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An Instrumental Variable Procedure for the Identification of Probabilistic Frequency Response Uncertainty Regions

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Abstract

A procedure is developed to identify probabilistic frequency response system uncertainty regions. The procedure utilizes time-domain measurement data and prior information about the system and the noise. There are no restrictions on the input signal, it may even be generated in closed loop. The system is assumed to be linear, time invariant, and a bound is assumed on the system's (generalized) pulse response parameters. The noise is assumed to be a realization of a stationary stochastic process, and independent of the input signal (in open loop operation) or an external reference signal (in closed loop operation). Frequency response confidence regions are constructed by explicitly evaluating the bias and variance errors of an instrumental variable estimate.

1 Introduction

For robust controller analysis and synthesis it is necessary to have available a bound on the model error, the difference between plant and nominal model. For example robust stability can be established if frequency response uncertainty regions are available. Many authors have considered the problem of deriving frequency response system uncertainty regions on the basis of measurement data and prior assumptions about system and noise. The two main different uncertainty bounding approaches are the deterministic and the stochastic approach.

Procedures to derive frequency response uncertainty regions based on deterministic prior assumptions are presented in for example [2, 4, 5, 7, 12, 18]. In particular the noise is assumed to behave worst-case deterministic. The resulting uncertainty regions are correct provided the prior information that is used is correct. Unfortunately in practice it is often rather difficult, if not impossible, to guarantee that the priors, such as assumed noise bounds, are correct.

The stochastic approach is represented by e.g. [1, 3, 16]. In this approach the noise is assumed to behave noisy, i.e. random and uncorrelated to the input signal. Besides that also prior assumptions about the system are made, which vary from deterministic, [3], to stochastic, [16]. Typically these procedures yield uncertainty regions which are correct with a certain specified probability, provided the prior assumptions that are made are correct.

In this paper a new procedure is presented to identify probabilistic frequency response uncertainty regions. The procedure involves the explicit calculation of the bias and variance errors of an IV (Instrumental Variable) estimate. A linear model parametrization in terms of general basis functions is used, see [9] and [15, Ch. 2]. In this way approximate knowledge about pole locations of the unknown system can be incorporated by the choice of proper basis functions. In fact the present procedure is the statistical counterpart of the deterministic uncertainty bounding procedure described in [5]. There are no restrictions on the input signal, it need for example not be sinusoidal. The basic assumption about the noise process is that it is stationary and independent of the input signal in open loop, or an external reference signal in closed loop. The probability density function of the noise process is arbitrary and not assumed to be known. Instead asymptotic results are derived with a central limit theorem.

The present approach is different from the one in [1], where a multisinusoidal input signal is needed, and the noise is assumed gaussian with known noise generating filter. Unlike in [16] no stochastic assumptions are made about the undermodelling part. In the approach of [3] a frequency domain approach is taken, and also a periodic input signal is needed.

The outline of the paper is as follows. In the next section the identification setting is described. Section 3 presents the instrumental variable estimate. In Section 4 the frequency response error of the IV model is evaluated, which leads to probabilistic frequency response system uncertainty regions. In Section 5 the results are discussed.

Because of space limitations all proofs have been omitted. These can be found in [6]. In this reference also simulations and an application of the identification procedure to a multivariable industrial process can be found.

2 Identification Setting

Consider the linear, time-invariant, discrete time, causal and ℓ_∞-stable SISO system $G_0(z)$ represented by

$$G_0(z) = \sum_{k=0}^{\infty} g_0(k) P_k(z),$$

where $\{P_k(z)\}_{k=0,\dots,\infty}$, is some specified set of basis functions given by

$$P_k(z) = \sum_{k'=0}^{\infty} p_k(k') z^{-k'},\ k = 0, \dots, \infty,$$

for given and known scalar pulse response parameters $p_k(k')$. These basis functions can for example chosen to be the pulse functions, or the Laguerre functions, or general orthonormal basis functions, see [9] and [15, Ch. 2]. The (unknown) coefficients $g_0(k)$ can be considered as generalized pulse response parameters of the system $G_0(z)$.

Consider given input data $\{u(t)\}_{t=1,\ldots,N}$ and measured output data $\{y(t)\}_{t=1,\ldots,N}$ and the following input-output relation of the data generating system,

$$y(t) = G_0(q)u(t) + e_0(t),\ t = 1,\ldots,N, \tag{1}$$

where N denotes the measurement time and $\{e_0(t)\}$ is an unknown additive output noise. There are no restrictions on the input signal, basically it may be determined in open loop as well as in closed loop.

It is assumed that a signal $\{r(t)\}_{t=1,\ldots,N}$ is available, which is highly correlated with the input signal $\{u(t)\}$, but independent of the noise process $\{e_0(t)\}$. Let by definition $r(t) = 0$ for $t \leq 0$. Typically in open loop operation the signal $\{r(t)\}$ is equal to the input $\{u(t)\}$. In a closed loop environment an external reference signal $\{\bar{r}(t)\}$ can be used, or a filtered version of this signal, $r(t) = F(q)\bar{r}(t)$.

The following assumptions are made about the noise process $\{e_0(t)\}$.

Assumption 2.1 *The noise process $\{e_0(t)\}$ is stationary with auto-covariance function $R_{e_0}(\tau) = Ee_0(t+\tau)e_0(t)$, and it satisfies $e_0(t) = H_0(q)w_0(t)$ for some ℓ_2-stable $H_0(q)$, and where $\{w_0(t)\}$ is a sequence of independent random variables with zero mean values, variances λ_0, and bounded fourth moments.*

Note that the distribution of the noise process is arbitrary, and not assumed to be known. The following assumptions about $\{r(t)\}$ are made.

Assumption 2.2 *The signal $\{r(t)\}$ is a bounded deterministic quasi-stationary signal, hence its auto-covariance function*

$$R_r(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} r(t+\tau)r(t)$$

exists $\forall\,\tau$.

In order to cope with unknown initial conditions the input signal in the past is assumed to be bounded by

$$|u(t)| \leq \bar{u},\ \forall\, t \leq 0, \tag{2}$$

for some given $\bar{u}$. This bound may result from actuator constraints and need not be very tight as its influence on the identification result is restricted.

The coefficients $g_0(k)$ are assumed to be bounded by

$$|g_0(k)| \leq \bar{g}(k),\ k = 0,\ldots,\infty, \tag{3}$$

for given $\bar{g}(k)$. Moreover it is assumed that the bound $\bar{g}(k)$ shows exponential decay rate for k larger than some k^*, i.e.

$$\bar{g}(k) \leq M\rho^k,\ \forall\, k > k^*,$$

for some given $M \geq 0$ and $\rho < 1$. In [9] it is discussed that such a bound exists when an arbitrary ℓ_∞-stable system is expanded in a general orthonormal basis.

The identification objective is to derive probabilistic uncertainty regions for the system's frequency response,

$$G_0(e^{i\omega}) = \sum_{k=0}^{\infty} g_0(k)P_k(e^{i\omega}).$$

The identification problem is tackled by splitting the transfer function $G_0(z)$ into two parts,

$$G_0(z) = \tilde{G}_0(z) + \bar{G}_0(z), \tag{4}$$

$$\tilde{G}_0(z) = \sum_{k=0}^{n} g_0(k)P_k(z),\ \bar{G}_0(z) = \sum_{k=n+1}^{\infty} g_0(k)P_k(z),$$

for some user-defined truncation value n.

Next deterministic uncertainty bounds will be determined for the tail $\bar{G}_0(e^{i\omega})$, using the deterministic prior bounds $\bar{g}(k)$ given in (3). And probabilistic uncertainty bounds will be derived for $\tilde{G}_0(e^{i\omega})$, using variance expressions of an instrumental variable estimate. These variance expressions are based on the stochastic noise assumption 2.1. In the variance expressions the influence of the undermodelling part $\bar{G}_0(z)$ is properly taken into account. The sum of the deterministic uncertainty bounds for $\bar{G}_0(e^{i\omega})$ and the probabilistic uncertainty bounds for $\tilde{G}_0(e^{i\omega})$ provides probabilistic uncertainty regions for the system $G_0(e^{i\omega})$.

Note that there generally is an optimal value for n. If it is chosen too small, the resulting bounds will be completely determined by the prior information (3), which is generally conservative. If it is chosen too large, the confidence regions for $\tilde{G}_0(z)$ will be large as the variance increases with the number of parameters to be estimated. More will be said about this later.

3 The Instrumental Variable Estimate

Consider the parametrized model

$$G(z) = \sum_{k=0}^{n} g(k)P_k(z),$$

where $\{g(k)\}_{k=0,\ldots,n}$ are the model parameters. Define the model input signal $\tilde{u}(t)$ as

$$\tilde{u}(t) := \begin{cases} 0, & t \leq 0, \\ u(t), & t = 1,\ldots,N. \end{cases}$$

The model input-output relation is given by

$$\begin{aligned} y(t) &= G(q)\tilde{u}(t) + e(t) = \sum_{k=0}^{n} g(k)P_k(q)\tilde{u}(t) + e(t) = \\ &= \sum_{k=0}^{n} g(k)w_k(t) + e(t), \end{aligned}$$

where $e(t)$ is the output error, and

$$w_k(t) := \sum_{k'=0}^{t-1} p_k(k')u(t-k'),\ k = 0,\ldots,n. \tag{5}$$

Next define the instrumental signals

$$v_k(t) := P_k(q)r(t) = \sum_{k'=0}^{t-1} p_k(k')r(t-k'),\ k = 0,\ldots,n,$$

and the matrices

$$W(t) := \begin{bmatrix} w_0(t) \\ \vdots \\ w_n(t) \end{bmatrix}, \; V(t) := \begin{bmatrix} v_0(t) \\ \vdots \\ v_n(t) \end{bmatrix}.$$

Also denote

$$\widetilde{N} = N - t_s + 1,$$

for some integer $t_s \in [1, N)$, which is user-defined. The integer t_s represents the starting sample used in the IV estimate, and can be used to reduce the influence of the unknown initial conditions, as will become clear later.

Consider the basic IV estimate ([17, p. 262], [13, p. 192/193]),

$$\begin{bmatrix} \widehat{g}(0) \\ \vdots \\ \widehat{g}(n) \end{bmatrix} = \mathrm{sol}\left\{ \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) e(t) = 0 \right\},$$

which is given by

$$\begin{bmatrix} \widehat{g}(0) \\ \vdots \\ \widehat{g}(n) \end{bmatrix} = \left[\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \right]^{-1} \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) y(t). \quad (6)$$

Notice that in case of open loop operation, $r(t) = u(t)$, this is just a FIR least squares estimate for general basis functions. The estimated IV model is given by

$$\widehat{G}(z) = \sum_{k=0}^{n} \widehat{g}(k) P_k(z).$$

This identified model is used to construct frequency response uncertainty regions. This is done by explicitly calculating the bias and variance errors of the IV estimate.

4 Frequency Response Uncertainty Regions

An analysis is made of the frequency response identification error of the instrumental variable estimate. This then leads to frequency response confidence regions for the system $G_0(z)$.

4.1 The Frequency Response Error of the IV Model

Consider some frequency ω_j chosen arbitrarily in the interval $[0, \pi]$. Substitution of the parameter estimate (6) yields the frequency response of the IV estimate,

$$\begin{aligned} \widehat{G}(e^{i\omega_j}) &= \sum_{k=0}^{n} \widehat{g}(k) P_k(e^{i\omega_j}) = \left[P_0(e^{i\omega_j}) \cdots P_n(e^{i\omega_j}) \right] \cdot \\ &\quad \cdot \left[\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \right]^{-1} \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) y(t). \quad (7) \end{aligned}$$

Define for $t = t_s, \ldots, N$ the signals $r_1(t)$ and $r_2(t)$ as

$$\begin{aligned} r_1(t) &:= \left[\mathrm{Re}\left(P_0(e^{i\omega_j}) \right) \cdots \mathrm{Re}\left(P_n(e^{i\omega_j}) \right) \right] \cdot \\ &\quad \cdot \left[\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \right]^{-1} V(t), \quad (8) \end{aligned}$$

$$\begin{aligned} r_2(t) &:= \left[\mathrm{Im}\left(P_0(e^{i\omega_j}) \right) \cdots \mathrm{Im}\left(P_n(e^{i\omega_j}) \right) \right] \cdot \\ &\quad \cdot \left[\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \right]^{-1} V(t). \quad (9) \end{aligned}$$

These signals $r_p(t)$, $p = 1, 2$, are filtered versions of the signal $r(t)$, and they can be computed, as they only depend on known quantities. They play an essential role throughout the following derivation of IV model error bounds. Note that they depend on the frequency ω_j that has been chosen, but for notational convenience this dependency is not explicitly mentioned all the time.

Using (1) and (4) the output $y(t)$ can be written as

$$\begin{aligned} y(t) &= G_0(q) u(t) + e_0(t) = \widetilde{G}_0(q) u(t) + \bar{G}_0(q) u(t) + e_0(t) \\ &= \sum_{k=0}^{n} g_0(k) P_k(q) u(t) + \sum_{k=n+1}^{\infty} g_0(k) P_k(q) u(t) + e_0(t) \\ &= \sum_{k=0}^{n} g_0(k) w_k(t) + a(t) + b(t) + e_0(t), \end{aligned}$$

where $w_k(t)$ is defined in (5) and

$$a(t) := \sum_{k=n+1}^{\infty} g_0(k) \sum_{k'=0}^{t-1} p_k(k') u(t-k'), \quad (10)$$

$$b(t) := \sum_{k=0}^{\infty} g_0(k) \sum_{k'=t}^{\infty} p_k(k') u(t-k'). \quad (11)$$

The signal $a(t)$ represents the response of the tail $\bar{G}_0(q)$. The signal $b(t)$ represents the response due to past input signals, the initial conditions. Using this the following alternative expression can be given for $\widehat{G}(e^{i\omega_j})$ given by (7),

$$\begin{aligned} \widehat{G}(e^{i\omega_j}) &= \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} (r_1(t) + i r_2(t)) y(t) = \\ &= \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} (r_1(t) + i r_2(t)) \cdot \\ &\quad \cdot \left(\sum_{k=0}^{n} g_0(k) w_k(t) + a(t) + b(t) + e_0(t) \right). \end{aligned}$$

The first term of this expression can be worked out as follows,

$$\begin{aligned} &\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} (r_1(t) + i r_2(t)) \sum_{k=0}^{n} g_0(k) w_k(t) = \\ &= \left[P_0(e^{i\omega_j}) \cdots P_n(e^{i\omega_j}) \right] \cdot \left[\frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \right]^{-1} \cdot \\ &\quad \cdot \frac{1}{\widetilde{N}} \sum_{t=t_s}^{N} V(t) W^T(t) \begin{bmatrix} g_0(0) \\ \vdots \\ g_0(n) \end{bmatrix} = \\ &= \left[P_0(e^{i\omega_j}) \cdots P_n(e^{i\omega_j}) \right] \begin{bmatrix} g_0(0) \\ \vdots \\ g_0(n) \end{bmatrix} = \widetilde{G}_0(e^{i\omega_j}). \end{aligned}$$

Next define for $p = 1, 2$,

$$d(p) := \sum_{t=t_s}^{N} r_p(t) a(t), \quad (12)$$

$$f(p) := \sum_{t=t_s}^{N} r_p(t) b(t), \quad (13)$$

which depend on the frequency ω_j as $r_p(t)$, $p=1,2$, depends on the frequency ω_j. Again using (4) this finally gives the following expression for the identification error,

$$\begin{aligned}&\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j}) = \\ &= \widehat{G}(e^{i\omega_j}) - \tilde{G}_0(e^{i\omega_j}) - \bar{G}_0(e^{i\omega_j}) = \\ &= \frac{1}{\widetilde{N}}(d(1) + id(2) + f(1) + if(2) + \\ &\quad + \sum_{t=t_s}^{N}(r_1(t) + ir_2(t))e_0(t)) - \bar{G}_0(e^{i\omega_j}). \qquad (14)\end{aligned}$$

Basically all terms at the right-hand side of this expression are unknown. However, it appears possible to derive a probabilistic distribution for the term containing $e_0(t)$, using assumption 2.1. And the terms with $d(p)$, $f(p)$ and $\bar{G}_0(e^{i\omega_j})$ can be bounded using the prior information (2) and (3).

4.2 Auxiliary Results

In this subsection the various terms appearing in (14) are evaluated. Consider any bounded signal $\{r_p(t)\}$ and consider $\bar{d}(p)$, $\bar{f}(p)$ defined by (12), (13) respectively, with $a(t)$, $b(t)$ defined by (10), (11) respectively. Making use of (2) and (3), the following bounds can be derived,

$$|d(p)| \le \bar{d}(p) := \sum_{k=n+1}^{\infty} \bar{g}(k) \left| \sum_{t=t_s}^{N} r_p(t) \sum_{k'=0}^{t-1} p_k(k') u(t-k') \right|, \qquad (15)$$

which represents a computable bound for the tail contribution. And,

$$|f(p)| \le \bar{f}(p) := \sum_{k=0}^{\infty} \bar{g}(k) \sum_{t'=0}^{\infty} \left| \sum_{t=t_s}^{N} r_p(t) p_k(t+t') \right| \bar{u}, \quad (16)$$

which represents a computable bound for the contribution of the unknown initial conditions. The actual computation of the expressions involve the evaluation of infinite sums. Due to the fact that $\bar{g}(k)$ shows exponential decay rate in k, and $p_k(k')$ shows exponential decay rate in k' the outcomes are finite. Computational aspects are considered in [6]. Clearly $\bar{d}(p)$ will be small if n is chosen large, and $\bar{f}(p)$ will be small if t_s is chosen large.

The real and imaginary part of the frequency response of the tail, $\bar{G}_0(e^{i\omega_j})$, can be bounded as follows,

$$\begin{aligned}\left|\mathrm{Re}\left(\bar{G}_0(e^{i\omega_j})\right)\right| &= \left|\mathrm{Re}\left(\sum_{k=n+1}^{\infty} g_0(k) P_k(e^{i\omega_j})\right)\right| \le \\ &\le \sum_{k=n+1}^{\infty} \bar{g}(k) \left|\mathrm{Re}\left(P_k(e^{i\omega_j})\right)\right| := \delta(1), \qquad (17)\end{aligned}$$

$$\left|\mathrm{Im}\left(\bar{G}_0(e^{i\omega_j})\right)\right| \le \sum_{k=n+1}^{\infty} \bar{g}(k) \left|\mathrm{Im}\left(P_k(e^{i\omega_j})\right)\right| := \delta(2). \quad (18)$$

Note that $\delta(1)$ and $\delta(2)$ are finite due to the exponential decay rate of $\bar{g}(k)$. Computational aspects of the evaluation of these infinite sums are considered in [6].

Next a key lemma is established with respect to the asymptotic distribution of $\sum_{t=t_s}^{N} r_p(t) e_0(t)$.

Lemma 4.1 *Suppose that $\{e_0(t)\}$ and $\{r(t)\}$ are independent and that they satisfy the assumptions 2.1 and 2.2 respectively. Consider the signals $\{r_1(t)\}$ and $\{r_2(t)\}$, given by $r_1(t) = F_1(q)r(t)$, $r_2(t) = F_2(q)r(t)$ for any ℓ_∞-stable linear filters $F_1(q)$ and $F_2(q)$. Denote*

$$\Lambda^N_{r_1 r_2} := E\frac{1}{\widetilde{N}} \begin{bmatrix} \sum_{t=t_s}^{N} r_1(t)e_0(t) \\ \sum_{t=t_s}^{N} r_2(t)e_0(t) \end{bmatrix} \left[\sum_{t=t_s}^{N} r_1(t)e_0(t) \quad \sum_{t=t_s}^{N} r_2(t)e_0(t) \right],$$

and

$$\Lambda_{r_1 r_2} := \lim_{N\to\infty} \Lambda^N_{r_1 r_2}.$$

Also denote for $i,j = 1,2$,

$$R^N_{r_i r_j}(\tau) := \frac{1}{\widetilde{N}+\tau} \sum_{t=t_s}^{N+\tau} r_i(t) r_j(t-\tau), \ \tau = -N+t_s, \dots, 0,$$

$$R^N_{r_i r_j}(\tau) := \frac{1}{\widetilde{N}-\tau} \sum_{t=t_s}^{N-\tau} r_i(t+\tau) r_j(t), \ \tau = 1, \dots, N-t_s,$$

and $R^N_{r_i}(\tau) := R^N_{r_i r_i}(\tau)$, $i = 1,2$. Then

(i) $$\Lambda^N_{r_1 r_2} = \sum_{\tau=-N+t_s}^{N-t_s} \frac{\widetilde{N}-|\tau|}{\widetilde{N}} R_{e_0}(\tau) \begin{bmatrix} R^N_{r_1}(\tau) & R^N_{r_1 r_2}(\tau) \\ R^N_{r_1 r_2}(\tau) & R^N_{r_2}(\tau) \end{bmatrix}$$

(ii) $$\Lambda_{r_1 r_2} = \sum_{\tau=-\infty}^{\infty} R_{e_0}(\tau) \begin{bmatrix} R_{r_1}(\tau) & R_{r_1 r_2}(\tau) \\ R_{r_1 r_2}(\tau) & R_{r_2}(\tau) \end{bmatrix}$$

(iii) $$\frac{1}{\sqrt{\widetilde{N}}} \begin{bmatrix} \sum_{t=t_s}^{N} r_1(t)e_0(t) \\ \sum_{t=t_s}^{N} r_2(t)e_0(t) \end{bmatrix} \xrightarrow{N\to\infty} \mathcal{N}(0, \Lambda_{r_1 r_2}),$$

where $\mathcal{N}(0, \Lambda_{r_1 r_2})$ denotes the Multivariate Normal distribution with mean 0 and covariance matrix $\Lambda_{r_1 r_2}$. Moreover, if $\Lambda^N_{r_1 r_2}$ is invertible,

(iv) $$\frac{1}{\widetilde{N}} \left[\sum_{t=t_s}^{N} r_1(t)e_0(t) \quad \sum_{t=t_s}^{N} r_2(t)e_0(t) \right] \left(\Lambda^N_{r_1 r_2}\right)^{-1} \cdot \begin{bmatrix} \sum_{t=t_s}^{N} r_1(t)e_0(t) \\ \sum_{t=t_s}^{N} r_2(t)e_0(t) \end{bmatrix} \xrightarrow{N\to\infty} \chi^2(2),$$

where $\chi^2(2)$ denotes the Chi-square distribution with 2 degrees of freedom.

The results given in (iii) and (iv) are asymptotic results, established using a central limit theorem. For finite N the given distributions are approximations of the true ones. However, extensive monte carlo simulations show that this approximation can be very good for small N already, see [6]. Note that the expression for the covariance matrix in part (i) is a non-asymptotic result, it is correct for any N.

4.3 Frequency Response Confidence Regions

Using the results of the previous subsection a computable bound for the IV model error $\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})$ is straightforwardly obtained. And as such a confidence region for the system's frequency response $G_0(e^{i\omega_j})$ is obtained. The bound is given in the following main theorem.

Theorem 4.2 *Consider the IV estimate (6) with frequency response $\widehat{G}(e^{i\omega_j})$ given by (7). Suppose that $\{e_0(t)\}$ and $\{r(t)\}$ are independent and that they satisfy the assumptions 2.1 and 2.2 respectively. Let $\bar{d}(p)$, $p = 1, 2$, and $\bar{f}(p)$, $p = 1, 2$, be given by (15) and (16) respectively, with $r_1(t)$ and $r_2(t)$ given by (8) and (9) respectively. Moreover, let $\delta(p)$, $p = 1, 2$, be given by (17) and (18).*

Let $c_{\mathcal{N},\alpha}$ correspond to a probability α in the standard Normal distribution, such that, if $x \in \mathcal{N}(0,1) \Rightarrow \text{prob}(|x| \leq c_{\mathcal{N},\alpha}) = \alpha$. Let $c_{\chi,\alpha}$ correspond to a probability α in the Chi-square distribution with 2 degrees of freedom, such that, if $x \in \chi^2(2) \Rightarrow \text{prob}(x \leq c_{\chi,\alpha}) = \alpha$.

Denote matrix-element (i,j) of $\Lambda^N_{r_1r_2}$ as given in part (i) of Lemma 4.1 by $\lambda^N_{r_ir_j}$. Moreover introduce $\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$ as the square-root of the inverse of $\Lambda^N_{r_1r_2}$, provided this matrix is invertible, i.e. $\Gamma^T\Gamma = \left(\Lambda^N_{r_1r_2}\right)^{-1}$. Then, if $N \to \infty$,

$$\textbf{(i)} \qquad \left|\text{Re}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right)\right| \leq$$
$$\leq c_{\mathcal{N},\alpha}\sqrt{\frac{\lambda^N_{r_1r_1}}{\widetilde{N}}} + \frac{\bar{d}(1)}{\widetilde{N}} + \frac{\bar{f}(1)}{\widetilde{N}} + \delta(1), \ \textit{w.p.} \geq \alpha,$$
$$\textbf{(ii)} \qquad \left|\text{Im}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right)\right| \leq$$
$$\leq c_{\mathcal{N},\alpha}\sqrt{\frac{\lambda^N_{r_2r_2}}{\widetilde{N}}} + \frac{\bar{d}(2)}{\widetilde{N}} + \frac{\bar{f}(2)}{\widetilde{N}} + \delta(2), \ \textit{w.p.} \geq \alpha.$$

And, if $\Lambda^N_{r_1r_2}$ is invertible,

$$\textbf{(iii)} \qquad \begin{bmatrix} \text{Re}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right) \\ \text{Im}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right) \end{bmatrix}^T \Gamma^T \cdot$$
$$\cdot\Gamma \begin{bmatrix} \text{Re}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right) \\ \text{Im}\left(\widehat{G}(e^{i\omega_j}) - G_0(e^{i\omega_j})\right) \end{bmatrix} \leq$$
$$\leq \left(\sqrt{\frac{c_{\chi,\alpha}}{\widetilde{N}}} + \sqrt{\gamma_{11}^2 + \gamma_{21}^2}\left(\frac{\bar{d}(1)}{\widetilde{N}} + \frac{\bar{f}(1)}{\widetilde{N}} + \delta(1)\right) + \right.$$
$$\left. + \sqrt{\gamma_{12}^2 + \gamma_{22}^2}\left(\frac{\bar{d}(2)}{\widetilde{N}} + \frac{\bar{f}(2)}{\widetilde{N}} + \delta(2)\right)\right)^2, \ \textit{w.p.} \geq \alpha,$$

The parts (i) and (ii) of this theorem provide probabilistic bounds for the real and imaginary parts of the IV model error, and as such for the frequency response of the system $G_0(z)$. These may be combined into rectangular system confidence regions in the complex plane using Bonferroni's inequality, [14, p. 49]. In particular, if any complex-valued random variable x has the property that $\text{Re}(x) \leq a$, w.p. $\geq \alpha$, and $\text{Im}(x) \leq b$, w.p. $\geq \beta$, then $\text{Re}(x) \leq a \wedge \text{Im}(x) \leq b$, w.p. $\geq 1 - (1-\alpha) - (1-\beta)$.

Ellipsoidal system confidence regions are obtained with part (iii) of the above theorem, provided the matrix $\Lambda^N_{r_1r_2}$ is invertible. Note that this is generally the case, except for frequencies $\omega_j = 0, \pi$. For these frequencies the signal $\{r_2(t)\}$ is identically zero, as $\text{Im}\left(P_k(e^{i\omega_j})\right)$ appearing in (9) is zero. This very naturally means that for frequencies 0 and π there is no imaginary system uncertainty.

The first contribution to the frequency response uncertainty regions as specified in Theorem 4.2, corresponds to the variance of the IV model, due to the noise $\{e_0(t)\}$. The second contribution, with $\bar{d}(p)$, is due to the response of the tail $\bar{G}_0(q)$, and represents a bias contribution. The third contribution, with $\bar{f}(p)$, is due to the unknown initial conditions. Finally, the fourth contribution, with $\delta(p)$, corresponds to the frequency response of the tail $\bar{G}_0(q)$, and also represents a bias contribution.

The different error sources in the IV estimate can be clearly distinguished and traded-off. In particular the truncation value n can be used to make a trade-off between bias and variance. A larger value n means a smaller bias, but a larger variance. By trying different values an optimal value can be determined. Similarly the integer t_s offers the possibility to trade-off the influence of initial conditions to the variance. A larger value t_s means a decrease of the error contribution $\bar{f}(p)$, but an increase of the variance, due to a decreasing $\widetilde{N} = N - t_s + 1$.

It is emphasized that the identification of the IV model is not a purpose in itself, but serves as a basis for the construction of system uncertainty regions. The design variables in the IV identification, such as the IV model order n, should not be used to obtain a tractable (low-order) nominal model, but should be tuned in such a way that the uncertainty regions are as small as possible. The identification of a good nominal model, suited for use in control design, is not the issue here.

Remark 4.3 *The probabilistic uncertainty regions given in Theorem 4.2 correspond to an explicit frequency domain variance and bias expression for an instrumental variable estimate $\widehat{G}(e^{i\omega_j})$. In case of open loop identification, if $r(t) = u(t)$, the IV estimate is identical to a FIR least squares estimate. The expressions have been derived for any set of basis functions, $\{P_k(z)\}_{k=0,\ldots,\infty}$. Also the contribution of the initial conditions and undermodelling are properly taken into account.*

In literature variance expressions are given for IV and FIR estimates, however mainly with respect to the parameter variance, assuming that the system is in the model set, and neglecting the influence of the initial conditions, see for example [13, Ch. 9] *and* [17, Ch. 8]. *Some progress has been made in* [10, 11], *where for a different identification setting a procedure is presented to incorporate the influence of the bias when computing the variance.*

Theorem 4.2 provides frequency response confidence regions for the unknown system $G_0(z)$. However, it appears that these can only be calculated if the auto-covariance function of the noise process is known, as $\Lambda^N_{r_1r_2}$ given in part (i) of Lemma 4.1 contains $R_{e_0}(\tau)$, $\tau = -N+t_s, \ldots, N-t_s$. In [8] a procedure is described to estimate the auto-covariance function $R_{e_0}(\tau)$ from measurement data. In [6] it is shown, by means of monte carlo simulations, that this estimate is quite accurate, even if it is based on a small amount of data.

5 Discussion

In this paper an identification procedure has been developed which yields confidence regions for the frequency response of some stable LTI system. The procedure involves the explicit calculation of bias and variance errors of an IV or FIR least squares estimate. Important features of the identification procedure are:

- Essentially the procedure is stochastic. Probabilistic uncertainty regions are calculated based on data, deterministic system priors, and stochastic noise priors.
- The actual computations can be performed quite efficiently. No nonlinear optimizations are involved, as use is made of a linear system parametrization, and consequently there is no problem with local optima.
- The required prior information can be reliably estimated from data.
- There are no restrictions on the input signal, it need for example not be periodic. It is even not necessary that the input is generated in open loop.
- No order assumption about the system is made.
- The procedure is easily extendable to MIMO systems.
- Rough prior knowledge about the system, or more specifically pole-locations, can be incorporated by using generalized orthonormal basis functions.
- Unknown initial conditions are properly taken into account.
- The identification procedure is robust for noise outliers, and small errors in the prior information. This means for example that if the system has a small nonlinearity (measured in terms of its ℓ_∞-induced norm), the resulting uncertainty regions are just slightly erratic, and hence are still (approximately) valid.

On the other hand some drawbacks of the probabilistic uncertainty bounding identificiation procedure developed in this paper, are:

- Although all computations can be carried out efficiently and accurately, the identification procedure requires a lot of computations. This means that on-line application of the procedure seems infeasible.
- The procedure makes use of results which are asymptotic in the number of data. As in applications there are always finite-data records, the results might not be valid in practice. On the other hand, monte carlo simulations ([6]) show that the error caused by the finiteness of the number of data can be very small, even for small values of N. The acccuracy of the finite-data approximation depends on several factors, such as the length of the pulse response of the noise generating filter, and the actual distribution of the noise process.

References

[1] D.S. Bayard, "Statistical plant set estimation using Schroeder-phased multisinusoidal input design", *Proc. Am. Contr. Conf.*, Chicago, pp. 2988–2995, 1992.

[2] D.K. de Vries and P.M.J. Van den Hof, "Quantification of model uncertainty from data: input design, interpolation and connection with robust control design specifications", *Proc. Am. Contr. Conf.*, Chicago, pp. 3170–3175, 1992.

[3] D.K. de Vries and P.M.J. Van den Hof, "Quantification of uncertainty in transfer function estimation: a mixed deterministic-probabilistic approach", *Prepr. Proc. 12th IFAC World Congress*, Sydney, Australia, Vol. 8, pp. 157–160, 1993.

[4] R.G. Hakvoort, "Worst-case system identification in H_∞: error bounds and optimal models", *Prepr. IFAC World Congress*, Sydney, Australia, Vol. 8, pp. 161–164, 1993.

[5] R.G. Hakvoort, "A linear programming approach to the identification of frequency domain error bounds", *Prepr. Proc. SYSID'94, 10th IFAC Symp. on Syst. Id.*, Copenhagen, Denmark, Vol. 2, pp. 195–200, 1994.

[6] R.G. Hakvoort, *System Identification for Robust Process Control — nominal models and error bounds*, PhD. Thesis, Mech. Eng. Syst. and Contr. Group, Delft Univ. of Techn., The Netherlands, 1994.

[7] R.G. Hakvoort and P.M.J. Van den Hof, "Identification of model error bounds in ℓ_1- and H_∞-norm", *The Modeling of Uncertainty in Control Systems, Proc. 1992 Santa Barbara Workshop, Lecture Notes in Contr. and Inf. Sciences*, Springer Verlag, London, Vol. 192, pp. 139–155, 1993.

[8] R.G. Hakvoort, P.M.J. Van den Hof and O.H. Bosgra, "Consistent parameter bounding identification using cross-covariance constraints on the noise", *Proc. 32nd IEEE Conf. Dec. and Contr.*, San Antonio, pp. 2601–2606, 1993.

[9] P.S.C. Heuberger, P.M.J. Van den Hof and O.H. Bosgra, "A generalized orthonormal basis for linear dynamical systems", to appear in *IEEE Trans. Autom. Contr.*, abbreviated version in *Proc. 32nd IEEE Conf. Dec. and Contr.*, San Antonio, pp. 2850–2855, 1993.

[10] H. Hjalmarsson, "A model variance estimator", *Prepr. Proc. 12th IFAC World Congress*, Sydney, Australia, Vol. 9, pp. 5–10, 1993.

[11] H. Hjalmarsson and L. Ljung, "Estimating model variance in the case of undermodelling", *IEEE Trans. Autom. Contr.*, Vol. AC-37, pp. 1004-1008, 1992.

[12] R.O. Lamaire, L. Valavani, M. Athans and G. Stein, "A frequency domain estimator for use in adaptive control systems", *Automatica*, Vol. 27, pp. 23–38, 1991.

[13] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, Englewood Cliffs, N.J., 1987.

[14] E.B. Manoukian, *Modern Concepts and Theorems of Mathematical Statistics*, Springer Verlag, New York, 1986.

[15] B.M. Ninness, *Stochastic and Deterministic Modelling*, PhD. Thesis, Univ. of Newcastle, Australia, 1993.

[16] B.M. Ninness and G.C. Goodwin, "Robust frequency response estimation accounting for noise and undermodelling", *Proc. Am. Contr. Conf.*, Chicago, pp. 2847–2851, 1992.

[17] T. Söderström and P. Stoica, *System Identification*, Prentice-Hall, U.K., 1989.

[18] B. Wahlberg and L. Ljung, "Hard frequency-domain model error bounds from least-squares like identification techniques", *IEEE Trans. Autom. Contr.*, Vol. AC-37, pp. 900–912, 1992.