



Asymptotically optimal orthonormal basis functions for LPV system identification[☆]

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ABSTRACT

A global model structure is developed for parametrization and identification of a general class of Linear Parameter-Varying (LPV) systems. By using a fixed orthonormal basis function (OBF) structure, a linearly parametrized model structure follows for which the coefficients are dependent on a scheduling signal. An optimal set of OBFs for this model structure is selected on the basis of local linear dynamic properties of the LPV system (system poles) that occur for different constant scheduling signals. The selected OBF set guarantees in an asymptotic sense the least worst-case modeling error for any local model of the LPV system. Through the fusion of the Kolmogorov n -width theory and Fuzzy c -Means clustering, an approach is developed to solve the OBF-selection problem for discrete-time LPV systems, based on the clustering of observed sample system poles.

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1. Introduction

In general, many physical systems and control problems exhibit parameter variations due to non-stationary or nonlinear behavior or dependence on external variables, such as space coordinates, in particular found in servo-mechanical applications. These systems vary in size and complexity, but they share the common need for accurate and efficient control of the relevant process variables. However, accurate modeling of such systems is in general a complex and tedious task, involving the use of nonlinear differential equations, leading to models with many parameters and high computational complexity. For processes with mild nonlinearities or dependence on external variables, the theory of *Linear Parameter-Varying* (LPV) systems offers an attractive modeling framework (Rugh & Shamma, 2000). Discrete-time LPV systems are generally described in either a *State-Space* (SS) or an *Input/Output* (I/O) representation (Tóth, Felici, Heuberger, & Van den Hof, 2007), where the parameters are functions of a time-varying *scheduling signal* $p(k) : \mathbb{Z} \rightarrow \mathbb{P}$, that schedules between local *Linear Time Invariant* (LTI) behaviors of

the system. The compact set $\mathbb{P} \subset \mathbb{R}^{n_p}$ denotes the *scheduling space*. Practical use of the LPV framework is stimulated by the fact that control design for LPV systems is well worked out. For this class of systems, application of LTI control theory via *gain scheduling* (Rugh & Shamma, 2000) and LPV control synthesis techniques like μ -*synthesis* (Zhou & Doyle, 1998) or *Linear Matrix Inequality* (LMI)-based optimal control (Scherer, 1996) offer fast and reliable controller design, proved by a wide range of applied LPV control solutions from aerospace applications (Marcos & Balas, 2004) to CD players (Dettori & Scherer, 2001). However, it still remains a problem how to develop LPV models in a systematic fashion.

Recently several methods have been worked out, aiming at global identification of discrete-time LPV models from given measured data. This comprises methods based on multiple-model approaches (Murray-Smith & Johansen, 1997; Steinbuch, van de Molengraft, & van der Voort, 2003; Wassink, van de Wal, Scherer, & Bosgra, 2004), set-membership methods (Mazzaro, Movsichoff, & Pena, 1999; Milanese & Vicino, 1991), subspace techniques (dos Santos, Ramos, & de Carvalho, 2007; Felici, van Wingerden, & Verhaegen, 2006, 2007; Verdult & Verhaegen, 2002), basis functions (Tóth, Heuberger, & Van den Hof, 2007), LMI-based optimization (Sznaier, Mazzaro, & Inanc, 2000), simple *Least Mean Squares* (LMS) approaches (Giarré, Bauso, Falugi, & Bamieh, 2006; Wei & Del Re, 2006), and parameter estimation based gradient searches (Lee & Poola, 1996; Verdult, Ljung, & Verhaegen, 2002). Most of these approaches build on the fact that an LPV system \mathcal{S} can always be viewed as a collection of “local” behaviors and p -dependent weighting functions, i.e. *scheduling functions* that

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schedule between them (Rugh & Shamma, 2000; Tóth, Heuberger et al., 2007). For any constant scheduling signal: $p(k) = \bar{p}$ for all $k \in \mathbb{Z}$ where $\bar{p} \in \mathbb{P}$, the LPV system \mathcal{S} is identical to an LTI system $\mathcal{F}_{\bar{p}}$. Thus, the set of local behaviors of \mathcal{S} is given as $\mathfrak{F}_{\mathbb{P}} = \{\mathcal{F}_{\bar{p}}\}_{\bar{p} \in \mathbb{P}}$. The p -dependent scheduling function set, that schedules on $\mathfrak{F}_{\mathbb{P}}$, is denoted by $\mathfrak{H}_{\mathbb{P}} = \{h_{\bar{p}}(\cdot)\}_{\bar{p} \in \mathbb{P}}$.

Identification of $\mathfrak{F}_{\mathbb{P}}$ is commonly accomplished in a sampled sense by LTI identification of \mathcal{S} for a set of constant scheduling signals, associated with (for instance equidistant) points in the scheduling space \mathbb{P} . Then, assuming that the scheduling functions $\{h_{\bar{p}}\}$ have a particular structure of dependence, like polynomial, an interpolation problem is formulated on \mathbb{P} to give a global approximation of \mathcal{S} . Recently it was exposed that this approach should be handled with care for several reasons (Tóth, Felici et al., 2007; Tóth, Heuberger et al., 2007). In Tóth, Felici et al. (2007) it was shown that for general discrete-time LPV systems each $h_{\bar{p}}$ is a function of time-shifted versions of p (dynamic dependence). Then, if the particular interpolation structure of $\{h_{\bar{p}}\}$ is chosen to be too simple (dependence only on $p(k)$ (static dependence), linear dependence, etc.) the interpolation based on state-space or I/O model parametrization can result in significantly different models (Tóth, Felici et al., 2007). An additional concern of interpolation is that the McMillan degree of the local systems $\{\mathcal{F}_{\bar{p}}\}$ may vary for different values of $\bar{p} \in \mathbb{P}$. This shows that the choice of an easily interpolatable model structure which can incorporate aspects of dynamical dependence and local order changes is a crucial point of this identification approach.

The *Orthonormal Basis Function* (OBF)-based model representation offers such a structure with a well worked-out theory in the context of LTI system approximation and identification (Heuberger, Van den Hof, & Wahlberg, 2005). The basis functions, that provide bases for the system space \mathcal{H}_2 (Hilbert space of complex functions that are squared integrable on the unit circle), are generated by a cascaded network of stable all-pass filters, whose pole locations represent the prior knowledge about the system at hand. This approach characterizes the transfer function of a strictly proper LTI system as

$$F(z) = \sum_{i=1}^{\infty} w_i \phi_i(z), \quad (1)$$

where $\{w_i\}_{i=1}^{\infty}$ is the set of coefficients and $\Phi_{\infty} = \{\phi_i\}_{i=1}^{\infty}$ represents the sequence of OBFs. This implies that every $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}}$ can be represented as a linear combination of a given Φ_{∞} , i. e. $\mathfrak{F}_{\mathbb{P}} \subset \text{span}\{\Phi_{\infty}\}$. In practice, only a finite number of terms is used in (1), like in *Finite Impulse Response* (FIR) models. In contrast with FIR structures, the OBF parametrization can achieve almost zero modeling error with a relatively small number of parameters, due to the infinite impulse response characteristics of the basis functions. In this way, it is always possible to find a finite $\Phi_n \subset \Phi_{\infty}$, with a relatively small number of functions $n \in \mathbb{N}$, such that the representation error for all $\mathcal{F}_{\bar{p}}$ is negligible. Using this idea in the time-domain (substitution of z with the forward time-shift operator q), it is possible to prove that LPV systems also have a series expansion representation in terms of LTI basis functions, but with coefficients $\{w_i\}_{i=1}^{\infty}$ dependent on p . Thus in terms of a finite OBF set $\Phi_n \subset \Phi_{\infty}$, the following approximation of the I/O map of \mathcal{S} can be introduced:

$$y \approx \sum_{i=1}^n w_i(p) \phi_i(q) u, \quad (2)$$

where $\{w_i\}_{i=1}^n$ is a set of *coefficient functions*, with dynamic dependence on p . Note that in this structure, Φ_n gives the basis set used to approximate each element of $\mathfrak{F}_{\mathbb{P}}$ while $\{w_i\}_{i=1}^n$ describes the scheduling functions $\mathfrak{H}_{\mathbb{P}}$. Thus for a given $\Phi_n = \{\phi_i\}$, identification of the LPV system based on (2) simplifies to the identification of

the scheduling functions. Assuming static dependence of $\{w_i\}_{i=1}^n$, such a task can be accomplished via two approaches:

- **Local approach:** Identify some $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}}$ for constant $p(t) = \bar{p}$ with the LTI OBF model structure

$$\hat{y} = \sum_{i=1}^n r_{\bar{p},i} \phi_i(q) u. \quad (3)$$

Based on a chosen functional dependence, e.g. polynomial, interpolate the resulting $\{r_{\bar{p},i}\}$ for an estimate of $\{w_i\}_{i=1}^n$ in (2), such that $w_i(\bar{p}) = r_{\bar{p},i}$.

- **Global approach:** Parametrize the functional dependence of $\{w_i\}_{i=1}^n$ linearly (e.g. polynomial). Then for a data record with varying p , the estimation of the parameters of $\{w_i\}_{i=1}^n$ reduces to linear regression based on (2) in a least-squares prediction error setting.

There are many beneficial properties of the structure (2). For instance, the obtained model simplifies control design (see Section 6) and this parametrization is not affected by local order changes. The problem that remains to be solved with the proposed OBF-based identification approaches is to choose the set of OBFs Φ_n , “sufficiently rich” to describe $\mathfrak{F}_{\mathbb{P}}$ with a predefined number of functions. Seeking the solution for this problem is the purpose of the present paper.

Even in the case of LTI systems, the choice of OBFs to approximate a given system \mathcal{F} in an “optimal” sense (based on some error measure) is a highly non-trivial task (Heuberger et al., 2005). For the LTI case, already quite some effort has been put into tackling the basis function selection problem resulting in methods of nonlinear optimization (Heuberger et al., 2005) and iterative search (Bodin, Villemoes, & Wahlberg, 1997). One of the concepts used for this purpose is the *Kolmogorov n-width* (KnW) theory for OBFs (Oliveira e Silva, 1996), which establishes optimality in the sense of the worst-case modeling error for any LTI system with pole locations in a given region of the complex plane. Denote by

$$\Omega_{\mathbb{P}} = \{\lambda \in \mathbb{C} \mid \lambda \text{ is a pole of } \mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}} \text{ for } \bar{p} \in \mathbb{P}\},$$

the collection of all pole locations belonging to the local behaviors of the LPV system \mathcal{S} . Then, based on $\Omega_{\mathbb{P}}$, the KnW theory can be evidently applied (e.g. by the approach of Heuberger et al. (2005)) to solve the optimal selection of OBFs with respect to $\mathfrak{F}_{\mathbb{P}}$. However, this approach is not applicable if $\mathfrak{F}_{\mathbb{P}}$ is unknown. This underlines the need for a mechanism that guarantees optimality of the OBF selection (selection of Φ_n) based on the available information.

In this paper, we assume as a starting point that a collection of pole locations, some samples of $\Omega_{\mathbb{P}}$, is available that are obtained from local linear behaviors of the LPV system \mathcal{S} . This set of pole samples $\bar{\Omega} \subset \Omega_{\mathbb{P}}$ can result – but not necessarily – from identification of the related local linear models. Based on $\bar{\Omega}$, we aim at the derivation of a basis function selection mechanism, that is capable of accomplishing the following objectives:

- Reconstruction of $\Omega_{\mathbb{P}}$ from $\bar{\Omega}$.
- Determination of the set of OBFs, which has the least possible worst-case modeling error for any LTI system with pole locations in $\Omega_{\mathbb{P}}$, therefore for all $\mathcal{F}_{\bar{p}} \in \mathfrak{F}_{\mathbb{P}}$.

This choice of model structure leads to the local and global identification methods. The proposed method is the joint application of the KnW theory and *Fuzzy c-Means* (FcM) clustering (Jain & Dubes, 1988). The contribution of this method is to provide a practical model structure selection tool for the local and global LPV identification methods based on globally fixed OBFs. Earlier work along this line is proposed in Tóth, Heuberger, and Van den Hof (2006a), Tóth, Heuberger, and Van den Hof (2006b) and Vergeer (2005).

The paper is organized as follows: Section 2 introduces the description and properties of OBFs while Section 3 describes the n -width result with respect to these functions; in Section 4, the mechanism of the KnW-based FcM pole clustering is given that solves simultaneously the determination of $\Omega_{\mathbb{P}}$ from sampled poles and the selection of optimal OBFs with respect to $\Omega_{\mathbb{P}}$; in Section 5, the OBF-based LPV system identification scheme is specified to provide a brief description how the selected basis functions are used in an identification scenario; in Section 6, the applicability of the introduced method is shown through an example; and finally, in Section 7, the main results of the paper are discussed.

2. Orthonormal basis functions

We consider only the case of real rational (finite-dimensional) discrete-time, SISO transfer functions. For details see Heuberger et al. (2005), Heuberger, Van den Hof, and Bosgra (1995) and Ninness and Gustafsson (1997). Let $G_0 \equiv 1$ and $\{G_i\}_{i=1}^{\infty}$ be a sequence of inner functions (i.e. stable transfer functions with $G_i(z)G_i(\frac{1}{z}) = 1$), and let $\{A_i, B_i, C_i, D_i\}$ be minimal balanced SS representations of G_i . Let $\{\xi_1, \xi_2, \dots\}$ denote the collection of all poles of the inner functions G_1, G_2, \dots . Under the (completeness) condition that $\sum_{i=1}^{\infty} (1 - |\xi_i|) = \infty$, the scalar elements of the sequence of vector functions

$$V_n(z) = (zI - A_n)^{-1} B_n \prod_{i=0}^{n-1} G_i(z), \quad (4)$$

constitute a basis for $\mathcal{H}_{2-}(\mathbb{E})$, the *Hardy space* of functions, which are 0 for $z = \infty$, analytic on \mathbb{E} , the exterior of the unit disk \mathbb{D} , and squared integrable on the unit circle \mathbb{T} with norm $\|\cdot\|_{\mathcal{H}_2}$. In this way $\mathcal{H}_{2-}(\mathbb{E})$ is the space of all stable strictly proper transfer functions. These functions (4) are often referred to as the *Takenaka-Malmquist functions*. The special cases when all G_i are equal, i.e. $G_i(z) = G_b(z), \forall i > 0$, where G_b has McMillan degree $n_b > 0$, are known as *Hambo functions* or *generalized orthonormal basis functions* (GOBFs) for arbitrary n_b , *2-parameter Kautz functions* for $n_b = 2$, and as *Laguerre functions* for $n_b = 1$. Note that for these cases the completeness condition is always fulfilled. In the remainder we will only consider the set of Hambo functions. Let G_b be an inner function with McMillan degree $n_b > 0$ and minimal balanced SS representation $\{A_b, B_b, C_b, D_b\}$. Define $V_1(z) = (zI - A_b)^{-1} B_b$ and $\phi_j = [V_1]_j, j \in \mathbb{I}_1^{n_b}$, where $\mathbb{I}_1^{s_2} = \{s_1, s_1 + 1, \dots, s_2\} \subset \mathbb{Z}$ is the index set. The Hambo basis then consists of the functions $\Phi_{n_b}^{\infty} = \{\phi_j G_b^i\}_{j=1, \dots, n_b}^{i=0, \dots, \infty}$. An important aspect of these bases is that the inner function G_b is, modulo the sign, completely determined by its poles $\mathcal{E}_{n_b} := \{\xi_1, \dots, \xi_{n_b}\}$:

$$G_b(z) = \pm \prod_{j=1}^{n_b} \frac{1 - z\xi_j^*}{z - \xi_j}, \quad (5)$$

where $*$ denotes complex conjugation, and it is immediate that the function V_1 has the same poles. Any $F \in \mathcal{H}_{2-}(\mathbb{E})$ can be decomposed as

$$F(z) = \sum_{i=0}^{\infty} \sum_{j=1}^{n_b} w_{ij} \phi_j(z) G_b^i(z), \quad (6)$$

and it can be shown that the rate of convergence of this series expansion is bounded by $\rho = \max_k |G_b(\lambda_k^{-1})|$, called the *decay rate*, where $\{\lambda_k\}$ are the poles of $F(z)$. In the “best” case, where the poles of F are the same (with multiplicity) as the poles of G_b , only the terms with $i = 0$ in (6) are non-zero. The I/O relation of the OBF parametrization (6) is illustrated in Fig. 1.

In practice, only a finite number of terms $\Phi_{n_b}^{n_e} = \{\phi_j(z) G_b^i\}_{j=1, \dots, n_b}^{i=0, \dots, n_e}$ with $n_e \geq 0$ is used in (6), like in *Finite Impulse Response* (FIR) models. In contrast with FIR structures, which are

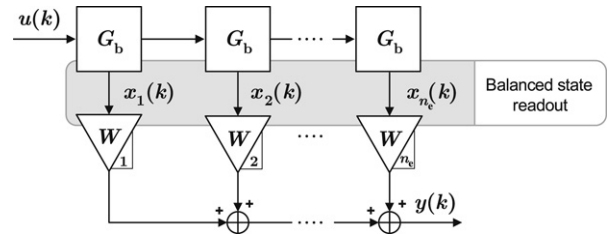


Fig. 1. I/O signal flow graph of the OBF model structure described by (6) for a finite n_e number of extensions of G_b and with $W_i = [w_{i1}, \dots, w_{in_b}]$.

described by (6) as a finite linear combination of *pulse basis functions* $\phi_1(z) = z^{-1}$ with $n_b = 1$ and $G_b^i(z) = z^{-i}$, the OBF parametrization uses a broad class of basis functions with infinite impulse responses. Therefore, OBF parametrization can achieve almost zero modeling error with a relatively small number of parameters due to the faster convergence of the series representation than in the FIR case. Moreover, the reduced number of parameters in the structure results in decreased variance of the final model estimate.

Identification of any $F \in \mathcal{H}_{2-}(\mathbb{E})$ based on a predefined set of OBFs $\Phi_{n_b}^{n_e}$ consisting of $n = (n_e + 1)n_b$ basis functions, is performed as a linear regression with respect to the basis coefficients $W_{n_b}^{n_e} = [w_{ij}]_{j=1, \dots, n_b}^{i=0, \dots, n_e}$ due to the linear parametrization of (6). The OBF-based identification has valuable properties. Non-asymptotic variance bounds of the estimates are computable through reproducing kernels and the identified models are unbiased if the input signal is uncorrelated to the noise. This is explained by the *Output Error* (OE) like structure of the OBF parametrization (Heuberger et al., 2005). However, selection of the basis function set has a major impact on the outcome of the identification process as the distance between basis poles and the original system poles determines the convergence rate of the coefficients, meaning that with a “better” basis function set a better approximation can be achieved.

As discussed, OBF-based parametrization can be effectively used for LTI system representation and in this way to describe each $\mathcal{F}_{\mathbb{P}} \in \mathfrak{F}_{\mathbb{P}}$ of an LPV system \mathcal{S} . However, if the same OBFs are used to compose each $\mathcal{F}_{\mathbb{P}}$, then it is required that the basis function set is “well chosen” with respect to the entire $\mathfrak{F}_{\mathbb{P}}$. In the next section, the concept of optimality of an OBF set with respect to $\mathfrak{F}_{\mathbb{P}}$ is established, giving the key theorem to solve the basis function selection problem of the proposed identification scheme.

3. Kolmogorov n -width for OBFs

In the proposed LPV identification approach, it is crucial to find an appropriate model set, i.e. set of basis functions $\Phi_{n_b}^{n_e}$ for the local behaviors $\mathfrak{F}_{\mathbb{P}}$, in the sense that $\Phi_{n_b}^{n_e}$ is sufficiently rich to describe the systems belonging to $\mathfrak{F}_{\mathbb{P}}$, with a relatively small number of statistically meaningful parameters. In LTI system identification, one approach to find appropriate model sets is based on the n -width concept (Pinkus, 1985), which was shown to result in appropriate model sets for robust modeling of linear systems (Mäkilä & Partington, 1993). Using this concept, Oliveira e Silva (1996), (Heuberger et al., 2005, Ch. 11) showed that OBF model structures are optimal in the n -width sense for specific subsets of systems. In the following, the basic ingredients of this theory for discrete-time, stable, SISO systems are described.

Let \mathfrak{F} denote a set of systems with transfer functions $\{F\} = \mathfrak{F} \subseteq \mathcal{H}_{2-}(\mathbb{E})$, that we want to approximate with the linear combination of n elements of $\mathcal{H}_{2-}(\mathbb{E})$. Let $\Phi_n = \{\phi_i\}_{i=1}^n$ be a sequence of n linearly independent elements of $\mathcal{H}_{2-}(\mathbb{E})$, and let $\Psi_n = \text{span}(\Phi_n)$. The distance $d_{\mathcal{H}_{2-}}(F, \Psi_n)$ between $F \in \mathcal{H}_{2-}(\mathbb{E})$ and Ψ_n is defined as

$$d_{\mathcal{H}_{2-}}(F, \Psi_n) = \inf_{G \in \Psi_n} \|F - G\|_{\mathcal{H}_2}. \quad (7)$$

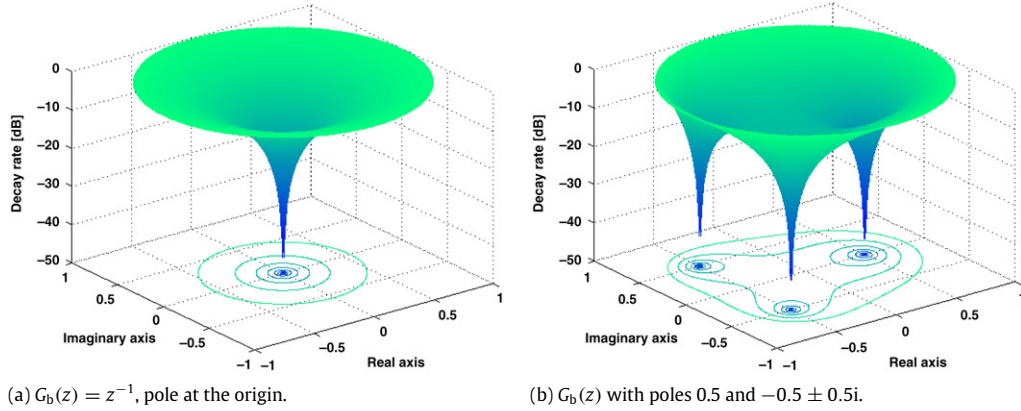


Fig. 2. The plot of the function $|G_b(z^{-1})|$ for different choices of the inner function G_b and the decay rate ρ (in dB). Level sets of $|G_b(z^{-1})|$ give the boundaries of the regions $\{z \in \mathbb{C}, |G_b(z^{-1})| \leq \rho\}$. Optimality of the G_b generated basis is ensured with a worst-case decay rate ρ^{n_e+1} for systems with pole locations inside the regions defined by the level set boundaries.

If \mathfrak{M}_n is the collection of all n -dimensional subspaces of $\mathcal{H}_{2-}(\mathbb{E})$, then the Kolmogorov n -width of \mathfrak{T} in $\mathcal{H}_{2-}(\mathbb{E})$ is

$$\pi_n(\mathfrak{T}, \mathcal{H}_{2-}(\mathbb{E})) = \inf_{\Psi_n \in \mathfrak{M}_n} \sup_{F \in \mathfrak{T}} d_{\mathcal{H}_{2-}}(F, \Psi_n), \quad (8)$$

which means the smallest possible approximation error for the worst-case F in \mathfrak{T} . The subspace $\Psi_n \in \mathfrak{M}_n$, for which π_n is minimal, is called the optimal subspace in the KnW sense. Now we can formulate this concept for OBFs:

Proposition 1 (Oliveira e Silva, 1996). Let G_b be an inner function with McMillan degree $n_b > 0$, with poles \mathcal{E}_{n_b} and $n_e \in \mathbb{N}$. Consider the subspace $\Psi_n = \text{span}\{\phi_j(z) G_b^i(z)\}_{j=1, \dots, n_b}^{i=0, \dots, n_e}$. Then the subspace Ψ_n is optimal in the Kolmogorov n -width sense, with $n = (n_e + 1)n_b$, for the set of systems with transfer functions analytic in the complement of the region

$$\Omega(\mathcal{E}_{n_b}, \rho) = \{z \in \mathbb{C}, |G_b(z^{-1})| \leq \rho\}, \quad (9)$$

and squared integrable on its boundary. The worst-case approximation error is proportional to ρ^{n_e+1} .

This remarkable result shows that for the specified region (9) one cannot improve on the worst-case error by adding new poles to the n_b basis poles. It also generalizes the well-known fact that the set of pulse functions $\{z^{-i}\}_{i=1}^n$ is optimal for the class of stable systems analytical outside the circular region $\Omega(0, \rho) = \{|z| \leq \rho\}$, $\rho > 0$. The boundary of $\Omega(0, \rho)$ is given in Fig. 2(a) as a function of the decay rate ρ . For a given $\rho > 0$, the boundary of the region results as the level set of this function, like the contour lines at the bottom of the figure. The worst-case approximation error in this case is proportional to ρ^n . This implies the optimality of FIR model structures with respect to the identification of such systems. However in case of arbitrary regions, like the regions in Fig. 2(b), the level sets are commonly non-circular containing separate regions that merge for increasing values of ρ . For these regions, the optimal choice of a basis has to be found among general basis functions (OBF model structures).

In the LPV identification scenario we are dealing with the opposite problem, referred to as the *inverse Kolmogorov problem*, where we are given a region of non-analyticity $\Omega_{\mathbb{P}} \subset \mathbb{D}$ and we want to find an inner function G_b to describe/approximate this region in the form $\Omega(\mathcal{E}_{n_b}, \rho)$ with ρ as small as possible. The reason is that in terms of Proposition 1, the inner function G_b , associated with the best fitting $\Omega(\mathcal{E}_{n_b}, \rho)$, generates the n -width optimal basis functions with respect to $\Omega_{\mathbb{P}}$. For a given number of poles n_b , this comes down to the following min-max problem:

$$\min_{\xi_1, \dots, \xi_{n_b}} \max_{z \in \Omega_{\mathbb{P}}} \prod_{j=1}^{n_b} \left| \frac{z - \xi_j}{1 - z \xi_j^*} \right|. \quad (10)$$

See Heuberger et al. (2005, Chapters 10 and 11) for details on this nonlinear optimization problem and solution methods.

The previous results show that the Kolmogorov n -width theory for OBFs provides an effective way to choose appropriate basis functions for the description¹ of $\mathfrak{F}_{\mathbb{P}}$ based on $\Omega_{\mathbb{P}}$. However, in an identification scenario we are facing the situation where $\Omega_{\mathbb{P}}$ is unknown. Thus, to enable the application of this theory, we will focus on the problem of reconstruction of $\Omega_{\mathbb{P}}$ based on some sample pole locations of this set. As we will see, the joint solution of this reconstruction problem and the optimization (10) can be found through a clustering approach.

4. Fuzzy-Kolmogorov c-Max clustering

4.1. The pole clustering algorithm

In the following we propose a particular data clustering algorithm, which by weighting-function-based separation, the so-called *fuzzy clustering* of sampled pole locations $\hat{\Omega}$ of the LPV system, can effectively handle the reconstruction of $\Omega_{\mathbb{P}}$ jointly with the solution of (10).

Objective-function-based fuzzy clustering algorithms, such as the *Fuzzy c-Means* (FcM), have been used in a wide collection of applications like pattern recognition, data analysis, image processing and fuzzy modeling (Bezdek, 1981; Kaymak & Setnes, 2002). Generally, FcM partitions the data into overlapping groups so called clusters, where each data element is associated with a set of membership levels with respect to these clusters. These indicate the strength of the association between that data element and a particular cluster. In this way, fuzzy clustering is a process of assigning these membership levels such that the resulting clusters describe the underlying structure within the data (Jain & Dubes, 1988). This enables the determination of the region $\Omega_{\mathbb{P}}$ on the basis of the observed poles by exploring the underlying data coherency. To exploit this fruitful property, in the following such a *Fuzzy-Kolmogorov c-Max* (FKcM) algorithm is presented, which provides an effective OBF selection approach based on the fusion of the KnW theory and the FcM technique.

Let $c > 1$ be the number of clusters or data groups and let $Z = [z_k]_{k=1}^N \in \mathbb{D}^N$, be the set of $N \in \mathbb{N}$ observed poles for clustering. A cluster is represented by its center (or *prototype*) $v_i \in \mathbb{D}$, $i \in \mathbb{I}_c^+$.

¹ Note that $\mathfrak{F}_{\mathbb{P}}$ is a set of LTI systems which can be represented in the frequency domain. However, it must be clear that the global LPV system \mathfrak{A} is not described in frequency domain terms.

Furthermore, membership functions $\mu_i : \mathbb{D} \rightarrow [0, 1]$ determine the “degree of membership” to the cluster for all $z \in \mathbb{D}$. By using a *threshold value* ε , we obtain a set

$$\Omega_\varepsilon = \{z \in \mathbb{D} \mid \exists i \in \mathbb{I}_1^c, \mu_i(z) \geq \varepsilon\}. \quad (11)$$

We can now formulate the problem we will consider.

Problem 2. For a set of sampled pole locations Z and for a given number of clusters c , find a set of cluster centers $\{v_i\}_{i=1}^c$, a set of membership functions $\{\mu_i\}_{i=1}^c$, and the maximum of ε , such that

- Ω_ε contains Z .
- With respect to Ω_ε , the OBFs, with poles \mathcal{E}_c in the cluster centers $\{v_i\}_{i=1}^c$, are optimal in the KnW sense, where $n = c$.

The solution is based on finding clusters in accordance with the KnW concept and subsequently finding a maximal value for ε , such that all sampled poles are inside Ω_ε . The latter is equivalent to minimizing ρ in the optimization problem of (10). Note that optimality of the OBFs is sought as $n_e = 0$. According to the principle of KnW theory, this might result in repetitive optimal poles and therefore similar clusters. In the following we will focus on finding n -width-based clusters.

Denote $V = [v_i]_{i=1}^c$ and introduce the membership matrix $U = [\mu_{ik}]_{c \times N}$, where μ_{ik} is the degree of membership of z_k to cluster i . To constrain the clustering it is required that $U \in \mathcal{U}_c^N$, where

$$\mathcal{U}_c^N = \left\{ U \in [0, 1]^{c \times N} \mid \sum_{i=1}^c \mu_{ik} = 1 \text{ for } \forall k \in \mathbb{I}_1^N, \right. \\ \left. 0 < \sum_{k=1}^N \mu_{ik} \text{ for } \forall i \in \mathbb{I}_1^c \right\} \quad (12)$$

characterizes the *fuzzy constraints*.

Furthermore, distances d_{ik} are introduced between v_i and z_k to measure dissimilarity of Z with respect to each candidate cluster. To derive an algorithmic solution of Problem 2, the Kolmogorov metric² (KM) of \mathbb{D} :

$$\kappa(x, y) := \left| \frac{x - y}{1 - xy^*} \right| : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}_0^+, \quad (13)$$

with $\mathbb{R}_0^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ is used, which is the 1-width version of the cost function of (10). As a notation, $d_{ik} = \kappa(v_i, z_k)$ is introduced. It will be shown that KM relates the FcM asymptotically to the KnW theory and to the solution of Problem 2.

Fuzzy clustering can be defined as the minimization of the FcM-functional (Bezdek, 1981), $J_m(U, V) : \mathcal{U}_c^N \times \mathbb{D}^c \rightarrow \mathbb{R}_0^+$. For Problem 2, J_m is formulated as

$$J_m(U, V) = \max_{k \in \mathbb{I}_1^N} \sum_{i=1}^c \mu_{ik}^m d_{ik}. \quad (14)$$

Here, the design parameter $m \in (1, \infty)$ determines the fuzziness of the resulting partition. It can be observed, that (14) corresponds to a *worst-case (max) sum-of-error* criterion, contrary to the *mean-squared-error* criterion of the original FcM, see Bezdek (1981). The exact relation of (14) with the KnW optimality of the partition (U, V) is explained later. The following theorem yields the ingredients for the approach to solve Problem 2:

Theorem 3 (Optimal Partition). Let $m > 1$, a data set $Z \in \mathbb{D}^N$, and a fuzzy partition $(U, V) \in \mathcal{U}_c^N \times \mathbb{D}^c$ be given. Denote $[V]_i = v_i$ and $[U]_{ij} = \mu_{ij}$. Define $\gamma_i(v, U)$ as the minimal value of $\gamma \in [0, 1]$ fulfilling the quadratic constraints:

$$\begin{bmatrix} |1 - z_k^* v|^2 & \mu_{ik}^m (z_k - v) \\ \mu_{ik}^m (z_k - v)^* & \gamma^2 \end{bmatrix} \geq 0, \quad \forall k \in \mathbb{I}_1^N, \quad (15)$$

where $v \in \mathbb{D}$. Additionally, let $d_{ik} = \kappa(v_i, z_k)$ be the dissimilarity measure of z_k with respect to V and $\mathcal{I}_s^{(k)} = \{i \in \mathbb{I}_1^c \mid d_{ik} = 0\}$ be the singularity set of z_k with $n_s^{(k)} = \text{card}(\mathcal{I}_s^{(k)})$ (number of elements). Then (U, V) is a local minimum of J_m , if for any $(i, k) \in \mathbb{I}_1^c \times \mathbb{I}_1^N$:

$$\mu_{ik} = \begin{cases} \left[\sum_{j=1}^c \left(\frac{d_{jk}}{d_{ik}} \right)^{\frac{1}{m-1}} \right]^{-1} & \text{if } \mathcal{I}_s^{(k)} = \emptyset, \\ \frac{1}{n_s^{(k)}} & \text{if } i \in \mathcal{I}_s^{(k)}, \\ 0 & \text{if } i \notin \mathcal{I}_s^{(k)} \neq \emptyset, \end{cases} \quad (16)$$

$$\text{and } v_i = \arg \min_{v \in \mathbb{D}} \gamma_i(v, U). \quad (17)$$

The proof is given in Appendix A. In the FcM case, minimization of (14) subject to (12) is usually tackled by alternating optimization (Picard iteration) (Bezdek, 1981), steering the solution towards a settling partition in the sense of Theorem 3. For the FKcM this yields Algorithm 1, formulated next, where V_l and U_l denote the actual fuzzy partition in iteration step l . In the following part, we will discuss the main properties of this algorithm and clarify each step in detail.

Algorithm 1. Fuzzy-Kolmogorov c -Max

- (1) *Initialization*
Fix c and m ; and initialize $V_0 \in \mathbb{D}^c, l = 0$.
- (2) *Membership update*
With (16), solve $U_{l+1} = \arg \min_{U \in \mathcal{U}_c^N} J_m(U, V_l)$.
- (3) *Cluster center update*
With (17), solve $V_{l+1} = \arg \min_{V \in \mathbb{D}^c} J_m(U_{l+1}, V)$.
- (4) *Check of convergence*
If $J_m(U_{l+1}, V_{l+1})$ has converged, then stop, else $l = l + 1$ and go to Step 2.

4.2. Properties of the FKcM

In order to explain the specific choices for the fuzzy functional (14) and the dissimilarity measure (13), we use the following theorem.

Theorem 4 (Limiting Property of J_m). Given a data set $Z \in \mathbb{D}^N, N > 0$, and a set of cluster centers $V \in \mathbb{D}^c, c > 0$, such that $d_{ik} = \kappa(v_i, z_k) \neq 0$ for all $(i, k) \in \mathbb{I}_1^N \times \mathbb{I}_1^c$ (no singularity). Define U_m as a membership matrix of V satisfying (16) for $m > 1$. Then

- a. $\lim_{m \rightarrow 1} J_m(U_m, V) = \max_{k \in \mathbb{I}_1^N} \min_{i \in \mathbb{I}_1^c} \{d_{ik}\}$, which corresponds to the hard partitioning of Z , i.e. $\mu_{ik} \in \{0, 1\}, \forall (i, k) \in \mathbb{I}_1^c \times \mathbb{I}_1^N$. Here, the optimal partition corresponds to a collection of 1-width optimal basis functions with respect to each reconstructed pole region.
- b. $J_2(U_2, V) = \max_{k \in \mathbb{I}_1^N} \left[\sum_{i=1}^c d_{ik} \right]^{-1}$, which is the maximum of the harmonic-means-based distance of each z_k with respect to the clusters.
- c. $J_m(U_m, V) = c^{1-m} \max_{k \in \mathbb{I}_1^N} \left[\prod_{i=1}^c d_{ik} \right]^{1/c} + \mathcal{O}(e^{-m})$. Furthermore, $J_m(U_m, V)$ decreases monotonically with m , and $J_\infty(U_\infty, V) = 0$.

² Note that KM is not a distance in \mathbb{D} , only $\text{arctanh}(\kappa(x, y))$, called the *Poincaré distance*, bears this property (Brannan, Esplen, & Gray, 1999). However in fuzzy clustering, the dissimilarity measure does not need to qualify as a distance.

The proof is presented in Appendix A. Based on Theorem 4, the minimization of J_m corresponds to a close approximation of (10) for large m , enabling the FKcM to solve Problem 2 directly. However, if $m \rightarrow \infty$, then in the optimal partition $\mu_{ik} \rightarrow 1/c$ for all $(i, k) \in \mathbb{I}_1^c \times \mathbb{I}_1^N$, which can cause numerical problems in the minimization of (17). Therefore, to obtain a well-approximating solution of Problem 2, an appropriately large value of $m \in (1, \infty)$ should be used. Based on experience, $m \in [5, 10]$ usually yields satisfactory results.

For $m > 1$, the FKcM-functional (14) is a bounded ($0 \leq J_m \leq 1$) monotonically descending function both in $\{d_{ik}\}$ and U , which allows Algorithm 1 to converge in practice. The convergence point, which is directly dependent on the initial V^0 , can either be a local minimum or a saddle point of J_m , fulfilling Theorem 3. Therefore, it is advisable to repeat the algorithm multiple times with different initial choices for V^0 and then select the best resulting set of OBFs by comparison of the achieved decay rate

$$\bar{\rho} = \max_{z \in Z} \prod_{i=1}^c \left| \frac{z - v_i}{1 - zv_i^*} \right|, \tag{18}$$

and by visual inspection of the region $\Omega(\mathcal{E}_c = V, \bar{\rho})$ with respect to Z . In practice, uniformly random choices for V^0 are suggested.

4.3. Optimization and numerical conditioning

While the membership update step in Algorithm 1 can be analytically computed through (16), the cluster center update step requires the solution of (17) which is a minimization problem with a Quadratic Constraint (QC) where γ is the optimization variable and v is the decision variable. Based on Scherer and Hol (2006), it is possible to derive Sum-of-Squares (SoS) relaxations of such constraints, through which (15) is turned into LMIs. The resulting LMIs constrained convex minimization of γ is a Linear Semi Definite Programming (LSDP) problem that can be efficiently solved by a variety of (interior-point-based) solvers like SeDuMi (Sturm, 1999) or CSDP etc. Alternatively, bisection-based recursive search can also be utilized to obtain the minimization of γ in (17). In each step of this bisection-based minimization, the QCs with a fixed γ are rewritten as LMI constraints. Checking feasibility of the constraints indicates how to proceed with the minimization of γ .

For high values of m , the QCs (15) become numerically ill-conditioned which can be overcome by the normalization of $\{\mu_{ik}^m\}_{k=1}^N$:

$$\bar{\mu}_{ik} = \frac{\mu_{ik}^m}{\mu_i}, \quad \text{with } \check{\mu}_i = \sum_{k=1}^N \mu_{ik}^m. \tag{19}$$

4.4. Termination criterion

In Algorithm 1, the cost function J_m flattens when m increases. This yields that for high values of m , J_m drastically drops in a local minimum, while J_m is almost constant for other points. To avoid unnecessary termination, the relative evolution of J_m , in each iteration step l , has to be checked in a windowed sense:

$$1 - \frac{\max_k [J_m(U_k, V_k) - J_m(U_{k-1}, V_{k-1})]}{\max_k J_m(U_k, V_k)} < \varepsilon_t \tag{20}$$

where $k \in \mathbb{I}_{l-n_w}^l$, $n_w \in \mathbb{N}$ is the length of the window, and $0 \ll \varepsilon_t < 1$ is a user defined termination constant. For $m \in [5, 10]$, $\varepsilon_t = 0.99$ with $n_w = 3$ usually works well.

4.5. Cluster merging

The determination of the number of “natural” groups in Z , i.e. the best suitable c for clustering, is important for the successful application of the FKcM method. Similarity-based adaptive cluster merging (ACM) is frequently used for this purpose (Kaymak & Setnes, 2002), but other strategies exist also. ACM is suitable for problems where little is known about the statistical properties of the data, like in the pole clustering case. The basic idea is the following: a measure of similarity is introduced with respect to cluster pairs. A cluster pair is merged when its similarity does not decrease between iterations and if also this pair is the most similar of all cluster pairs. However, merging is only applied if the similarity measure exceeds a certain threshold value, $\varepsilon_a \in [0, 1]$ arbitrarily chosen by the user. In FcM clustering, most commonly the following similarity measure is applied:

Definition 5 (Inclusion Similarity Measure (Kaymak & Setnes, 2002)). The fuzzy-inclusion-similarity measure (given point-wise on Z) for two fuzzy clusters i and j is defined as

$$s_{ij} = \frac{\sum_{k=1}^N \min(\mu_{ik}, \mu_{jk})}{\min\left(\sum_{k=1}^N \mu_{ik}, \sum_{k=1}^N \mu_{jk}\right)}. \tag{21}$$

This measure takes into account the contribution to similarity from all $\{z_k\}_{k=1}^N$. For the theoretical details see Kaymak and Setnes (2002). Then, in the l th iteration of Algorithm 1, the most similar cluster pair can be selected as

$$(\hat{i}, \hat{j}) = \underset{(i,j) \in \mathbb{I}_1^c \times \mathbb{I}_1^c, i > j}{\text{arg max}} \{s_{ij}^{(l)}\}. \tag{22}$$

Merging is applied if $|s_{\hat{i}\hat{j}}^{(l-1)} - s_{\hat{i}\hat{j}}^{(l)}| < \varepsilon_s$, where $0 < \varepsilon_s \ll 1$ is a threshold value to judge the significance of decrease of cluster similarity between iterations. However as the partition converges, similarity changes a little between iterations, therefore merging is only applied if $s_{\hat{i}\hat{j}}^{(l)} > \varepsilon_a^{(l)}$ where $\varepsilon_a^{(l)} \in [0, 1]$ is an adaptive threshold. In Kaymak and Setnes (2002), it is suggested to use $\varepsilon_a^{(l)} = (c^{(l)} - 1)^{-1}$ which is observed empirically to work well if the initial number of clusters c^0 satisfy $c^0 < \frac{1}{2}N$.

In this way, the FKcM algorithm with ACM provides the possibility to automatically choose the number of required OBFs for the model structure based on Z . So by starting from a large c , the algorithm converges to a partition which contains only the necessary number of clusters representing the data. However in terms of Proposition 1, the setting of Problem 2 implies that repetitive basis poles can be part of the optimal solution. With the ACM, these solutions are not accessible, since repetitive poles result in perfectly similar clusters which are immediately joined. Therefore, ACM only provides convergence to partitions with distinct cluster centers.

5. LPV system identification

In the previous section, an OBF selection algorithm has been proposed to obtain an adequate selection of the model structure (2) with respect to an unknown LPV system \mathcal{S} . The fact that an LPV system can be viewed as a set of local LTI behaviors $\mathfrak{F}_{\mathbb{P}}$ which are combined by a set of scheduling functions $\mathfrak{S}_{\mathbb{P}}$ is the motivation to select the optimal model structure based on $\mathfrak{F}_{\mathbb{P}}$. In the following, according to the LPV system identification approach of Section 1, it is briefly shown how these OBFs can be used for identifying a discrete-time \mathcal{S} efficiently, i.e. how the scheduling functions can be

Fig. 3. I/O signal flow graph of the W-LPV OBF model structure.

estimated. Based on (2), we introduce a model structure presented in Fig. 3, where the selected OBFs are set up as a filter bank followed by a p -dependent weighting function (Tóth, Heuberger et al., 2007). Due to the similarity of this model structure to Wiener models we will call it a Wiener-LPV (W-LPV) OBF model.

Let $\Phi_{n_b}^{n_e}$ be a set of OBFs in $\mathcal{H}_{2-}(\mathbb{R})$. Denote by $\{A_b, B_b, C_b, D_b\}$ a minimal balanced SS realization of $G_b^{n_e}$, where G_b is the inner function associated with $\Phi_{n_b}^0$. Let \mathcal{S} be a data generating SISO LPV system without a feedthrough term. By applying the W-LPV model of \mathcal{S} in an OE setting and with static coefficient dependence, leads to the following 1-step ahead predicted output

$$\hat{y}(k) = \sum_{i=0}^{n_e} \sum_{j=1}^{n_b} w_{ij}(p(k)) \underbrace{\phi_j(q)G_b^i(q)}_{\hat{y}_{ij}(k)} u(k), \quad (23)$$

where q denotes the forward time-shift operator and $\{w_{ij}\}$ is a set of functions. The SS equivalent representation of (23), is defined as

$$x(k+1) = A_b x(k) + B_b u(k), \quad (24a)$$

$$\hat{y}(k) = W(p(k)) x(k), \quad (24b)$$

where $x^T = [\check{y}_{01} \dots \check{y}_{n_e n_b}]$ and $W(p) = [w_{01}(p) \dots w_{n_e n_b}(p)]$. Note that using the model structure (23) in an OE setting has many attractive properties. For instance, it is linear in the coefficient functions $\{w_{ij}\}$, the noise model is independently parametrized from the process part, and this model structure has a direct state-space realization via (24a) and (24b) where

only the output equation has dependence on p . The latter implies that LPV control design simplifies for the obtained model estimate. Furthermore, it can be shown based on series expansion representation of LPV systems, that the W-LPV OBF structure can represent the general class of LPV systems with arbitrary precision in case the weighting functions W have dynamic dependence (dependence on the past/future of p) (Tóth, Heuberger et al., 2007). In case of static dependence, i.e. dependence on $p(k)$ (the instantaneous value of p), approximation of a wide class of LPV systems is available by this model structure. This class however is hard to characterize as the approximation error results both from the finite number/quality of the chosen OBFs and the assumption of static dependence. Identification of an LPV system \mathcal{S} by this model class can be accomplished through the local and global approach presented in Section 1. The schematic view of these approaches is given in Fig. 4.

5.1. Local approach

In this method, the identification is based on data records $D_l = \{y(k), u(k), \bar{p}_1\}_{k=1}^{N_d}$ gathered from \mathcal{S} for constant scheduling points $P_n = \{\bar{p}_1, \dots, \bar{p}_n\}$ with persistently exciting (in the LTI sense) u . Then local LTI models of \mathcal{S}

$$\hat{y}(k) = \sum_{i=0}^{n_e} \sum_{j=1}^{n_b} r_{\bar{p}_l, i, j} \phi_j(q) G_b^i(q) u(k), \quad (25)$$

in terms of the OBFs $\Phi_{n_b}^{n_e}$ with $r_{\bar{p}_l, i, j} \in \mathbb{R}$ are estimated, e.g. by linear regression, based on D_l . The obtained coefficients $\{\hat{r}_{\bar{p}_l, i, j}\}$ are the estimated samples of $\{w_{ij}\}$ in (23) for the applied constant scheduling functions, i.e. $w_{ij}(\bar{p}_l) \approx \hat{r}_{\bar{p}_l, i, j}$. Then by choosing a particular structure of the static functional dependence of each w_{ij} , interpolation (by arbitrary method) of $\{\hat{r}_{\bar{p}_l, i, j}\}$ gives an estimate $\{\hat{w}_{ij}\}$ of $\{w_{ij}\}$ for which $\hat{w}_{ij}(\bar{p}_l) = \hat{r}_{\bar{p}_l, i, j}$. In this way, the local approach simply provides an extension of the classical LTI OBF identification approaches with all their beneficial properties to the LPV case.

Note that an adequate choice of P_n is required for an efficient model estimate and adequateness of P_n depends on the variation

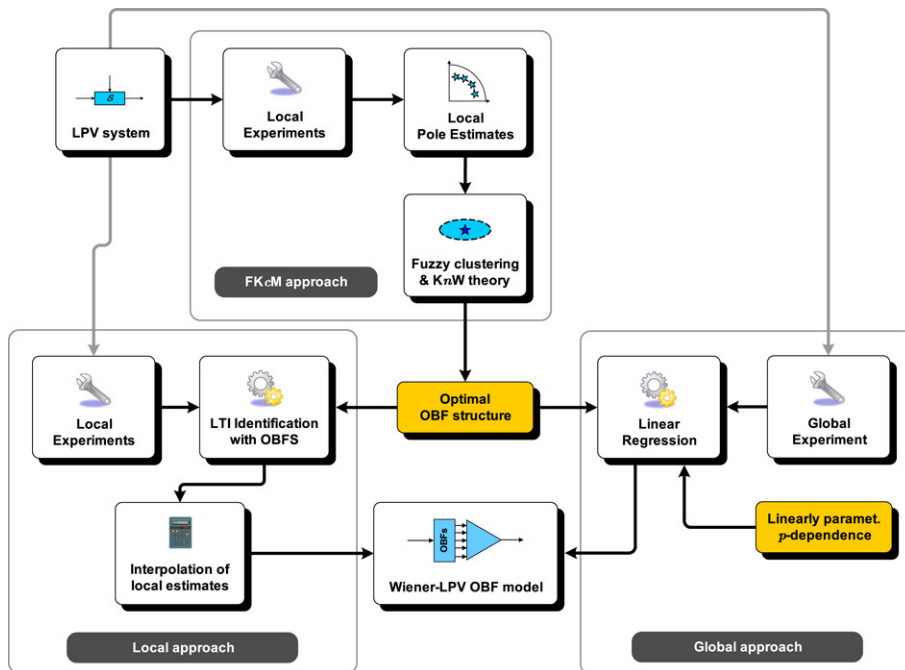


Fig. 4. Block diagram of local and global identification methods.

