# A general transform theory of rational orthonormal basis function expansions*

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**Abstract**

In this paper a general transform theory is presented that underlies expansions of stable discrete-time transfer functions in terms of rational orthonormal bases. The types of bases considered are generated by cascade connections of stable all-pass functions. If the all-pass sections in such a network are all equal, this gives rise to the Hambo basis construction. In this paper a more general construction is studied in which the all-pass functions are allowed to be different, in terms of choice and number of poles that are incorporated in the all-pass functions. It is shown that many of the interesting properties of the so-called Hambo transform that underlies the Hambo basis expansion carry over to the general case. Especially the recently developed expressions for the computation of the Hambo transform on the basis of state-space expressions can be extended to the general basis case. This insight can for instance be applied for the derivation of a recursive algorithm for the computation of the expansion coefficients, which are then obtained as the impulse response coefficients of a linear time-varying system.

## 1 Introduction

Rational orthonormal basis functions have since long been applied for the purpose of system approximation and identification. The earliest basis constructions date back to the work of Takenaka and Malmquist in the twenties [14]. In the thirties Lee and Wiener studied the application of the well-known Laguerre functions in the context of network theory [8]. A few decades later Kautz proposed a class of basis functions that have come to be known as the Kautz functions [7]. In more recent years there has been a resurgence of the interest in the application of the Laguerre and Kautz functions, mainly in the field of system identification [12, 13].

In a parallel development several generalizations of the classical Laguerre and Kautz constructions have been proposed [6, 9]. The common principle that connects all these constructions is that they are are generated by a series connection of stable all-pass transfer functions. In the construction proposed by Heuberger, known as the Hambo basis [6, 11], the sections in the cascade are all equal to the same all-pass transfer function $G_b(z)$ which can have arbitrary (finite) McMillan degree. The poles of $G_b(z)$ determine the characteristics of the basis expansion. In the construction that was proposed by Ninness and Gustafsson [9] the sections are taken to be all-pass functions of McMillan degree 1, but the poles of the all-pass functions are allowed to be different.

In this paper a construction is considered that encompasses these bases, in the sense that the all-pass functions in the cascade are allowed to be different and the all-pass functions are allowed to have different McMillan degree. The only restriction that is made is that the all-pass functions are stable. This type of basis construction was first suggested by Roberts and Mullis [10]. In section 2 the particularities of this basis construction will be reviewed.

For the Hambo basis a general transform theory was developed that underlies the expansion of signals and systems in terms of these basis functions [5]. This theory has been applied successfully in the context of system identification for the calculation of asymptotic bias and variance expressions. More recently it has been applied for the solution of the minimal partial realization problem for expansions in terms of Hambo basis functions [3].

The main subject of this paper is the generalization of the Hambo transform theory to the general basis construction. It turns out that many of the interesting properties of the Hambo transform theory carry over to the general case. The main difference is that, while the Hambo transform of a linear time-invariant system is again a linear time-invariant system, the generalized Hambo transform of this system is a linear *time-varying* system.

The outline of this paper is as follows. In section 2 the general basis construction that is studied in this paper is explained. In section 3 definitions are given of the signal and operator transforms that underlie these expansions. In section 4 it is shown how the generalized operator transform of a system $G(z)$ is obtained by applying a variable substitution in the Laurent expansion of $G(z)$. In section 5 expressions

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are derived by which, on the basis of a minimal state-space realization of $G(z)$, a corresponding minimal state-space realization of its operator transform can be computed. It is also shown how this last result can be applied for the calculation of the expansion coefficients of the system $G(z)$.

### Notation

- $H_2$ The space of scalar, discrete-time, stable *proper* transfer functions.
- $H_{2-}$ The space of scalar, discrete-time, stable, *strictly-proper* transfer functions.
- $\ell_2^n(J)$ The space of square summable vector sequences of dimension $n$, with $J$ denoting the index set.
- $[\![X,Y]\!]$ $\sum_{k=1}^{\infty} X(k)Y^T(k) = \frac{1}{2\pi i}\oint X(z)Y^*(1/z)\frac{dz}{z}$, with $X \in \ell_2^{n_x}$, $Y \in \ell_2^{n_y}$.
- $\prod_{j=l}^{k} X_j$ $X_k X_{k-1} \cdots X_{l+1} X_l$. For $l > k$ it is equal to identity.
- $\mathbf{P}_X$ Orthogonal projection onto the space $X$.

## 2 Basis construction

The starting point of the general orthonormal basis construction considered in this paper is the selection of a sequence of stable all-pass functions $\{G_{b,k}(z)\}_{k\in\mathbb{N}}$. Given a set of $n_{b,k}$ stable poles (with $n_{b,k}$ some finite number) an all-pass function $G_{b,k}$ is defined as

$$G_{b,k}(z) = \prod_{j=1}^{n_{b,k}} \frac{1-\xi_{k,j}^* z}{z-\xi_{k,j}}.$$

For the construction of the basis, balanced realizations $(A_{b,k}, B_{b,k}, C_{b,k}, D_{b,k})$ of the all-pass functions $G_{b,k}(z)$ are used. Defining the input-to-state transfer functions $\phi_k(z) = (zI - A_{b,k})^{-1} B_{b,k}$, we construct the vector valued functions $V_k(z)$:

$$V_k(z) = \phi_k(z) \prod_{j=1}^{k-1} G_{b,j}(z). \qquad (1)$$

**Proposition 1** *The functions $V_k(z)$ with $k \in \mathbb{N}$ constitute an orthonormal basis of $H_{2-}$ in the sense that $[\![V_i(z), V_j(z)]\!] = 0$ if $i \neq j$ and $I_{n_i}$ if $i = j$, and any system $G(z) \in H_{2-}$ can be uniquely represented by a sequence of coefficient vectors $\{L_k\}_{k\in\mathbb{Z}}$ as in*

$$G(z) = \sum_{k=1}^{\infty} L_k^T V_k(z).$$

The mutual orthonormality of the basis functions can easily be verified by means of residue calculus. Completeness in $H_{2-}$ can be proved along the lines of the analysis in [9].

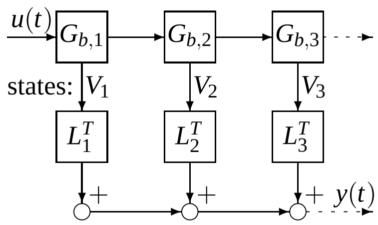


**Figure 1:** Representation of the system $G(z)$ as a cascade network of generalized orthonormal basis functions.

The representation of a system $G(z)$ in terms of these basis functions can be viewed as a cascade network of balanced all-pass functions as is visualized in Figure 1. The output of the system is then equal to the sum of the states of the sections $G_{b,k}$ multiplied by the coefficient vectors $L_k^T$.

By the isomorphic property of the $z$-transform, an orthonormal basis of the signal space $\ell_2(\mathbb{N})$ is obtained by applying the inverse $z$-transform to the functions $\{V_k(z)\}_{k\in\mathbb{Z}}$. The corresponding time-domain basis functions are written as $V_k(t)$ with index $t \in \mathbb{N}$ denoting time.

Consider the series connection of a sequence of $k$ balanced all-pass functions $S_k$. A state-space realization of $S_k$ can be constructed from the realizations $(A_{b,k}, B_{b,k}, C_{b,k}, D_{b,k})$ such that the basis function vectors $V_k(z)$, stacked in one large column, form precisely the input-to state transfer function of $S_k$. It can further be shown that this realization of $S_k(z)$ is minimal and balanced.

**Proposition 2** *[10]*
*Consider the system $S_k = \prod_{j=1}^{k} G_{b,j} = G_{b,k} \cdots G_{b,2} G_{b,1}$. Then for any $k > 1$ it holds that $S_k$ is an all-pass function that has a minimal balanced state-space realization $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$ which satisfies the recursive relation:*

$$\mathcal{A}_k = \begin{bmatrix} \mathcal{A}_{k-1} & 0 \\ B_{b,k}\mathcal{C}_{k-1} & A_{b,k} \end{bmatrix} \qquad \mathcal{B}_k = \begin{bmatrix} \mathcal{B}_{k-1} \\ B_{b,k}\mathcal{D}_{k-1} \end{bmatrix}, \qquad (2)$$

$$\mathcal{C}_k = \begin{bmatrix} D_{b,k}\mathcal{C}_{k-1} & C_{b,k} \end{bmatrix} \qquad \mathcal{D}_k = D_{b,k}\mathcal{D}_{k-1}, \qquad (3)$$

*with $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1) = (A_{b,1}, B_{b,1}, C_{b,1}, D_{b,1})$.*

By construction the input-to-state transfer function of the system $S_k$ is equal to

$$\begin{bmatrix} V_1^T(z) & V_2^T(z) & \cdots & V_k^T(z) \end{bmatrix}^T,$$

with $V_j(z)$ as defined in (1). In time domain notation it therefore holds that

$$V_k(t+1) = \begin{bmatrix} B_{b,k}\mathcal{C}_{k-1} & \big| A_{b,k} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ \hline V_k(t) \end{bmatrix} + B_{b,k}\mathcal{D}_{k-1}\delta(t),$$

where $\delta(t)$ represents the unit pulse signal. This can also be stated as

$$V_k(t+1) = B_{b,k}\sum_{j=1}^{k-1}\prod_{i=j+1}^{k-1} D_{b,i}C_{b,j}V_j(t) + A_{b,k}V_k(t) + B_{b,k}\prod_{j=1}^{k-1} D_{b,j}\delta(t). \tag{4}$$

This equation can be used to calculate the basis functions $V_k$ in an efficient manner.

The basis construction of equation (1) generalizes the constructions that were mentioned in the introduction. Obviously, it closely resembles the Hambo basis construction [6, 5]. The main difference is that the all-pass functions in the cascade are allowed to be different. It should be noted that the Laguerre construction, which is basically a Hambo basis construction with $n_b = 1$ is also a special case of (1). The generalized construction also subsumes the Ninness construction [9] which can be viewed as the special case in which the all-pass sections are single pole functions (McMillan degree 1). Supposing that the all-pass functions $G_{b,k}$ used in the construction have poles $\xi_k$, then the Ninness basis can be generated according to (1) using the balanced realizations $\left(\xi_k, \sqrt{1-|\xi_k|^2}, \sqrt{1-|\xi_k|^2}, -\xi_k^*\right)$. For the case where $n_{b,k} = 2\ \forall k$ the construction (1) becomes equivalent to the general Kautz construction [7].

In the remainder of this paper it is assumed for notational convenience that the all-pass functions $G_{b,k}$ and their realizations are real. The theory can be adapted to deal with complex all-pass functions without any difficulty [2].

## 3 Signal and operator transforms

In this section the fundamentals of the transform theory that underlies expansions in the generalized basis are given. This theory has its roots in the Hambo transform theory that was presented in [5].

Consider an arbitrary signal $x \in \ell_2(\mathbb{N})$. Because the functions $\{V_k(t)\}$ constitute an orthonormal basis of the signal space $\ell_2(\mathbb{N})$ the signal $x$ can be expanded as,

$$x(t) = \sum_{k=1}^{\infty} \mathcal{X}^T(k)V_k(t),$$

with $\mathcal{X}(k)$ denoting the expansion coefficient vectors. The vector sequence $\{\mathcal{X}(k)\}_{k\in\mathbb{N}}$, denoted by $\mathcal{X}$ for ease of notation, is the so called *signal transform* of the signal $x$ taken with respect to the generalized basis. Note that the dimension of the coefficient vectors $\mathcal{X}(k)$ can change with index $k$, depending on the McMillan degree of the corresponding all-pass function $G_{b,k}$. The coefficients $\mathcal{X}(k)$ are obtained by taking inner products of $x$ with the basis functions $V_k$:

$$\mathcal{X}(k) = \sum_{t=1}^{\infty} V_k(t)x(t) = [\![V_k, x]\!]\,. \tag{5}$$

As in the "classical" Hambo setting [5], the *operator transform*, induced by the generalized basis in this case, is defined as follows.

**Definition 3** *The operator transform of a system $G \in H_2$, denoted by $\tilde{G}$, is defined as the operator that maps the input coefficient sequence $\mathcal{U}$ into the output coefficient sequence $\mathcal{Y}$, for all inputs $u \in \ell_2(\mathbb{N})$ with $y$ the corresponding outputs.*

It is not difficult to see that $\tilde{G}$ is a causal linear operator.

**Proposition 4** *Given $G \in H_2$ and $u, y \in \ell_2(\mathbb{N})$ such that $y = Gu$, it holds that*

$$\mathcal{Y}(k) = \sum_{j=1}^{k} M_{k,j}\mathcal{U}(j), \tag{6}$$

*with*

$$M_{k,j} = \left[\!\!\left[\phi_k(z), \phi_j(z)\prod_{i=j}^{k-1} G_{b,i}(1/z)G(z)\right]\!\!\right] \tag{7}$$

**Proof:** $\mathcal{Y}(k)$ can be expressed as

$$\mathcal{Y}(k) = \left[\!\!\left[V_k, G\sum_{j=1}^{\infty}\mathcal{U}^T(j)V_j\right]\!\!\right] = \sum_{j=1}^{\infty}[\![V_k, V_jG]\!]\,\mathcal{U}(j).$$

Using (1) one finds

$$\mathcal{Y}(k) = \sum_{j=1}^{\infty}\left[\!\!\left[\phi_k\prod_{i=1}^{k-1}G_{b,i}, \phi_j\prod_{i=1}^{j-1}G_{b,i}G\right]\!\!\right]\mathcal{U}(j).$$

Consider the inner product term for the case where $j \le k$. Use is made of the fact that the adjoint of $G_{b,k}(z)$ is equal to $G_{b,k}(1/z)$ which by its all-pass property is equal to $G_{b,k}(z)^{-1}$. Therefore the inner product can be written as $\left[\!\left[\phi_k(z), \phi_j(z)\prod_{i=j}^{k-1}G_{b,i}(1/z)G(z)\right]\!\right]$. Now consider the inner product term for the case $j > k$. Then with the same argument one finds $\left[\!\left[\phi_k(z), \phi_j(z)\prod_{i=k}^{j-1}G_{b,i}(z)G(z)\right]\!\right]$. This latter expression is equal to zero. This follows from the fact that the elements of the transfer function $\phi_k(z)$ constitute an orthonormal set which exactly spans the orthogonal complement in $H_{2-}$ of the shift-invariant subspace $G_{b,k}(z)H_{2-}$. The right argument of the inner product is an element of that subspace. ∎

Since $\tilde{G}$ is a *causal* linear operator it can be represented by a block triangular matrix (of infinite dimension), in which the block indexed $(k, j)$ has dimension $n_{b,k} \times n_{b,j}$. It can also be viewed as a causal (linear) time-varying system for which also the dimension of the input and output vectors can change in time, see [4] for a treatment of linear time-varying systems. In the Hambo basis case it even holds that $\tilde{G}$ is a shift-invariant operator (with respect to the index $k$). This means that in that case it holds that $M_{k,j} = M_{k+m,j+m}$ for all $k, j$ and $m$. Stated otherwise, $M_{k,j}$ depends only on the difference $k-j$. This can easily be verified using the fact that in that case $\prod_{i=j}^{k-1} G_{b,i}(1/z) = G_b(1/z)^{k-j}$, where $G_b$ is the single all-pass function generating the basis. In the Hambo case one therefore usually writes $M_{k-j}$ instead of $M_{k,j}$. It should be noted that in that case the matrix representation of $\tilde{G}$ becomes block lower triangular Toeplitz. Also in that case all the blocks have dimension $n_b \times n_b$.

For the Hambo basis it was shown in [5] that $\tilde{G}(\lambda)$ which is defined as $\tilde{G}(\lambda) = \sum_{k=1}^{\infty} M_k \lambda^{-k}$, is obtained from $G(z)$ by applying a simple variable substitution. Also it was shown how a minimal state-space realization of $\tilde{G}(\lambda)$ can be derived on the basis of a minimal realization of $G(z)$ by means of solving a set of Sylvester equations in [3]. In the following sections it will be shown that very similar results hold for the generalized basis.

## 4 Variable substitution property

It will be shown that the operator transform, as defined previously, can be obtained from the original transfer function $G(z) \in H_2$ by applying a variable substitution in its Laurent expansion, which is given by

$$G(z) = \sum_{\tau=0}^{\infty} g_\tau z^{-\tau}. \tag{8}$$

The parameters $g_\tau$ represent the impulse response parameters of the system $G$. The variable substitution consists of a replacement of the shift operation $z^{-1}$ by a causal linear time-varying operator, denoted $\mathcal{N}$, to be defined shortly.

In order to demonstrate this property, use is made of a dual orthonormal basis that spans the space of coefficient sequences.

**Proposition 5** *Denote $W_t(k) = V_k(t)$. Then the sequences $\{W_t(k)\}_{k\in\mathbb{N}}$ with $t \in \mathbb{N}$, constitute an orthonormal basis of the space of coefficient sequences that is related to the generalized orthonormal basis generated by the sequence of balanced all-pass functions $G_{b,k}$.*

For a proof the reader is referred to [5] in which a similar proposition is given for the Hambo basis case.

With (4) it immediately follows that the basis functions $W_t$ can be constructed recursively using the following set of equations.

$$W_1(k) = B_{b,k} \prod_{j=1}^{k-1} D_{b,j}, \tag{9}$$

$$W_{t+1}(k) = B_{b,k} \sum_{j=1}^{k-1} \prod_{i=j+1}^{k-1} D_{b,i} C_{b,j} W_t(j) + A_{b,k} W_t(k). \tag{10}$$

These expressions imply the following.

**Proposition 6** *It holds for all $t \in \mathbb{N}$ that $W_{t+1} = \mathcal{N} W_t$ where $\mathcal{N}$ is a causal linear time-varying operator that has state-space realization $(D_{b,k}, C_{b,k}, B_{b,k}, A_{b,k})$.*

Defining $\mathcal{N}^m$ as $\prod_{j=1}^{m} \mathcal{N}$ proposition 6 may alternatively be expressed as $W_t = \mathcal{N}^{t-1} W_1$. Note that the operator $\mathcal{N}$ can be thought of as the block lower triangular matrix $\mathcal{A}_\infty$ defined as the limiting case of $\mathcal{A}_k$ in (2).

The operator $\mathcal{N}$ can be used to derive an expression for $\tilde{G}$ according to the following proposition.

**Proposition 7** *(Variable substitution property)*
*The operator transform $\tilde{G}$ of a system $G \in H_2$ is given by*

$$\tilde{G} = \sum_{\tau=0}^{\infty} g_\tau \mathcal{N}^\tau. \tag{11}$$

**Proof:** Take any input $u \in \ell_2(\mathbb{N})$ and corresponding output $y$ of the system $G$. It follows from (5) and the definition of $W_t$ that the expansion coefficients of these signals satisfy

$$\mathcal{U}(k) = \sum_{t=1}^{\infty} \mathcal{N}^{t-1} W_1(k) u(t), \; \mathcal{Y}(k) = \sum_{t=1}^{\infty} \mathcal{N}^{t-1} W_1(k) y(t).$$

Supposing $G$ has Laurent expansion (8) the expression for $\mathcal{Y}$, can be phrased as

$$\begin{aligned}\mathcal{Y}(k) &= \sum_{t=1}^{\infty} \mathcal{N}^{t-1} W_1(k) \sum_{\tau}^{t} g_\tau u(t-\tau) \\ &= \sum_{\tau=0}^{\infty} g_\tau \sum_{t=\tau+1}^{\infty} u(t-\tau) \mathcal{N}^{t-1} W_1(k).\end{aligned}$$

Substituting $t' = t - \tau$ one gets

$$\mathcal{Y}(k) = \sum_{\tau=0}^{\infty} g_\tau \sum_{t'=1}^{\infty} u(t') \mathcal{N}^{t'+\tau-1} W_1(k).$$

By linearity of the operator $\mathcal{N}$ this can be expressed as

$$\mathcal{Y}(k) = \sum_{\tau=0}^{\infty} g_\tau \left( \mathcal{N}^\tau \sum_{t'=1}^{\infty} u(t') \mathcal{N}^{t'-1} W_1 \right)(k).$$

Since it holds that $\sum_{t'=1}^{\infty} u(t') \mathcal{N}^{t'-1} W_1 = \mathcal{U}$ this expression can more concisely be written as (11). ∎

Expression (11) can be interpreted as the equivalent of the variable substitution property of the original Hambo transform [5]. In that case it holds that the operator transform $\tilde{G}(\lambda)$ is obtained according to $\tilde{G}(\lambda) = \sum_{\tau=0}^{\infty} g_\tau N(\lambda)^\tau$, with the operator $N$ being the time-invariant version of $\mathcal{N}$. With $G_b(z)$ with balanced state-space realization $(A_b, B_b, C_b, D_b)$ is the all-pass function generating the Hambo basis, then the corresponding $N(\lambda)$ has state-space realization $(D_b, C_b, B_b, A_b)$.

## 5 State-space expressions

In [3] it was shown how using a realization theory approach a set of Sylvester equations could be derived by which on the basis of a state-space realization of a system $G(z)$, a state-space realization of its Hambo operator transform can be determined. In this section it is shown that the very same Sylvester equations can be used for the computation of a state-space realization of the operator transform that is induced by the generalized basis construction of (1).

The expressions for the state-space realization of the generalized operator transform, denoted by $\tilde{G}$ are derived on the basis of equation (7). Due to lack of space we cannot give a complete derivation but an idea of the proof is given in the appendix.

**Proposition 8** *Given a stable linear-time invariant system $G$ with minimal state-space realization $(A,B,C,D)$, it holds, with $M_{k,j}$ as defined by* (6)*, that*

$$M_{k,j} = \tilde{D}_k, \qquad j = k, \tag{12}$$

$$M_{k,j} = \tilde{C}_k \prod_{i=j+1}^{k-1} \tilde{A}_i \tilde{B}_j, \qquad j < k, \tag{13}$$

*with $\tilde{A}_k, \tilde{B}_k, \tilde{C}_k$ and $\tilde{D}_k$ satisfying*

$$A\tilde{B}_k A_{b,k} + BC_{b,k} = \tilde{B}_k, \tag{14}$$

$$A_{b,k}\tilde{C}_k A + B_{b,k}C = \tilde{C}_k, \tag{15}$$

$$A^T X_o \tilde{A}_k A + C^T \left(C_{b,k}\tilde{C}_k A + D_{b,k}C\right) = X_o \tilde{A}_k, \tag{16}$$

$$A_{b,k}\tilde{D}_k A_{b,k}^T + \left(B_{b,k}D + A_{b,k}\tilde{C}_k B\right) B_{b,k}^T = \tilde{D}_k, \tag{17}$$

*where $X_o$ represents the observability Gramian associated with the pair $(A,C)$. Hence the operator transform $\tilde{G}$, as defined in definition 3 has state-space realization $\left(\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k\right)$.*

An idea of the proof is given in the appendix. It can be shown that equations (14) through (17) can be written as one Sylvester equation [2]:

$$\begin{bmatrix} A^T & C^T C_{b,k} \\ 0 & A_{b,k} \end{bmatrix} \begin{bmatrix} X_o\tilde{A}_k & X_o\tilde{B}_k \\ \tilde{C}_k & \tilde{D}_k \end{bmatrix} \begin{bmatrix} A & BB_{b,k}^T \\ 0 & A_{b,k}^T \end{bmatrix} + \begin{bmatrix} C^T D_{b,k} \\ B_{b,k} \end{bmatrix} \begin{bmatrix} C & DB_{b,k}^T \end{bmatrix} = \begin{bmatrix} X_o\tilde{A}_k & X_o\tilde{B}_k \\ \tilde{C}_k & \tilde{D}_k \end{bmatrix}. \tag{18}$$

Note that equation (18) can be simplified further by first applying an output balancing transformation to the state-space realization of $G(z)$ so that $X_o = I$. It should also be mentioned that a dual formulation of (18) involving the controllability Gramian associated with the pair $(A,B)$ is also possible.

Equation (16) reveals that the state-space dimension of the resulting realization of $\tilde{G}$ is equal to the state-space dimension of the realization $(A,B,C,D)$ of $G$. It can be shown that the Gramians associated with $\left(\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k\right)$ are equal to the Gramians associated with $(A,B,C,D)$ [2]. Since $(A,B,C,D)$ is assumed to be minimal this, among other things, implies that $\left(\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k\right)$ is minimal as well, which in turn implies that McMillan degree is invariant under operator transformation.

In the special case of Hambo bases, the operator transform $\tilde{G}(\lambda)$ is a linear time-invariant system. In order to obtain a minimal state-space realization of $\tilde{G}(\lambda)$ equation (18) needs to be solved only once, instead of for every $k$ as in the general case.

In the context of Hambo transform theory formula (18) has been derived before in [3]. Also, for that specialized case, a dual formula for the inverse operator transform was given. In this paper we will not pursue the derivation of an inverse expression for the generalized case (see [2] for a treatment of that problem). Instead we will present one possible application of the generalized operator transform theory, namely for the computation of the expansion coefficients $L_k$.

**Proposition 9** *Suppose $G \in H_{2-}$ has a minimal state-space realization $(A,B,C)$. Then the expansion coefficients $L_k$ of the expansion of $G$ in terms of the generalized orthonormal basis functions as defined by* (1) *satisfy*

$$L_k = \tilde{C}_k \prod_{j=1}^{k-1} \tilde{A}_j B, \tag{19}$$

*with $\tilde{A}_j$ and $\tilde{C}_k$ obtained as the solutions to equations* (16) *and* (15).

A proof is given in the appendix.

This proposition shows that the coefficients can be computed as the impulse response of a linear time-varying system with state-space realization $(\tilde{A}_k, B, \tilde{C}_k)$. In the Hambo basis case it holds that $\tilde{A}_i = \tilde{A}_j \; \forall i,j$ and then the coefficients can be computed as the impulse response of a linear *time-invariant* system, i.e. in that case one gets

$$L_k = \tilde{C}\tilde{A}^{k-1}B.$$

The result of proposition 9 can be used to devise an efficient recursive algorithm for the computation of expansion coefficients for generalized orthonormal bases. At the $k$-th step only the Sylvester equations for computation of $\tilde{A}_k$ and $\tilde{C}_k$ need to be solved. Such an algorithm is of course very reminiscent of similar recursive algorithms that were presented in the literature, e.g. [1] for the generalized case and [6] for the Hambo basis case.

## 6 Conclusion

In this paper a general transform theory was presented that underlies expansions of stable linear time-invariant systems in terms of general rational orthonormal basis functions. The basis functions considered are generated by a cascade network of internally balanced all-pass functions. This construction closely resembles the Hambo basis construction of [6] but it is more general in the sense that different all-pass sections, in terms of choice and number of poles, can be incorporated in the cascade. It was shown that much of the operator transform theory that underlies the Hambo basis expansion can be extended to the generalized case. The notable difference is that while the Hambo operator transform is again a linear time-invariant system, the generalized operator transform is a linear time-varying system. It was shown how the generalized operator transform can be obtained by means of a variable substitution in the Laurent expansion of the system considered. Furthermore, expressions were presented by which, on the basis of a given minimal state-space realization, a corresponding minimal state-space realization of the operator transform can be computed. Finally, as an application of the presented theory, it was shown how the expansion coefficients of a given stable LTI system with respect to the generalized basis can be computed as the impulse response of a linear time-varying system that involves the state-space matrices of the LTI system and its generalized operator transform.

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## Appendix

**Idea of the proof of proposition 8**

The expression of $M_{k,j}$ for $j=k$ follows by straightforward evaluation (7) with $G(z)=D+\sum_{t=1}^{\infty}CA^{t-1}B\,z^{-t}$ and $\phi_k(z)=\sum_{l=1}^{\infty}A_{b,k}^{l-1}B_{b,k}\,z^{-l}$ . The expression for $M_{k,j}$ for $j<k$ follows similarly making use of the following lemmata.

**Lemma 10** *Given $G(z)\in H_2$ with state-space realization $(A,B,C,D)$, it holds that*

$$\left[\!\left[\phi_k(z),\phi_j(z)G_{b,j}(1/z)G(z)\right]\!\right]= \sum_{t=1}^{\infty}A_{b,k}^{t-1}B_{b,k}CA^{t-1}\sum_{l=0}^{\infty}A^{l}BC_{b,j}A_{b,j}^{l}.$$

**Lemma 11** *Given $G(z)\in H_2$ with a minimal state-space realization $(A,B,C,D)$, it holds for all $j$ that $\mathbf{P}_{H_{2-}}\prod_{i=j+1}^{k-1}G_{b,i}(1/z)G(z)$ has a state-space realization $\left(A,\prod_{i=j+1}^{k-1}\tilde{A}_iB,C\right)$, with*

$$\tilde{A}_i=\sum_{l=0}^{\infty}A^{l}g_{b,i_l}, \tag{20}$$

*and $g_{b,i_l}$ the impulse response coefficients of the all-pass function $G_{b,i}$.*

Equation (20) is equivalent to (16) in combination with (15).

**Proof of proposition 9:** The coefficients satisfy

$$L_k=[\![V_k(z),G(z)]\!]=\left[\!\!\left[\phi_k(z)\prod_{i=1}^{k-1}G_{b,i}(z),G(z)\right]\!\!\right]= \left[\!\!\left[\phi_k(z),\prod_{i=1}^{k-1}G_{b,i}(1/z)G(z)\right]\!\!\right].$$

The right side term of the inner product can be replaced by its projection onto $H_{2-}$ without changing the outcome. Using lemma 11 and taking the inner product one finds

$$L_k=\sum_{t=1}^{\infty}A_{b,k}^{t-1}B_bCA^{t-1}\prod_{i=1}^{k-1}\tilde{A}_iB=\tilde{C}_k\prod_{i=1}^{k-1}\tilde{A}_iB.$$

■