representations

Raymond A. de Callafon[‡] Paul M.J. Van den Hof Peter M.M. Bongers[§]

Mechanical Engineering Systems and Control Group
Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands
Tel. +31-15-784703; Fax: +31-15-784717;
E-mail: callafon@tudw03.tudelft.nl

Abstract

The powerful standard representation for uncertainty descriptions in a basic perturbation model based on a standard plant representation can be used to attain necessary and sufficient conditions for stability robustness within various uncertainty descriptions. In this paper these results are employed to formulate necessary and sufficient conditions for stability robustness of several uncertainty sets based on unstructured additive coprime factor uncertainty, gap-metric uncertainty as well as the recently introduced Λ-gap uncertainty.

1 Introduction

In a model-based control design paradigm, the design is based on a (necessarily) approximative model $\hat{P}$ of a plant to be controlled. An apparently successful control design leads to a controller C, having some desired closed loop properties for the feedback controlled model $\hat{P}$, but due to the mismatch between the actual plant P_o and the model $\hat{P}$, a verification of these desired closed loop properties is preferred before implementing the controller C on the actual plant P_o. In this paper the discussion is directed towards the verification of one of the most important closed loop properties: stability.

To evaluate stability when the controller C is being applied to the plant P_o, a characterization of the mismatch between the plant P_o and the model $\hat{P}$ is indispensable. Since the real plant P_o is *unknown*, the discrepancy in general is characterized by a so called *uncertainty set*, denoted with $\mathcal{P}$. Typically an uncertainty set $\mathcal{P}$ is defined by the (nominal) model $\hat{P}$, which is found by physical modelling or identification techniques, and some bounded 'area' around it [4]. The uncertainty set $\mathcal{P}$ itself reflects all possible perturbations of the (nominal) model $\hat{P}$ that may occur.

By defining the uncertainty set in such a way that at least the plant $P_o \in \mathcal{P}$, stability robustness results for the set $\mathcal{P}$ will reflect sufficient conditions under which the plant P_o will be stabilized by C, see [4] or [5]. In this perspective, special attention will be given in this paper to an uncertainty set $\mathcal{P}_{CF}$ which is characterized by additive perturbations on a coprime factor description of the nominal model $\hat{P}$. The specific application of such an uncertainty set description will be motivated by the favourable properties it has over a standard additive or multiplicative uncertainty set description.

Using the simple and powerful stability robustness results for a basic perturbation model in a standard plant configuration, [4, 5, 16], several different uncertainty sets employing weighted and unstructured additive perturbations on a coprime factorization, gap-metic based uncertainty sets and the recently introduced Λ-gap uncertainty sets will be shown to be closely related to each other. The contribution of this paper is in the unified treatment of these different uncertainty sets. While stability robustness results for uncertainty sets using additive perturbations on normalized (left) coprime factorizations [11] and gap-metric based uncertainty sets [10] have separately been derived before, this paper amplifies their relation, as well as the extension to a less conservative Λ-gap uncertainty set description [1, 2].

The outline of this paper will be as follows. In Section 2 some preliminary notations and definitions will be given, while in Section 3 the basic stability robustness result using a perturbation model based on a standard plant description [4, 6, 16] will be summarized. This perturbation model gives rise to an unified approach to handle stability robustness for various uncertainty descriptions, including additive weighted perturbations on a coprime factorization. Section 4 contains the results of applying this unified approach to additive uncertainty descriptions on fractional model representations and favourable properties are illuminated. The link with gap and Λ-gap based stability robustness results is discussed in sections 5 and 6, the latter one being less conservative than the former one, as shown in section 7. The paper ends with some concluding remarks.

2 Preliminaries

Throughout this paper, the feedback configuration of a plant P and a controller C is denoted by $\mathcal{T}(P,C)$ and defined by the feedback connection structure depicted in Figure 1.

A plant P and a controller C are assumed to be given by real rational transfer function matrices, and it will be assumed that the feedback connection is well-posed, i.e. that $\det[I + CP] \not\equiv 0$. Then the feedback system $\mathcal{T}(P,C)$ is

[‡]The work of Raymond de Callafon is sponsored by the Dutch Systems and Control Theory Network.

[§]Now with Unilever Research, Olivier van Noortlaan 120, 3133 AT Vlaardingen, The Netherlands

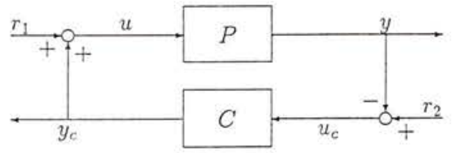

Fig. 1: Feedback connection structure $\mathcal{T}(P, C)$ of a plant P and a controller C

defined to be internally stable if the mapping from $col(r_2, r_1)$ to $col(u_c, u)$ is BIBO stable, i.e. if the corresponding transfer function is in $\mathbb{R}\mathcal{H}_\infty$, where $\mathbb{R}\mathcal{H}_\infty$ denotes the Hardy space of real rational transfer function matrices with bounded $\mathcal{H}_\infty$-norm [6]:

$$\|G\|_\infty := \sup_{\omega\in[0,\pi)} \bar{\sigma}\{G(e^{i\omega})\} \tag{1}$$

with $\bar{\sigma}$ the maximum singular value. Furthermore, the dynamics of the closed loop system $\mathcal{T}(P, C)$ will also be described by the mapping from $col(r_2, r_1)$ to $col(y, u)$ which is given by the transfer function matrix $T(P, C)$:

$$T(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} [I + CP]^{-1} \begin{bmatrix} C & I \end{bmatrix}. \tag{2}$$

Using the theory of fractional representations, as e.g. presented in [20], a plant P is expressed as a ratio of two stable transfer function matrices N and D. For two transfer functions $N, D \in \mathbb{R}\mathcal{H}_\infty$, the pair (N, D) is called right coprime over $\mathbb{R}\mathcal{H}_\infty$ if there exist $X, Y \in \mathbb{R}\mathcal{H}_\infty$ such that $XN + YD = I$. The pair (N, D) is a right coprime factorization (*rcf*) of P if (N, D) is right coprime and $P = ND^{-1}$. A right coprime factorization (N, D) is called normalized (*nrcf*) if it satisfies $N^*N + D^*D = I$, where * denotes complex conjugate transpose. For (normalized) left coprime factorizations (*lcf*) dual definitions exist. With respect to internal stability of the feedback system $\mathcal{T}(P, C)$ as mentioned above, the following lemma will be used.

Lemma 2.1 *Let P have a* rcf *(N, D) and let C have a* lcf *$(\tilde{D}_c, \tilde{N}_c)$. Then the following statements for internal stability of the feedback system $\mathcal{T}(P, C)$ are equivalent.*

i. The feedback system $\mathcal{T}(P, C)$ given in Figure 1 is internally stable.

ii. $T(P, C) \in \mathbb{R}\mathcal{H}_\infty$, with $T(P, C)$ defined in (2).

iii. $\Lambda^{-1} \in \mathbb{R}\mathcal{H}_\infty$, with $\Lambda := \tilde{D}_c D + \tilde{N}_c N$

Proof: $i \Leftrightarrow ii$: Follows directly from (2), and its direct relation with the transfer function from $col(r_2, r_1)$ to $col(u_c, u)$. $i \Leftrightarrow iii$: See [20], [2] or [17]. □

Fractional representations have a close relation with approximation in the graph topology. The graph topology is the weakest topology[1] in which a variation of the elements of a stable feedback configuration around their nominal values, preserves stability of that closed loop system [21]. The graph topology is known to be induced by several metrics, as e.g. the graph metric introduced in [19] or the gap metric introduced in [23], being expressed in the following way.

[1]Given two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$. $\mathcal{O}_1$ is said to be weaker than $\mathcal{O}_2$ if $\mathcal{O}_1$ is a subcollection of $\mathcal{O}_2$, see also [21]

Definition 2.2 [8] *Consider two plants P_1, P_2 with a* nrcf *(N_1, D_1), (N_2, D_2) respectively. Then the gap between P_1 and P_2 is expressed by*

$$\begin{aligned} \delta(P_1, P_2) &:= \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\} \text{ with} \\ \vec{\delta}(P_i, P_j) &:= \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} D_i \\ N_i \end{bmatrix} - \begin{bmatrix} D_j \\ N_j \end{bmatrix} Q \right\|_\infty \end{aligned}$$

3 Stability robustness in standard form

For analyzing the stability robustness of several uncertainty sets based on fractional model representations, standard results on stability robustness for a rather general interconnection structure as depicted in Figure 2(a) will be employed. Here the mismatch between $\hat{P}$ and P_o, an uncertainty on $\hat{P}$ or a perturbation of P_o has been isolated and represented in the Δ-block using the artificial signals d and z [4, 16].

The internal stability of the feedback system of Figure 2(a) is reflected by the map from $col(r_2, r_1)$ onto $col(u_c, u)$. For notational convenience, this map will be represented by the basic perturbation model given in Figure 2(b), to which standard stability robustness results will be applied. This basic perturbation model is denoted by the upper linear fractional transformation $\mathcal{F}(M, \Delta) := M_{22} + M_{21}\Delta[I - M_{11}\Delta]^{-1}M_{12}$, where the decomposition of $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is in accordance with Figure 2(b).

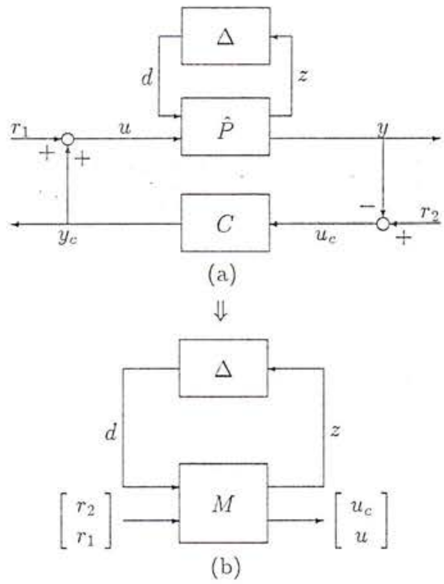

Fig. 2: (a) Feedback connection structure of a (perturbed) plant P_o and the controller C. (b) Basic perturbation model $\mathcal{F}(M, \Delta)$.

The elements of the transfer function M in Figure 2(b) can be expressed in terms of the model $\hat{P}$ and the controller C. If the controller C internally stabilizes $\hat{P}$, the transfer function M is BIBO stable and the small gain theorem can be applied to characterize stability results for the connection structure of Figure 2(b). This result is summarized in the following lemma.

Lemma 3.1 *Let the stable transfer functions $M, \Delta \in \mathbb{R}\mathcal{H}_\infty$ construct a feedback connection $\mathcal{F}(M, \Delta)$. Then* *(a) a sufficient condition for BIBO stability of $\mathcal{F}(M,\Delta)$ is given by*

$$\|M_{11}\Delta\|_\infty < 1 \tag{3}$$

(b) provided that for all Δ with $\|\Delta\|_\infty < \gamma$ the transfer function $M_{21}\Delta[I - M_{11}\Delta]^{-1}M_{12}$ does not exhibit unstable pole/zero cancellations[2], $\mathcal{F}(M,\Delta)$ is BIBO stable for all Δ with $\|\Delta\|_\infty < \gamma$ if and only if

$$\|M_{11}\|_\infty \leq \gamma^{-1} \tag{4}$$

Proof: Since $M \in \mathbb{R}\mathcal{H}_\infty$, and thus M_{11}, M_{12}, M_{21}, $M_{22} \in \mathbb{R}\mathcal{H}_\infty$, the small gain theorem [22] directly leads to result (a). Additionally also necessary conditions can be formulated on the stability of $[I - M_{11}\Delta]^{-1}$ for all Δ with $\|\Delta\|_\infty < \gamma$. Provided that unstable poles of $[I - M_{11}\Delta]^{-1}$ are not cancelled in M, this leads to the necessary condition of (4). For a complete proof see [11], [13] or [20]. □

4 Stability robustness for uncertainty descriptions based on fractional model representations

The result of Section 3 on stability robustness can be applied to various H_∞-norm bounded uncertainty sets by rewriting the uncertainty description into the basic perturbation model $\mathcal{F}(M,\Delta)$. In this section this is done for uncertainty sets based on coprime factor uncertainties.

A crucial aspect in the result of Lemma 3.1 is the assumption that $\Delta \in \mathbb{R}\mathcal{H}_\infty$. In case of an additive or multiplicative uncertainty set in the basic perturbation model, this assumption implies the condition that the locations of all unstable poles of the plant P_o are assumed to be fixed. Additive perturbations on coprime factorizations are more flexible and allow changes in both the number and the locations of poles and zeros anywhere in $\mathbb{C}$ [3]. Moreover, fractional representations have a close relation with approximation in the graph topology. Firstly, an uncertainty set based on additive perturbations on a coprime factorization will be discussed.

Corollary 4.1 *Consider a plant $\hat{P}$ with* rcf *$(\hat{N},\hat{D})$, stabilized by a given controller C, and consider the uncertainty set*

$$\mathcal{P}_{CF}(\hat{N},\hat{D},V_D,V_N,W,\gamma) = \{P \mid P = [\hat{N}+\Delta_N][\hat{D}+\Delta_D]^{-1} \text{ with } \left\| \begin{bmatrix} V_D & 0 \\ 0 & V_N \end{bmatrix} \begin{bmatrix} \Delta_D \\ \Delta_N \end{bmatrix} W \right\|_\infty < \gamma\}.$$

for stable and stably invertible filters V_D, V_N, W. Then the feedback system $\mathcal{T}(P,C)$ is internally stable for all $P \in \mathcal{P}_{CF}$ if and only if

$$\left\| W^{-1}[\hat{D}+C\hat{N}]^{-1} \begin{bmatrix} I & C \end{bmatrix} \begin{bmatrix} V_D^{-1} & 0 \\ 0 & V_N^{-1} \end{bmatrix} \right\|_\infty \leq \gamma^{-1}$$

Proof: Defining

$$\Delta := \begin{bmatrix} V_D & 0 \\ 0 & V_N \end{bmatrix} \begin{bmatrix} \Delta_D \\ \Delta_N \end{bmatrix} W, \text{ such that } \|\Delta\|_\infty < \gamma, \tag{5}$$

[2] This additional condition which is often discarded in literature, excludes trivial situations as e.g. $M_{21} = 0$ or $M_{12} = 0$. It can be shown to be satisfied for the common uncertainty classes based on additive, multiplicative or coprime factor uncertainty.

where V_D, V_N and W are stable, the basic perturbation structure of the uncertainty set $\mathcal{P}_{CF}$ can be written into a form that corresponds to Figure 2(b). A sufficient condition for the transfer function M to be stable is that C internally stabilizes $\hat{P}$ and V_D, V_N and W are also stably invertible. The map M_{11} is given by

$$M_{11} = -W^{-1}[\hat{D}+C\hat{N}]^{-1} \begin{bmatrix} I & C \end{bmatrix} \begin{bmatrix} V_D^{-1} & 0 \\ 0 & V_N^{-1} \end{bmatrix} \tag{6}$$

which proves the result by application of Lemma 3.1. □

Corollary 4.1 can alternatively be proven by employing stability results directly in terms of coprime factor representations of plant and controller. Here we have stressed the fact that the considered uncertainty set allows a description in terms of a standard perturbation model as depicted in Figure 2.

In the next section it will be shown how these results can be exploited to derive stability robustness conditions for gap-metric uncertainty sets as well as for uncertainty sets based on further generalizations of the gap-metric. To this end, the result on the equivalence between several formulations of the same uncertainty sets will be presented first.

Proposition 4.2 *The set $\mathcal{P}_{CF}(\hat{N},\hat{D},V_D,V_N,W,\gamma)$ as defined in Corollary 4.1 can alternatively be written in the following equivalent forms:*

(a) $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma) =$

$$\{P \mid P = (\hat{N}W + V_N^{-1}\Delta_N)(\hat{D}W + V_D^{-1}\Delta_D)^{-1} \text{ with } \left\| \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \right\|_\infty < \gamma\} \tag{7}$$

(b) $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma) =$

$$\{P \mid P = N_n D_n^{-1},\ (N_n,D_n) \text{ a nrcf and } \exists\, Q \in \mathbb{R}\mathcal{H}_\infty \text{ such that } \left\| \begin{bmatrix} V_N & 0 \\ 0 & V_D \end{bmatrix} \left[\begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} W - \begin{bmatrix} N_n \\ D_n \end{bmatrix} Q \right] \right\|_\infty < \gamma\}. \tag{8}$$

Proof: Part (a) follows by simple calculation. The proof of part (b) is more involved and is based on the fact that any right, but not necessarily coprime, fractional representation (N,D) can be written as a right fractional representation (N_nQ, D_nQ) with $Q \in \mathbb{R}\mathcal{H}_\infty$ and (N_n,D_n) a *nrcf*. In this way the right, but not necessarily coprime, fractional representation of P in (7) can be written as $(\hat{N}W + V_N^{-1}\Delta_N) = N_nQ$ and $(\hat{D}W + V_D^{-1}\Delta_D) = D_nQ$ with (N_n,D_n) a *nrcf* and $Q \in \mathbb{R}\mathcal{H}_\infty$. It follows then that $\Delta_N = V_N[N_nQ - \hat{N}W]$ and $\Delta_D = V_D[D_nQ - \hat{D}W]$ which proves the result. Note that the factor Q cancels in the representation of P. □

5 Stability robustness based on distance measures

In this section stability robustness results for gap-metric uncertainty sets are formulated. The main result of this section is not new, but already proven separately in [10]. The close relation of the stability robustness result here with the formulation in the previous section concerning general coprime factor uncertainty sets will be illuminated. This relation will be employed in the next section to formulate similar results for uncertainty sets based on the so-called Λ-gap, as recently introduced in [1] and [2].

The following uncertainty sets are being considered

$$\begin{aligned}\mathcal{P}_{\vec{\delta}}(\hat{P},\gamma) &:= \{P \mid \vec{\delta}(\hat{P},P) < \gamma\}\\ \mathcal{P}_{\delta}(\hat{P},\gamma) &:= \{P \mid \delta(\hat{P},P) < \gamma\},\end{aligned}$$

for which the following relation with the coprime factor uncertainty sets can be shown, as presented before.

Lemma 5.1 *Let a plant $\hat{P}$ and a controller C constitute an internally stable feedback system $\mathcal{T}(\hat{P},C)$. Consider the uncertainty set $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma)$ under the additional conditions that $(\hat{N},\hat{D})$ is a* nrcf *of $\hat{P}$, and $V_D = I$, $V_N = I$, and $W = I$. Then*

(a) $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma) = \mathcal{P}_{\vec{\delta}}(\hat{P},\gamma)$

(b) *For $\gamma < 1$, $\mathcal{P}_{\vec{\delta}}(\hat{P},\gamma) = \mathcal{P}_{\delta}(\hat{P},\gamma)$.*

Proof: *Part (a).* According to Proposition 4.2(b) and taking into account the specific choice of weighting functions in the lemma, it follows that

$$\begin{gathered}\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma) = \\ \{P \mid P = N_n D_n^{-1},\ (N_n,D_n) \text{ a } nrcf \text{ and } \exists\, Q \in \mathbb{R}\mathcal{H}_\infty \\ \text{such that } \left\| \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} - \begin{bmatrix} N_n \\ D_n \end{bmatrix} Q \right\|_\infty < \gamma\}.\end{gathered}$$

Since $(\hat{N},\hat{D})$ is chosen to be a *nrcf* of $\hat{P}$ it is straightforward to verify that $\mathcal{P}_{CF} = \mathcal{P}_{\vec{\delta}}$.

Part (b). This is proven in [10]. The restriction to $\gamma < 1$ is caused by the fact that these sets with $\gamma \geq 1$ can not be stabilized by a single controller. □

Lemma 5.1 relates the set defined by a gap metric bound with the set of coprime factor perturbations by a special choice of the weighting functions V_D, V_N, W and the coprime factorization $(\hat{N},\hat{D})$ of the model $\hat{P}$. This gives rise to an unified approach to handle sets of plants that are bounded by a gap metric, and the stability robustness result for these sets follows now directly from Corollary 4.1.

Corollary 5.2 *Consider the situation of Lemma 5.1 with $\gamma < 1$. Then for each of the three sets of plants $\mathcal{P}_{CF}$, $\mathcal{P}_{\vec{\delta}}$ and $\mathcal{P}_{\delta}$, $\mathcal{T}(P,C)$ is internally stable for all $P \in \mathcal{P}$ if and only if*

$$\|T(\hat{P},C)\|_\infty \leq \gamma^{-1}. \tag{9}$$

Proof: The proof follows simply by substituting the specific weightings in the result of Corollary 4.1, employing the fact that premultiplication of the expression within the norm by $[\hat{N}^T\ \hat{D}^T]^T$ leaves the norm invariant, due to the normalization of the *rcf*. □

Note that the result of Corollary 5.2 is not new. It was shown already in [10], where a complete proof of the stability robustness result is given. Similar results on the interrelation between uncertainty sets based on distance measures and based on additive perturbations on coprime factor descriptions can also be found in [18]. It has been shown here that the stability robustness results in the standard form can simply be exploited, as formulated in section Section 3. Restricting attention to the situation that $\gamma < 1$ is natural, as $\|T(\hat{P},C)\|_\infty \geq \|[I + C\hat{P}]^{-1}\|_\infty > 1$, according to Bode's sensitivity integral, showing that stability robustness can only be achieved for sets with $\gamma < 1$.

Finally it should be noted that the gap and graph metric are induced by the same topology and are uniformly equivalent [8]. Therefore stability robustness in the graph metric yields a similar result as mentioned in Corollary 5.2 and their interrelation is discussed in [14].

6 Stability robustness in the Λ-gap

The results obtained in the previous section for gap-based stability robustness can be further extended for uncertainty sets based on the recently introduced Λ-gap, [1, 2]. This Λ-gap is a distance measure that adds an additional frequency weighting in the expression that is utilized in the gap-metric, while the frequency weighting is controller-dependent.

Definition 6.1 *Let two plants P_1, P_2 have* nrcf*'s (N_1,D_1), (N_2,D_2) respectively. Let C be a controller with* lrcf *$(\tilde{D}_c,\tilde{N}_c)$ such that $\mathcal{T}(P_1,C)$ is internally stable. Then the Λ-gap between the plants P_1, P_2 is defined to be expressed by*

$$\vec{\delta}_\Lambda(P_1,P_2) = \inf_{\bar{Q}\in\mathbb{R}\mathcal{H}_\infty} \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} \Lambda^{-1} - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} \bar{Q} \right\|_\infty$$

with $\Lambda = [\tilde{D}_c D_1 + \tilde{N}_c N_1]$.

The difference between $\vec{\delta}(P_1,P_2)$ and $\vec{\delta}_\Lambda(P_1,P_2)$ is the additional shaping of the *nrcf* of P_1 with Λ^{-1} into a *rcf* $(\bar{N},\bar{D})$. In this way $\bar{\Lambda} := \tilde{D}_c\bar{D} + \tilde{N}_c\bar{N} = I$, with $\bar{N} = N_1\Lambda^{-1}$, $\bar{D} = D_1\Lambda^{-1}$, which is used to consider the closed loop operation of P_1 induced by the controller C being employed. This makes the distance between P_1 and P_2 dependent on the *nrcf* of the controller C. Note that the distance measure $\vec{\delta}_\Lambda(P_1,P_2)$ is not a metric since $\vec{\delta}_\Lambda(P_1,P_2) \neq \vec{\delta}_\Lambda(P_2,P_1)$ due to the influence of the controller C [2].

Accordingly, an uncertainty set based on Λ-gap uncertainty can be defined as:

$$\mathcal{P}_{\vec{\delta}_\Lambda}(\hat{P},\gamma) := \{P \mid \vec{\delta}_\Lambda(\hat{P},P) < \gamma\}.$$

This uncertainty set can also be shown to be equivalent to an uncertainty set of coprime factor uncertainties, provided appropriate weighting functions are chosen.

Lemma 6.2 *Let a plant $\hat{P}$ and a controller C with a* lrcf *$(\tilde{D}_c,\tilde{N}_c)$ constitute an internally stable feedback system $\mathcal{T}(\hat{P},C)$. Consider the set $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma)$ under the additional conditions that $(\hat{N},\hat{D})$ is a* nrcf *of $\hat{P}$, and $V_D = I$, $V_N = I$, and $W = \Lambda^{-1}$ with $\Lambda = [\tilde{D}_c\hat{D} + \tilde{N}_c\hat{N}]$. Then*

(a) $\mathcal{P}_{CF}(\hat{N},\hat{D},V_N,V_D,W,\gamma) = \mathcal{P}_{\vec{\delta}_\Lambda}(\hat{P},\gamma)$;

(b) $\mathcal{T}(P,C)$ is internally stable for all $P \in \mathcal{P}_{CF}$ if and only if $\gamma \leq 1$.

Proof: The proof of (a) is straightforward, along the same lines as the proof of Lemma 5.1(a). Result (b) then follows directly from Corollary 4.1, employing the fact that $\Lambda[\hat{D} + C\hat{N}]^{-1}[I\ C] = [\tilde{D}_c\ \tilde{N}_c]$ having an ∞-norm of 1 due to the fact that it is a normalized left coprime factorization. □

As said before, in case of the Λ-gap, the uncertainty set defined accordingly considers perturbations of the nominal plant $\hat{P}$ that are controller dependent.

The introduction of weightings in the gap metric has also been studied in [7], [9] or [15]. In [7] a multiplicative uncertainty description on the *nrcf* $(\hat{N}, \hat{D})$ of the model $\hat{P}$ is being used, leading to an uncertainty structure Δ having a diagonal form. Due to the diagonal form only necessary and sufficient conditions based on the structured singular value $\mu\{\cdot\}$ can be obtained. The weightings in the weighted gap of [9] have to be defined *a posteriori* which makes the choice of the weighting functions, to access robustness issues on the basis of a weighted gap, not a trivial task. Information on the size of the coprime factor perturbations can be used in the weighted pointwise gap metric defined in [15], but still an efficient computational method for pointwise gap metric is not available yet. The Λ-gap can simply be calculated. Controller synthesis in the Λ-gap however is more complicated and is a problem that is not solved yet.

7 Conservatism issues

All stability robustness results in this paper reflect necessary and sufficient conditions of an uncertainty set to be stabilized by a single controller. As such no conservatism is introduced in the test for checking stability robustness itself. However, for a single given controller, different of such uncertainty sets contain a different portion of the set of all systems that is stabilized by the controller. In this perspective the concept of conservatism is an intrinsic property of the uncertainty set being used. As a result an uncertainty set will be called more conservative than another if one controller stabilizes both sets, while the former set is contained in the latter.

Theorem 7.1 ([1]) *Consider a plant $\hat{P}$ and a stabilizing controller C with* lrcf $(\tilde{D}_c, \tilde{N}_c)$. *Consider the following two uncertainty sets resulting from the stability robustness results in the previous sections:*

$$\begin{aligned} \mathcal{S}_\delta(\hat{P}, C) &:= \{\cup\, \mathcal{P}_\delta(\hat{P}, b),\ b < \|T(\hat{P}, C)\|_\infty^{-1}\} \\ \mathcal{S}_{\bar{\delta}_\Lambda}(\hat{P}, C) &:= \{\cup\, \mathcal{P}_{\bar{\delta}_\Lambda}(\hat{P}, c),\ c < 1\} \end{aligned}$$

then

$$\mathcal{S}_\delta(\hat{P}, C) \subset \mathcal{S}_{\bar{\delta}_\Lambda}(\hat{P}, C). \tag{10}$$

Proof: The following implication will be proven:

$$\bar{P} \in \mathcal{S}_\delta(\hat{P}, C) \Rightarrow \bar{P} \in \mathcal{S}_{\bar{\delta}_\Lambda}(\hat{P}, C). \tag{11}$$

As $\bar{P} \in \mathcal{S}_\delta(\hat{P}, C)$ there exists a $\bar{U} \in \mathbb{RH}_\infty$ such that

$$\left\| \begin{bmatrix} \hat{D}_n \\ \hat{N}_n \end{bmatrix} - \begin{bmatrix} \bar{D} \\ \bar{N} \end{bmatrix} \bar{U} \right\|_\infty \leq \frac{1}{\|T(\hat{P}, C)\|_\infty}. \tag{12}$$

This implies that

$$\left\| \begin{bmatrix} \hat{D}_n \\ \hat{N}_n \end{bmatrix} - \begin{bmatrix} \bar{D} \\ \bar{N} \end{bmatrix} \bar{U} \right\|_\infty \cdot \|\Lambda^{-1}\|_\infty \leq \frac{\|\Lambda^{-1}\|_\infty}{\|T(\hat{P}, C)\|_\infty}. \tag{13}$$

As $\|T(\hat{P}, C)\|_\infty = \|\Lambda^{-1}\|_\infty$, this implies that

$$\left\| \begin{bmatrix} \hat{D}_n \\ \hat{N}_n \end{bmatrix} - \begin{bmatrix} \bar{D} \\ \bar{N} \end{bmatrix} \bar{U} \right\|_\infty \cdot \|\Lambda^{-1}\|_\infty \leq 1. \tag{14}$$

Lower bounding the left hand term of this expression implies that

$$\left\| \begin{bmatrix} \hat{D}_n \\ \hat{N}_n \end{bmatrix} \Lambda^{-1} - \begin{bmatrix} \bar{D} \\ \bar{N} \end{bmatrix} \bar{U} \Lambda^{-1} \right\|_\infty \leq 1 \tag{15}$$

which proves the result. □

The gap-metric uncertainty set can exhibit severe conservatism, as very well illustrated in e.g. [12]. As the gap-metric does not take into account the closed loop operation of the plant P in the set, induced by the controller C being used, this conservatism can intuitively be understood. The above result shows that in almost all cases the Λ-gap uncertainty set is less conservative; the proof of the above Theorem shows that the two sets in (10) are equal only in the situation that $\Lambda = \alpha V$, with $\alpha \in \mathbb{R}$ and V a unitary matrix. In all other cases, the Λ-gap set is strictly less conservative. The controller-relevant weighting within the Λ-gap is the basic reason for this.

Conclusions

The powerful standard representation for uncertainty descriptions in a basic perturbation model based on a standard plant configuration can be used to attain necessary and sufficient conditions for stability robustness within various uncertainty descriptions. In this paper these results are applied to uncertainty descriptions based on fractional model representations, leading to necessary and sufficient conditions for stability robustness in case of additive coprime factor uncertainties.

In this way a unified approach to handle additive coprime factor perturbations can be derived which yields a manageable and comprehensive way to relate gap and Λ-gap based uncertainty sets to (weighted) additive coprime factor perturbations. Based on this framework necessary and sufficient conditions for gap and Λ-gap based uncertainty sets are presented and it is shown that in terms of stability robustness, the Λ-gap uncertainty set is less conservative than the gap uncertainty set.

Acknowledgement

The authors would like to thank Carsten Scherer and Ruud Schrama for fruitful discussions that contributed to the results of this paper.

References

[1] P.M.M. Bongers. On a new robust stability margin. *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing, Proc. of the Int. Symposium MTNS-91*, pp. 377–382, 1991.

[2] P.M.M. Bongers. *Modeling and Identification of Flexible Wind Turbines and a Factorizational Approach to Robust Control.* PhD thesis, Delft Univ. of Techn., Mech. Eng. Systems and Control Group, 1994.

[3] C.T. Chen and C.A. Desoer. Algebraic theory for robust stability of interconnected systems: necessary and sufficient conditions. In *Proc. 21th IEEE Conf. on Decision and Control*, pp. 491–494, 1982.

[4] J.C. Doyle, B.A. Francis, and A.R. Tannenbaum. *Feedback Control Theory.* MacMillan Publishing Company, NY, USA, 1992.

[5] J.C. Doyle and G. Stein. Multivariable feedback design: concepts for a classical/modern synthesis. *IEEE Trans. on Autom. Control, AC-26*, pp. 4–16, 1981.

[6] B.A. Francis. *A Course in H_∞ Control Theory*. Lect. Notes Contr. and Inform. Sc., Vol. 88, Springer Verlag, Berlin, 1987.

[7] E.J.M. Geddes and I. Postlethwaite. The weighted gap metric and structured uncertainty. In *Proc. Amer. Control Conf.*, pp. 1138–1142, 1992.

[8] T.T. Georgiou. On the computation of the gap metric. *Syst. & Control Lett., 11*, pp. 253–257, 1988.

[9] T.T. Georgiou and M.C. Smith. Robust control of feedback systems with combined plant and controller uncertainty. In *Proc. Amer. Control Conf.*, pp. 2009–2013, 1990.

[10] T.T. Georgiou and M.C. Smith. Optimal robustness in the gap metric. *IEEE Trans. on Autom. Control, AC-35*, pp. 673–686, 1990.

[11] K. Glover and D. McFarlane. Robust stabilization of normalized coprime factor plant descriptions with H_∞-bounded uncertainty. *IEEE Trans. on Autom. Control, AC-34*, pp. 821–830, 1989.

[12] G.C. Hsieh and M.G. Safonov. Conservatism of the gap metric. *IEEE Trans. on Autom. Control, AC-38*, pp. 594–598, 1993.

[13] J.M. Maciejowski. *Multivariable Feedback Design*. Addison–Wesley Publishing Company, Wokingham, UK, 1989.

[14] A. Packard and M. Helwig. Relating the gap and graph metrics via the triangle inequality. *IEEE Trans. on Autom. Control, AC-34*, pp. 1296–1297, 1989.

[15] L. Qui and E.J. Davidson. Pointwise gap metrics on transfer matrices. *IEEE Trans. on Autom. Control, AC-37*, pp. 741–758, 1992.

[16] M.G. Safonov and M. Athans. A multiloop generalization of the circle criterion for stability margin analysis. *IEEE Trans. on Autom. Control, AC-26*, pp. 415–422, 1981.

[17] R.J.P. Schrama. *Approximate Identification and Control Design with Application to a Mechanical System*. PhD thesis, Delft Univ. of Techn., Mech. Eng. Systems and Control Group, 1992.

[18] J.A. Sefton and R.J. Ober. On the gap metric and coprime factor perturbations. *Automatica Vol. 29, No. 3*, pp. 723-734, 1993.

[19] M. Vidyasagar. The graph metric for unstable plants and robustness estimates for feedback stability. *IEEE Trans. on Autom. Control, AC-29*, pp. 403–418, 1984.

[20] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. Cambridge, MIT Press, 1985.

[21] M. Vidyasagar, H. Schneider, and B.A. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. on Autom. Control*, pp. 880–894, 1982.

[22] G. Zames. Functional analysis applied to nonlinear feedback systems. *IEEE Trans. Circuit Theory*, Vol. *CT-10*, pp. 392–404, 1963.

[23] G. Zames and A.K. El-Sakkary. Unstable systems and feedback: The gap metric. In *Proc. Allerton Conf.*, pp. 380–385, 1980.