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Frequency Domain Identification with Generalized Orthonormal Basis Functions

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Abstract

A method is considered for identification of linear parametric models based on a least squares identification criterion that is formulated in the frequency domain. To this end use is made of the empirical transfer function estimate (ETFE), identified from time-domain data. As a parametric model structure use is made of a finite expansion sequence in terms of recently introduced generalized basis functions, being generalizations of the classical pulse, Laguerre and Kautz types of bases. An asymptotic analysis of the estimated models is provided and conditions for consistency are formulated. Explicit and transparent bias and variance expressions are established, the latter ones also valid in a situation of undermodelling.

Keywords: System identification; transfer function estimation; frequency domain method; asymptotic analysis.

1 Introduction

The identification of models on the basis of frequency domain data is a subject that attracts a growing number of researchers and engineers. Especially in application areas where experimental data of a process with (partly) unknown dynamics can be taken relatively cheaply, the excitation of the process with periodic signals (as e.g. sinusoids), is an attractive way of extracting accurate information of the process dynamics from experiments. Due to the commercial availability of frequency analyzers that can handle huge amounts of data by special purpose hardware, the experimental determination of frequency responses of dynamical systems has gained increasing interest in application areas as the modelling of e.g. mechanical servo systems and flexible space structures.

Identification on the basis of frequency domain data can have a number of advantages when compared to the "classical" time-domain approach. For a very nice overview of these arguments the reader is referred to [17]. Here we would like to limit attention to the fact that dealing with frequency domain data allows us to achieve a substantial data reduction, thus enabling us to handle huge amounts of time-domain data, with a corresponding reduced variance of the model estimates. Additionally, the formulation of an identification criterion in the frequency domain can be beneficial especially in those situations where the application of the model dictates a performance evaluation in terms of frequency domain properties. This last situation occurs often when the identified models are used as a basis for model-based control design.

On the other hand it should be stressed that any frequency domain data is obtained by some data handling/processing mechanism that starts off with time-domain data. This is reason not to overestimate the difference between time-domain and frequency-domain identification, see e.g. also [13].

The common way of formulating an identification problem in the frequency domain is by assuming the availability of the exact frequency response of the (unknown) linear system, disturbed by some additive (frequency-domain) noise with specific properties, as e.g. independence among the several frequencies. For this situation a large number of identification methods exist mostly dealing with least squares criteria [9, 18, 24, 20]. Maximum amplitude (ℓ_∞) criteria are considered in [21, 7], while special purpose multivariable algorithms are discussed e.g. in [10, 1]. Recently also subspace algorithms have been analyzed for frequency domain identification [15]. Many more references and techniques can be found in [19, 17]. A related approach to the problem based on the discrete Fourier transforms of input and output data in [13] shows a close resemblance of results with the standard time-domain approach.

In this paper we will formulate and analyse an identification problem with an identification criterion formulated in the frequency domain, for a situation in which we consider time-domain data of input and output samples to be available from measurements. These time-domain signals generate a (frequency-domain) empirical transfer function estimate (ETFE) which represents the data in a least squares identification criterion. As a parametric model structure we will use a linear regression form, using a finite series expansion of the model transfer function in terms of very flexible orthonormal basis functions as recently introduced in [8, 22]. This model structure is very powerful in accurately describing system's transfer functions with only few parameters. It generalizes the situation of Laguerre functions, for which a frequency domain identification analysis has been provided in [3]. The resulting identification scheme will be analyzed and expressions for bias and variance will be formulated, relating the identification results to the properties of the original excitation and disturbance signals in the time domain. This method will build on the standard identification framework as developed in [12].

The remainder of this paper is organized as follows. First,

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we will formalize our problem setting. Subsequently we will analyse the estimation results for an identification criterion over a finite number of frequencies. Next the situation of an infinite number of frequencies will be considered. Asymptotic bias and asymptotic variance expressions will be provided, for infinite and for finite model order (undermodelling). For all proofs of presented results the reader is referred to the extended paper [6].

2 Problem setting

It is assumed that a data generating system, and the measurement data that is obtained from this system, allow a description

$$y(t) = G_o(q)u(t) + v(t) \tag{1}$$

where $G_o(z)$ is a rational transfer function, analytic in $|z| \geq 1$, $v(t)$ is a realization of a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, satisfying $v(t) = H_o(q)e(t)$ where $e(t)$ is a sequence of independent identically distributed random variables, having zero mean, variance σ_e^2, and bounded moments of all orders. H_o is a stable proper minimum-phase transfer function. $u(t)$ will be a bounded quasi-stationary deterministic signal. q is the shift operator, $qu(t) = u(t+1)$.

As further notational convention, let T^N denote the integer interval $T^N := \mathbb{Z} \cap [0, N-1]$. For a signal $x(t)$ being defined on T^N, we will denote

$$X_N(e^{i\omega_k}) := \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-i\omega_k t} \quad \omega_k \in \Omega_N. \tag{2}$$

Specific sets of frequency points that arise are denoted as

$$\begin{aligned} \Omega_N &:= \{\frac{2\pi k}{N}, k \in T^N\} \\ \Omega_N^* &:= \{\omega_k \in \Omega_N \mid |U_N(e^{i\omega_k})| > 0\}. \end{aligned} \tag{3}$$

The identification problem that we consider now can be stated as follows.

Frequency domain identification problem:
Given measurements of input and output data $\{u(t), y(t)\}$, $t \in T^N$, given a frequency grid $\mathcal{W}_{N_p}^*$

$$\mathcal{W}_{N_p}^* = \{\omega_1, \cdots, \omega_{N_p}\} \subset \Omega_N^*, \tag{4}$$

then we consider the identification problem:

$$\hat{\theta} := \arg\min_\theta \frac{1}{N_p} \sum_{k=1}^{N_p} \left| \frac{Y_N(e^{i\omega_k})}{U_N(e^{i\omega_k})} - G(e^{i\omega_k}, \theta) \right|^2 \tag{5}$$

where $G(e^{i\omega_k}, \theta)$ is the parametrized frequency response of the model, and θ ranges over an appropriate parameter space $\Theta \in \mathbb{R}^{n_p}$ while $n_p \leq N_p$. The specific (linear) parametrization of the model is considered in the next section.

Note that the least squares criterion (5) implies that more attention is paid to those frequency regions which have a relatively denser frequency grid within $\mathcal{W}_{N_p}^*$.

3 Model Parametrization with Orthonormal Basis Functions

In the identification problem (5) we will employ a linear model parametrization

$$G(e^{i\omega_k}, \theta) = \phi(e^{i\omega_k})\theta \tag{6}$$

that is very flexible, being based on a recently introduced generalized orthonormal basis. The related theory as presented in [8], shows that for any scalar stable all-pass transfer function G_b with balanced realization (A, B, C, D) the sequence of functions

$$\mathcal{V}_k(z) = (zI - A)^{-1} B \cdot G_b(z)^k \tag{7}$$

generates an orthonormal basis for the space of stable systems $\mathcal{RH}_2$. As a result there exist unique D, and $\{L_k\}_{k=0,\cdots,\infty}$ such that

$$G_o(z) = D + \sum_{k=0}^{\infty} L_k \mathcal{V}_k(z). \tag{8}$$

Note that $\mathcal{V}_k(z) \in \mathcal{RH}_2^{n_b \times 1}$ with n_b the McMillan degree of G_b (dimension of A), and $L_k \in \mathbb{R}^{1 \times n_b}$.
The advantage of this generalized basis is that if a proper choice of dynamics (set of poles) is incorporated into the basis functions $\mathcal{V}_k$, then the series expansion (8) shows an increasing speed of convergence. Consequently the accuracy of a finite expansion model will substantially increase. For more details on the use of these basis functions see [8, 22]. Here we limit the discussion by remarking that for specific choices of all-pass functions, well known "classical" basis functions result, as the standard shift $V_k(z) = z^{-k}$, and the Laguerre functions induced by a first order all-pass function $G_b(z)$.

With respect to our formulated frequency domain identification problem, it follows that - for a specifically chosen basis $\mathcal{V}_k(z)$ - the regression vector $\phi(e^{i\omega_k})$ in (6) is given by

$$\phi(e^{i\omega_k}) = \left[1 \;\; \mathcal{V}_0^T(e^{i\omega_k}) \; \cdots \; \mathcal{V}_{n-1}^T(e^{i\omega_k})\right], \tag{9}$$

with $n_p = n_b n + 1$.
The estimated parameter vector is denoted by

$$\hat{\theta} = \left[\hat{D} \;\; \hat{L}_0 \; \cdots \; \hat{L}_{n-1}\right]^T. \tag{10}$$

4 Parameter Estimate

Building upon the relation (8) for the data generating system, we will write

$$G_o(e^{i\omega_k}) = \phi(e^{i\omega_k})\theta_o + Z(e^{i\omega_k}) \tag{11}$$

with $Z(e^{i\omega_k}) = \sum_{k=n}^{\infty} L_k \mathcal{V}_k(e^{i\omega_k})$, and

$$\theta_o = [D \;\; L_0 \; \cdots \; L_{n-1}]^T. \tag{12}$$

Using the system's equations, similar as in [12], we now can express the ETFE by

$$\frac{Y_N(e^{i\omega_k})}{U_N(e^{i\omega_k})} = G_o(e^{i\omega_k}) + S(e^{i\omega_k}) + F(e^{i\omega_k}) \tag{13}$$

where

$$F(e^{i\omega_k}) = \frac{V_N(e^{i\omega_k})}{U_N(e^{i\omega_k})} \tag{14}$$

and where $S(e^{i\omega_k})$ is a term due to the past of the input signal, $\{u(t)\}_{t<0}$. Written in vector notation we can rewrite this into:

$$\begin{aligned} \hat{\mathsf{G}} &= \mathsf{G}_o + \mathsf{S} + \mathsf{F} \\ &= \Phi\theta_o + \mathsf{Z} + \mathsf{S} + \mathsf{F} \end{aligned} \tag{15}$$

with

$$
\begin{aligned}
\mathsf{G} &= \left[\frac{Y_N(e^{i\omega_1})}{U_N(e^{i\omega_1})} \cdots \frac{Y_N(e^{i\omega_{N_p}})}{U_N(e^{i\omega_{N_p}})}\right]^T \\
\mathsf{G}_o &= \left[G_o(e^{i\omega_1}) \cdots G_o(e^{i\omega_{N_p}})\right]^T \\
\mathsf{S} &= \left[S(e^{i\omega_1}) \cdots S(e^{i\omega_{N_p}})\right]^T \\
\mathsf{F} &= \left[F(e^{i\omega_1}) \cdots F(e^{i\omega_{N_p}})\right]^T \\
\mathsf{Z} &= \left[Z(e^{i\omega_1}) \cdots Z(e^{i\omega_{N_p}})\right]^T \\
\Phi &= \begin{bmatrix} 1 & \mathcal{V}_0^T(e^{i\omega_1}) & \cdots & \mathcal{V}_{n-1}^T(e^{i\omega_1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathcal{V}_0^T(e^{i\omega_{N_p}}) & \cdots & \mathcal{V}_{n-1}^T(e^{i\omega_{N_p}}) \end{bmatrix}
\end{aligned} \tag{16}
$$

where Z reflects a term due to undermodelling, S represents the effect of unknown past inputs, and F reflects a term due to the noise.
The parameter estimate can be constructed from matrix operations according to

$$\hat{\theta} = \Psi \mathsf{G} \tag{17}$$

where

$$\Psi = (\Phi^* \Phi)^{-1} \Phi^* \tag{18}$$

and provided $\Phi^*\Phi$ is nonsingular. Given (15), (17) and (18) it follows that the least squares parameter estimate (5) can be expressed as

$$\hat{\theta} = \theta_o + \Psi \mathsf{Z} + \Psi \mathsf{S} + \Psi \mathsf{F}$$

For the bias and covariance of the estimated parameters (5) it can simply be verified that

$$
\begin{aligned}
\mathbb{E}[\hat{\theta}] &= \theta_o + \Psi \mathsf{Z} + \Psi \mathsf{S} && (19) \\
\mathrm{cov}[\hat{\theta}] &= \Psi \mathbb{E}[\mathsf{F}\mathsf{F}^*] \Psi^*. && (20)
\end{aligned}
$$

The transfer function estimate of the identified model will be given by

$$\hat{G}^f(e^{i\omega}) = \phi(e^{i\omega})\hat{\theta}. \tag{21}$$

Invertibility of the (block-Toeplitz) matrix $\Phi^*\Phi$ in (18) is guaranteed if a sufficient number of different frequencies is selected in the frequency grid, formalized in the following lemma[1].

Lemma 4.1 *If $N_p \geq n_p$ then Φ has full column rank, and so Ψ (18) is well defined.*

As a result of this lemma, the parameter estimate (5),(17) will be unique, provided that a sufficient number of sinusoids is present in the input signal. This situation is similar to the corresponding time-domain least squares problem. Note that in the case considered here, the regression vector ϕ is composed of basis functions only.

5 Analysis for finite number N_p of frequencies

We will first restrict the anaylsis of the identified model to the situation where the number of frequencies N_p in the criterion is finite. In order to relate the parameter estimate to the real dynamic properties of the underlying system, we have to impose additional conditions on the input signal. To this end the following assumption is formulated.

[1]The authors are indebted to Peter Heuberger for his contribution to providing a proof of this Lemma.

Assumption 5.1 *The input signal $u(t)$ satisfies the condition that there exists a bounded function $\Phi_u(\omega)$ such that*

$$\lim_{N\to\infty} \frac{N_p}{N}|U_N(e^{i\omega_k})|^2 = \Phi_u(\omega_k) \quad \forall\, \omega_k \in \Omega_N^u, \tag{22}$$

where $\Phi_u^{-1}(\omega)$, $\omega \in [0, 2\pi)$, is continuous, except for at most a finite number of points, and bounded. □

Whenever $N_p/N \to 0$, the condition mentioned in Assumption 5.1, implies that the input signals has to be periodic. In the situation of a periodic input signal with period length N_p and assuming that N/N_p is integer valued, it can be verified that

$$U_N(e^{i\omega_k}) = \sqrt{\frac{N}{N_p}} \cdot U_{N_p}(e^{i\omega_k}) \tag{23}$$

with $U_{N_p}(e^{i\omega_k})$ the DFT of the input signal over one period, and thus illustrating the boundedness of the expression in (22).
The next lemma is a classical result for ETFE estimates.

Lemma 5.2 (Brillinger [2]) *Let the input signal satisfy Assumption 5.1. Then*

$$\frac{N}{N_p}\mathbb{E}[\mathsf{F}\mathsf{F}^*] \to Q = \begin{bmatrix} \frac{N\Phi_v(\omega_1)}{N_p|U_N(e^{i\omega_1})|^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{N\Phi_v(\omega_{N_p})}{N_p|U_N(e^{i\omega_{N_p}})|^2} \end{bmatrix} \tag{24}$$

as $N \to \infty$, where the rate of convergence is $1/N$ element wise. Additionally, there exists a constant $c_1 \in \mathbb{R}$ such that

$$\|\frac{N}{N_p}\mathbb{E}[\mathsf{F}\mathsf{F}^*] - Q\|_2 \leq c_1\frac{N_p}{N}.$$

Using Lemma 5.2 we can state the following result for the estimated parameters in the asymptotic case when $N \to \infty$, where we will address the i-th element of the vector of estimated parameters $\hat{\theta}$ (10) as $\hat{\theta}_i$.

Proposition 5.3 *Consider Assumption 5.1 and the expressions (19), (20) of the estimated parameters $\hat{\theta}$ (5), while $N_p \geq n_p$. Then*

(a) $\lim_{N\to\infty} \mathbb{E}[\hat{\theta}] = \theta_o + \Psi\mathsf{Z}$;

(b)

$$N\,\mathrm{cov}[\hat{\theta}_i, \hat{\theta}_k] \to N_p \sum_{\ell=1}^{N_p} \Psi_{i\ell}\Psi_{k\ell}^* \frac{\Phi_v(\omega_\ell)}{\Phi_u(\omega_\ell)} \qquad as\ \frac{N_p}{N} \to 0. \tag{25}$$

Part (a) of the proposition shows that in case there is no undermodelling, i.e. $\mathsf{Z} \equiv 0$, the parameter estimate is asymptotically unbiased. In this situation, consistency of the parameter estimate follows from part (b), by realizing that the right-hand side expression of (25) is bounded, and thus the covariance will tend to 0 for increasing N. A similar consistency result will hold for the estimated frequency response of the model.

Proposition 5.3 provides an asymptotic expression for the covariance of the estimated parameters. However, (25) does not provide much insight in the underlying mechanisms. Structurally more simple expressions can be obtained when we let the number N_p of frequencies tend to infinity. This will be discussed in the following sections.

6 Asymptotic bias expression

In this section we will derive expressions for the bias in both the parameter estimate (5) as well as the transfer function estimate (21), for the asymptotic situation that the identification criterion is calculated over an infinite number N_p of frequencies.

In these asymptotic results we have to restrict the choice of frequency grid in such a way that the frequency points are equidistantly spaced.

Assumption 6.1 *The frequency grid $\mathcal{W}_{N_p}^u \subset \Omega_{N_p}^u$ (4), (3) satisfies*

$$\omega_k = \frac{2\pi(k-1)}{N_p} \quad \forall \quad N_p > 0$$

□

We start the asymptotic bias analysis by evaluating the properties of the matrix $\Phi^*\Phi$.

Lemma 6.2 *Consider Assumption 6.1. Then*

$$\frac{1}{N_p}\Phi^*\Phi \to I \quad as \ \frac{n}{N_p} \to 0$$

where the rate of convergence is n/N_p element wise. Additionally, there exists a constant $c_4 \in \mathbb{R}$ such that

$$\|\frac{1}{N_p}\Phi^*\Phi - I\|_2 \leq c_4\frac{n_p n}{N_p}$$

□

This lemma shows that the parameter estimate is numerically very well conditioned in the asymptotic case.
The following theorem addresses the asymptotic bias in the estimated parameters.

Theorem 6.3 *Let the frequency grid $\mathcal{W}_{N_p}^u$ satisfy Assumption 6.1. Then*

$$\lim_{\substack{N_p\to\infty \\ \frac{n_p n}{N_p}\to 0}} \mathbb{E}[\hat{\theta}] = \theta_o$$

□

This theorem shows that asymptotically in the number of time-domain data samples ($N_p \leq N$) and in the number of frequencies over which the identification criterion is defined, the parametrized part of the model can be estimated unbiasedly, despite the possible presence of undermodelling.
For the asymptotic bias in the transfer function estimate we have the following result.

Corollary 6.4 *Consider the situation of Theorem 6.3. Then for all $\omega \in [0, 2\pi)$,*

$$\lim_{\substack{N_p\to\infty \\ \frac{n_p n}{N_p}\to 0}} \left(G_o(e^{i\omega}) - \mathbb{E}[\hat{G}^f(e^{i\omega})]\right) = \sum_{k=n}^{\infty} L_k\mathcal{V}_k(e^{i\omega}).$$

□

Additionally it can be shown that the amplitude of this asymptotic bias term can be bounded by an expression

$$|\sum_{k=n}^{\infty} L_k\mathcal{V}_k(e^{i\omega})| \leq \mathcal{K}|\mathcal{V}_0(e^{i\omega})|\frac{\eta^n}{1-\eta} \qquad (26)$$

with $\mathcal{K} \in \mathbb{R}^{1\times n_b}$ and $1 > \eta \in \mathbb{R}$, while η becomes smaller when the poles that are present in the all-pass function G_b approach the poles of the actual system G_o. In other words, for an appropriately chosen basis, the series expansion (8) will have a high speed of conversion, and so the asymptotic bias in Corollary 6.4 will be accordingly small. For more details on this bound see [22].
Note that the above results are obtained for an infinite number of time and frequency domain data points. Since $N_p = N$ is allowed, it is not required that the input signal is periodic. The condition on the input signal is that it contains N_p different frequency components unequal to zero, while $N_p \to \infty$.

7 Asymptotic variance expressions

For deriving asymptotic variance expressions in the given situation use is made of Lemma 6.2. In the follwoing results we will denote the grouped elements (block elements) of the vector of estimated parameters (10) as

$$\begin{aligned} \hat{\theta}_0 &:= \hat{D} \\ \hat{\theta}_i &:= \hat{L}_{i-1}^T \quad i = 1, \cdots, n. \end{aligned}$$

Theorem 7.1 *Consider Assumptions 5.1, 6.1 and the estimated parameters (5). Then, for $i, k \geq 1$*

$$\begin{aligned} &\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N},\frac{n_p n}{N_p}\to 0}} N \operatorname{cov}[\hat{\theta}_i, \hat{\theta}_k] = \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{V}_0(e^{i\omega})\mathcal{V}_0^*(e^{i\omega})G_b^{i-k}(e^{i\omega})\frac{\Phi_v(\omega)}{\Phi_u(\omega)}d\omega. \quad (27) \end{aligned}$$

For $i \geq 1$ and $k = 0$

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N},\frac{n_p n}{N_p}\to 0}} N \operatorname{cov}[\hat{\theta}_i, \hat{\theta}_k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{V}_0(e^{i\omega})G_b^{i-1}(e^{i\omega})\frac{\Phi_v(\omega)}{\Phi_u(\omega)}d\omega, \qquad (28)$$

and for $i = k = 0$

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N},\frac{n_p n}{N_p}\to 0}} N \operatorname{cov}[\hat{\theta}_i, \hat{\theta}_k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_v(\omega)}{\Phi_u(\omega)}d\omega. \qquad (29)$$

□

The condition $\frac{N_p}{N} \to 0$ in Theorem 7.1, together with Assumption 5.1 or 6.1, imply that the input signal $u(t)$ has to be periodic for $t \in T^N$, with a period length of N_p points. Note that (27), (28) and (29) show that the asymptotic covariance of the estimated parameters does not depend on the number of periods N/N_p in the input signal. Also, the asymptotic covariance does not depend on the number of estimated parameters $n_p = n_b n + 1$, as opposed to the asymptotic covariance of the transfer function estimate given below. Furthermore, it follows directly from Theorem 7.1, and orthonormality of the basis functions $\mathcal{V}_k(e^{i\omega})$, that the estimated parameters are uncorrelated when $\Phi_v(\omega)/\Phi_u(\omega)$ is constant over frequency.
Theorem 6.3 and 7.1 show that the estimated parameters are asymptotically in N_p unbiased and that the variance converges to zero as N goes to infinity. Hence, the parameter estimate is consistent, also in the case of undermodelling. For time domain parameter estimation using a FIR model structure a white noise input signal is necessary to obtain this result, see [16].
For the covariance of the frequency response estimate we have the following result.

Theorem 7.2 *Consider Assumption 5.1, 6.1 and the estimated transfer function* (21). *Let* $\alpha, \beta \in [0, 2\pi)$. *Then*

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N}, \frac{n_p n}{N_p}\to 0}} \frac{N}{n_p} \operatorname{cov}[\hat{G}^f(e^{i\alpha}), \hat{G}^f(e^{i\beta})]$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{P}_n(\alpha,\zeta)\mathcal{P}_n^*(\beta,\zeta)\frac{\Phi_v(\zeta)}{\Phi_u(\zeta)}d\zeta \qquad (30)$$

where

$$\mathcal{P}_n(\omega,\zeta) := \frac{1}{\sqrt{n_p}}\left(1+\mathcal{V}_0^T(e^{i\omega})\mathcal{V}_0(e^{-i\zeta})\frac{\left(\frac{G_b(e^{i\omega})}{G_b(e^{i\zeta})}\right)^n - 1}{\frac{G_b(e^{i\omega})}{G_b(e^{i\zeta})}-1}\right). \qquad (31)$$

Before discussing this result, we will first present the following Corollary.

Corollary 7.3 *Consider the situation of Theorem 7.2 and in addition let* $\Phi_v(\zeta)/\Phi_u(\zeta) = 1$ *for all* $\zeta \in [0, 2\pi)$. *Then*

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N}, \frac{n_p n}{N_p}\to 0}} \frac{N}{n_p} \operatorname{cov}[\hat{G}^f(e^{i\alpha}), \hat{G}^f(e^{i\beta})] = \frac{1}{\sqrt{n_p}}\mathcal{P}_n(\alpha,\beta), \qquad (32)$$

and

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N}, \frac{n_p n}{N_p}\to 0}} \frac{N}{n_p} \operatorname{var}[\hat{G}^f(e^{i\alpha})] = \frac{1+n\|\mathcal{V}_0(e^{i\alpha})\|_2^2}{n_p} \qquad (33)$$

□

Theorem 7.2 provides an expression for the covariance of the estimated transfer function for the situation of a finite model order. At first sight the expressions (30) and (31) may seem a bit complicated; however one has to realize that when given a basis generating system G_b and a model order, the expression (31) can be calculated directly. The fact that an accessible expression is provided for the variance also for finite model orders is opposed to the situation for time-domain prediction error methods, see e.g. [11, 14, 12, 23]. However, in order to be able to establish these expressions, we need a periodic input signal, instead of a quasi-stationary one which is required in the time domain identification case. Note that (30) shows that the asymptotic covariance of the transfer function estimate does not depend on the number of periods N/N_p in the input signal, and that the asymptotic covariance between transfer function estimates at different frequencies is nonzero for finite model orders.

Corollary 7.3 shows that the variance of the transfer function estimate will be relatively large where $\|\mathcal{V}_0(e^{i\omega})\|_2^2$ is large (provided that $\Phi_v(\zeta)/\Phi_u(\zeta)$ is not relatively small for those frequencies). Thus, in general the variance will be relatively large near the poles of the basis generating system, and hence near the poles of the system itself for a properly chosen basis generating system (see [8, 22]). Additionally, noise at frequencies not situated near the poles of the basis generating system (e.g. high frequency noise) usually will be relatively harmless.

The frequency region (with regard to ζ and a fixed ω) where $\mathcal{P}_n(\omega,\zeta)$ is relatively large becomes more concentrated around $\zeta = \omega$ when the model order n increases. If $\Phi_v(\omega)/\Phi_u(\omega)$ is relatively small, then an increase of the model order can very well lead to a reduction of the variance in this frequency region. It follows that, up to a certain model order, variance considerations do not conflict with bias considerations for those frequency regions where $\Phi_v(\zeta)/\Phi_u(\zeta)$ is relatively small. This is a somewhat surprising and (to the authors' best knowledge) new result. The existing literature only addresses the situation where the model order n goes to infinity also. In that case the variance always increases with the model order, see e.g. [12] and Theorem 7.6 below.

Remark 7.4 *The results obtained in this and the previous section are in accordance with and actually are further generalizations of the results obtained in* [3], *where a similar identification problem is analysed for an integral identification criterion* ($N_p \to \infty$) *and Laguerre basis functions, and where no undermodelling is considered.*

Similar to the situation for the time-domain identification of finite expansion models, we can arrive at simple expressions for the asymptotic variance of the estimated parameters and transfer function in the situation that the model order n goes to infinity also.

In order to formulate this result we will build upon the expression (27) and we will show that this can be formulated in a simple way, utilizing an orthonormal transformation that is induced by the basis functions that we apply. This transformation is introduced in [22].

Definition 7.5 (Hambo transform [22].) *Let* $\Phi(w)$ *be a rational spectral density function with stable spectral factor* $H(e^{i\omega})$, *and let* G_b *be a scalar inner function with balanced realization* (A, B, C, D) *that induces an orthonormal basis* $V_k(z)$ *for* $\mathcal{H}_2$. *Then the Hambo transformed spectrum* $\widetilde{\Phi}(\omega)$ *is defined by*

$$\begin{aligned} \widetilde{\Phi}(\omega) &:= \widetilde{H}^T(e^{-i\omega})\widetilde{H}(e^{i\omega}) \quad \textit{with} \\ \widetilde{H}(\lambda) &= H(z)|_{z^{-1}=N(\lambda)} \quad \textit{and} \\ N(\lambda) &:= A + B(\lambda - D)^{-1}C. \end{aligned}$$

□

Applying this orthonormal transformation to the result (27), then an analysis similar to the one applied in the time-domain case ([22]), leads to

$$\lim_{\substack{N,N_p\to\infty \\ \frac{N_p}{N}, \frac{n_p n}{N_p}\to 0}} N \operatorname{cov}[\hat{\theta}_i, \hat{\theta}_k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{\Phi}_v(\omega)\widetilde{\Phi}_u(\omega)^{-1}e^{i\omega(i-k)}d\omega,$$

showing that this expression equals $R(i-k)$, with $R(\tau)$ the (matrix) covariance function related to the $n_b \times n_b$ spectral density function $\widetilde{\Phi}_v(\omega)\widetilde{\Phi}_u(\omega)^{-1}$.
Based on this result we can now formulate the following theorem, showing an asymptotic variance expression that is similar to the related time-domain result.

Theorem 7.6 *Consider Assumptions 5.1, 6.1 and the estimated transfer function* (21). *Let* $\alpha, \beta \in [0, 2\pi)$. *Then*

$$\lim_{\substack{N,N_p,n\to\infty \\ \frac{N_p}{N}, \frac{n_p n}{N_p}\to 0}} \frac{N}{n_p} \; cov[\hat{G}^f(e^{i\alpha}) - \hat{D}, \hat{G}^f(e^{i\beta}) - \hat{D}] =$$
$$\begin{cases} 0 & \textit{for } G_b(e^{i\alpha}) \neq G_b(e^{i\beta}), \\ \frac{1}{n_b}V_0^T(e^{i\alpha})V_0(e^{-i\alpha})\frac{\Phi_v(\alpha)}{\Phi_u(\alpha)} & \textit{for } \alpha = \beta. \end{cases}$$

□

The asymptotic-in-order result shows a very simple expression for the asymptotic variance of the estimated transfer function, where the asymptotic variance at a specific frequency is equal to a frequency weighted noise-to-signal ratio at that frequency. The frequency weighting is completely determined by the basis functions that are used in our model structure. This result exactly matches the equivalent form that has been derived for time-domain identification [22]. However in the current situation a periodic input signal is required instead of a general quasi-stationary one.

8 Conclusions

We have obtained asymptotic bias and variance expressions for the estimated parameters and transfer function in an identification problem that is based on time-domain data and an identification criterion that is formulated in the frequency domain. First the situation is considered of infinite time-domain data and a criterion over a finite number of frequencies. Additionally, the number of frequencies is taken to go to infinity also. As a model structure, a linear parametrization in terms of recently introduced generalized basis functions is employed, being a generalization of the classical pulse, Laguerre and Kautz type bases.

For the infinite-frequency criterion, it is shown that the parametrized part of the model can be estimated consistently, despite of the presence of neglected dynamics (undermodelling). Variance expressions are derived for situations of finite and infinite order models. Especially the finite order results are of considerable practical siginifance. For this situation there is no counterpart result known in time-domain prediction error identification. The results that are derived for the infinite order case, exactly match the available results for the time-domain case.

Note that the model error bounds established in [4, 5] are based on exactly the same identification setup as the one used in this paper. The results presented here therefore can be used to guide the crucial design choices mentioned above in order to minimize the error bounds.

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