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SYSTEM IDENTIFICATION WITH GENERALIZED ORTHONORMAL BASIS FUNCTIONS

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Abstract

A least squares identification method is studied that estimates a finite number of expansion coefficients in the series expansion of a transfer function, where the expansion is in terms of generalized basis functions. The basis functions are orthogonal in $\mathcal{H}_2$ and generalize the pulse, Laguerre and Kautz bases. The construction of the basis is considered and bias and variance expressions of the identification algorithm are discussed. The basis induces a new transformation (Hambo transform) of signals and systems, for which state space expressions are derived.

1 Introduction

The use of orthogonal basis functions for the Hilbert space $\mathcal{H}_2$ of stable systems has a long history in modelling and identification of dynamical systems. The main part of this work dates back to the classical work of Lee and Wiener.

In the past decades orthogonal basis functions, as e.g. the Laguerre functions, have been employed for the purpose of system identification in e.g. [8, 11]. In these works the input and output signals of a dynamical system are transformed to a (Laguerre) transform domain, being induced by the orthogonal basis for the signal space. Consecutively, more or less standard identification techniques are applied to the signals in this transform domain. The main motivation for this approach has been directed towards data reduction, as the representation of the measurement data in the transform domain becomes much more efficient once an appropriate basis is chosen.

In [15, 16] orthogonal functions are applied for the identification of a finite sequence of expansion coeffients. Given the fact that every stable system has a unique series expansion in terms of a prechosen basis, a model representation in terms of a finite length series expansion can serve as an approximate model, where the coefficients of the series expansion can be estimated from data. As the accuracy of the finite expansion models is limited by the basis functions, the development of appropriate basis functions is a topic that has gained considerable interest. The issue here is that it is profitable to design basis functions that reflect the dominant dynamics of the process to be modelled. Laguerre functions (see e.g. [12]), exhibit the choice of a scalar design variable a that has to be chosen in a range that matches the dominating (first order) dynamics of the process to be modelled. For moderately damped systems, Kautz functions have been employed, which actually are second order generalizations of the Laguerre functions, see [7, 16, 17].

Recently a generalized set of orthonormal basis functions has been developed that is generated by inner (all pass) transfer functions of any prechosen order, [4, 5, 6]. This type of basis functions generalizes the Laguerre and Kautz-type bases, which appear as special cases when chosing first order and second order inner functions. An alternative generalization in [10] is closely related.

The resulting identification method employs a model structure determined by the following expression for the prediction error

$$\varepsilon(t,\theta) = y(t) - \sum_{k=0}^{n-1} L_k(\theta)V_k(q)u(t-1), \qquad (1)$$

where y, u are the (measured) output, input of the system, L_k is a sequence of unknown coefficients, and V_k reflects the specific basis functions chosen. q^{-1} is the delay operator.

The corresponding least squares identification method has some favourable properties. Firstly it is a linear regression scheme, which leads to a simple analytical solution; secondly it is of the type of *output-error-methods*, which has the advantage that the input/output transfer function can be estimated consistently whenever the unknown noise disturbance on te output is uncorrelated with the input signal (Ljung, 1987).

However, it is well known that for moderately damped systems, and/or in situations of high sampling rates, it may take a large value of n, the number of coefficients to be estimated, in order to capture the essential dynamics of the system G into its model. If we are able to improve the basis functions, we can arrive at an accurate description of the model with only few coefficients to be estimated. This is beneficial for both aspects of bias and variance of the model estimate.

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In section 3 we will first present the general class of orthogonal basis functions that we will employ. After the formulation of the identification problem in section 4, we will introduce and discuss the *Hambo*-transformation of signals and systems that is induced by the new basis. This transformation is used in section 6 to present asymptotic bias and variance results of the model estimates. A simulation example in section 7 is added as illustration of the identification method.

2 Notation

We will use the usual notation with $\mathbb{R}^{p\times m}$ the set of real-valued $(p\times m)$-matrices, $\ell_2^{p\times m}[0,\infty)$ the space of matrix sequences $\{F_k\in\mathbb{R}^{p\times m}\}_{k=0,1,2,\cdots}$ such that $\sum_{k=0}^{\infty} tr(F_k^T F_k)$ is finite; $\mathcal{H}_2^{p\times m}$ the set of real $p\times m$ matrix functions that are squared integrable on the unit circle, $\|\cdot\|_p$ the p-norm of a vector, and $\|\cdot\|_{\mathcal{H}_2}$ the $\mathcal{H}_2$-norm of a stable transfer function; $\bar{E}$ is the operator $\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E$, and e_i is the i-th Euclidian basis vector in $\mathbb{R}^n$.
The transfer function $G(z)$ has a state space realization (A,B,C,D), if $G(z)=C(zI-A)^{-1}B+D$. The realization is balanced if the controllability and observability Gramians are equal and diagonal. A system $G\in\mathcal{H}_2$ is called inner if it is stable and it satisfies $G(z^{-1})G(z)=1$.

3 Generalized orthonormal basis functions

We will consider the generalized orthogonal basis functions that were introduced in [5], based on the preliminary work [4].

Theorem 3.1 *Let $G_b(z)$ be a scalar inner function with McMillan degree $n_b>0$, having a minimal balanced realization (A,B,C,D). Denote*

$$V_k(z) := z(zI-A)^{-1}BG_b^k(z) \qquad (2)$$

Then the sequence of scalar rational functions $\{e_i^T V_k(e^{i\omega})\}_{i=1,\cdots,n_b;k=0,\cdots\infty}$ forms an orthonormal basis for the Hilbert space $\mathcal{H}_2$. □

Note that these basis functions exhibit the property that they can incorporate systems dynamics in a very general way. One can construct an inner function G_b from any given set of poles, combining e.g. both fast and slow dynamics in damped and resonant modes.
A direct result is that for any specifically chosen $V_k(z)$, any strictly proper transfer function $G(z)\in\mathcal{H}_2$ has a unique series expansion $G(z)=z^{-1}\sum_{k=0}^{\infty} L_k V_k(z)$ with $L_k\in\ell^{1\times n_b}[0,\infty)$.

For specific choices of $G_b(z)$ well known classical basis functions can be generated. The choice $G_b(z)=z^{-1}$ leads to the standard pulse basis $V_k(z)=z^{-k}$. The first order inner function $G_b(z)=(1-az)/(z-a)$, $|a|<1$, generates the Laguerre basis

$$V_k(z)=\sqrt{1-a^2}\,z\frac{(1-az)^k}{(z-a)^{k+1}},$$

whereas a second order inner function $G_b(z)=\dfrac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}$ with some real-valued b,c satisfying $|c|,|b|<1$, induces the Kautz basis [7, 16].
Dually, when writing $V_k(z)=\sum_{\ell=0}^{\infty}\phi_k(\ell)z^{-\ell}$ it is straightforward that $\{e_i^T\phi_k(\ell)\}_{i=1,\cdots,n_b;k=0,\cdots,\infty}$ is an orthonormal basis for the signal space $\ell_2[0,\infty)$. These ℓ_2 basis functions can also be constructed directly from G_b and its balanced realization (A,B,C,D), see [5].

4 Identification of expansion coefficients

Following the standard framework of Ljung [9] we will consider a data generating system:

$$y(t)=G_0(q)u(t)+v(t)$$

with G_0 a scalar stable transfer function, and v a stochastic process, uncorrelated with u. The prediction error that results from applying the appropriate model structure is given in (1). We will assume that the input signal $\{u(t)\}$ is a quasi-stationary signal [9] having a rational spectral density $\Phi_u(\omega)$, with a stable spectral factor $H_u(e^{i\omega})$, i.e. $\Phi_u(\omega)=H_u(e^{i\omega})H_u(e^{-i\omega})$.
The unknown parameter θ is written as:

$$\theta := [L_0\cdots L_{n-1}]^T \quad \in\mathbb{R}^{n_b\cdot n}. \qquad (3)$$

We will further denote $x_k(t):=V_k(q)u(t-1)$ and

$$\psi(t):=[x_0^T(t)\;x_1^T(t)\;\cdots\;x_{n-1}^T(t)]^T \qquad (4)$$

and consequently $\varepsilon(t,\theta)=y(t)-\psi^T(t)\theta$.
Following [9], under weak conditions the least squares parameter estimate $\hat{\theta}_N(n) := \arg\min_\theta \frac{1}{N}\sum_{t=0}^{N-1}\varepsilon(t,\theta)^2$ will converge with probability 1 to the asymptotic estimate

$$\theta^*(n)=R(n)^{-1}F(n) \qquad (5)$$

with $R(n)=\bar{E}\psi(t)\psi^T(t)$ and $F(n)=\bar{E}\psi(t)y(t)$.
For the analysis of bias and variance errors of this identification scheme, we will further use the following notation:

$$G_0(z) = z^{-1}\sum_{k=0}^{\infty} L_k^{(0)}V_k(z)$$
$$\theta_0=[L_0^{(0)}\cdots L_{n-1}^{(0)}]^T;\qquad \theta_e=[L_n^{(0)}\;L_{n+1}^{(0)}\cdots]^T$$
$$\psi_e(t) = [x_n^T(t)\;x_{n+1}^T(t)\cdots]^T$$

leading to the following expression for $y(t)$:

$$y(t)=\psi^T(t)\theta_0+\psi_e^T(t)\theta_e+v(t). \qquad (6)$$

For the analyis of the statistical properties of the identified model, fruitful use can be made of a signal and system transformation, induced by the new set of basis functions. This transformation is presented next.

5 The Hambo transform of signals and systems

The presented generalized orthonormal basis for $\mathcal{H}_2/\ell_2$ induces a transformation of signals and systems to a transform domain. Next to the intrinsic importance of signal and systems analysis in this transform-domain (for some of these results see [4]), we can fruitfully use these transformations in the analysis of statistical properties of the identified models.

Definition 5.1 *(Hambo-transform). Let $\{\phi_k(t)\}_{k=0,\cdots\infty}$ be an orthonormal basis for ℓ_2, being generated by an inner function G_b with McMillan degree n_b as presented in section 3. Then we define the Hambo-transform as the mapping: $\ell_2^m \to \mathcal{H}_2^{n_b\times m}$, determined by*

$$\tilde{x}(\lambda) := \sum_{k=0}^{\infty} \mathcal{X}(k)\lambda^{-k} \tag{7}$$

with $\mathcal{X}(k) := \sum_{t=0}^{\infty} \phi_k(t)x^T(t)$, where x is an arbitrary signal in ℓ_2^m. □

Through the transformation, the ℓ_2-behaviour or graph of a dynamical system is transformed to a transform domain. This induces a corresponding transformation of the system's transfer function.

Proposition 5.2 [14] *Let $G \in \mathcal{H}_2$ and let $u, y \in \ell_2$ such that $y(t) = G(q)u(t)$, and consider the Hambo transform as defined in definition 5.1, induced by an inner function G_b with balanced realization (A, B, C, D). Then*

a. There exists a $\tilde{G} \in \mathcal{H}_2^{n_b\times n_b}$ satisfying $\tilde{y}(\lambda) = \tilde{G}(\lambda)\tilde{u}(\lambda)$.

b. Let G be written as $G(z) = \sum_{k=0}^{\infty} g_k z^{-k}$. Then $\tilde{G}$ in (a) is determined by

$$\tilde{G}(\lambda) = \sum_{k=0}^{\infty} g_k N(\lambda)^k \tag{8}$$

with $N(\lambda) := A + B(\lambda I - D)^{-1}C$. □

The Hambo-transform of any system G can be obtained by a simple variable-transfomation on the original transfer function, where the variable transformation concerned is given by $z^{-1} = N(\lambda)$. In the case of a Laguerre basis this transformation becomes $z = (\lambda + a)/(1 + a\lambda)$ (see also [15]). Note that, as G is scalar, $N(\lambda)$ is an $n_b \times n_b$ rational transfer function matrix of McMillan degree 1 (since D is scalar). Note also the appealing symmetric structure of this result. Whereas G_b has a balanced realization (A, B, C, D), $N(\lambda)$ has a realization (D, C, B, A) which also can be shown to be balanced. One of the typical properties of this transformation is formulated next.

Proposition 5.3 [14] *Consider a scalar inner transfer function $G_b(z)$ generating an orthonormal basis as discussed before. Then $\tilde{G}_b(\lambda) = \lambda^{-1}I_{n_b}$, with I_{n_b} the $n_b \times n_b$ identity matrix.*

The basis generating inner function transforms to a simple shift in the Hambo-domain.

The transformation discussed also generalizes to the situation of (quasi-)stationary stochastic processes. Let v be a scalar valued stochastic process or quasi-stationary signal (Ljung, 1987), having a rational spectral density $\Phi_v(\omega)$. Let $H_v(e^{i\omega})$ be a stable spectral factor of $\Phi_v(\omega)$, then the Hambo-transform of the spectral density $\Phi_v(\omega)$ will be defined as

$$\tilde{\Phi}_v(\omega) := \tilde{H}_v^T(e^{-i\omega})\tilde{H}_v(e^{i\omega}). \tag{9}$$

A closed form construction of the Hambo transform of any stable system G is given in the next Proposition.

Proposition 5.4 *Let G_b be an inner function with McMillan degree n_b and balanced realization (A, B, C, D), inducing a corresponding Hambo transform. Let $G \in \mathcal{H}_2$ be given by:*

$$G(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{n_g} z^{-n_g}}{1 + a_1 z^{-1} + \cdots + a_{n_g} z^{-n_g}}. \tag{10}$$

Then the Hambo transform of G has a state space realization $(A_{ort}, B_{ort}, C_{ort}, D_{ort})$ given by

$$\begin{bmatrix} A_{ort} & B_{ort} \\ C_{ort} & D_{ort} \end{bmatrix} = \begin{bmatrix} A_e - B_e(F_2D_e)^{-1}F_2C_e & B_e(F_2D_e)^{-1} \\ F_1C_e - F_1D_e(F_2D_e)^{-1}F_2C_e & F_1D_e(F_2D_e)^{-1} \end{bmatrix}$$

with $A_e \in \mathbb{R}^{n_g\times n_g}$, $B_e \in \mathbb{R}^{n_g\times n_b}$, $C_e \in \mathbb{R}^{n_b(n_g+1)\times n_g}$, $D_e \in \mathbb{R}^{n_b(n_g+1)\times n_b}$ given by:

$$A_e = \begin{bmatrix} D & 0 & 0 & . & . \\ CB & D & 0 & . & . \\ CAB & CB & D & . & 0 \\ . & . & . & . & . \\ CA^{n_g-2}B & . & . & . & D \end{bmatrix} \quad B_e = \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{n_g-1} \end{bmatrix}$$

$$C_e = \begin{bmatrix} 0 & 0 & 0 & . & 0 \\ B & 0 & 0 & . & . \\ AB & B & 0 & . & . \\ A^2B & AB & B & . & 0 \\ . & . & . & . & . \\ A^{n_g-1}B & . & . & . & B \end{bmatrix} \quad D_e = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \\ . \\ A^{n_g} \end{bmatrix}$$

$$\begin{aligned} F_1 &= [b_0I_{n_b} \;\; b_1I_{n_b} \;\cdots\; b_{n_g}I_{n_b}] \in \mathbb{R}^{n_b\times n_b(n_g+1)} \\ F_2 &= [I_{n_b} \;\; a_1I_{n_b} \;\cdots\; a_{n_g}I_{n_b}] \in \mathbb{R}^{n_b\times n_b(n_g+1)}. \end{aligned}$$

Proof: The transform can be written as $G_{ort}(z) = G_1(z)G_2^{-1}(z)$ where

$$\begin{aligned} G_1(z) &= b_0 \times I + b_1 \times N(z) + \cdots + b_{n_g} \times N^{n_g}(z) \\ G_2(z) &= I + a_1 \times N(z) + \cdots + a_{n_g} \times N^{n_g}(z) \end{aligned}$$

It is straightforward that $G_j(z)$ has realization $(A_e, B_e, F_jC_e, F_jD_e)$. It can be readily veryfied ([4]) that $G_1(z)G_2^{-1}(z)$ is a fractional representation of G_{ort} and that $(A_{ort}, B_{ort}, C_{ort}, D_{ort})$ is a realization of G_{ort}. Here the stability of G and G_b ensures the invertibility of F_2D_e. □

Corollary 5.5 *Consider the situation of the previous proposition. Then the eigenvalues of A_{ort}, denoted as $\{\lambda_i\}_{i=1,\cdots,n_g}$, satisfy $\lambda_i = G_b(\mu_i^{-1})$, with $\{\mu_i\}_{i=1,\cdots,n_g}$ the poles of G.*

Proof: Analysing Proposition 5.4 for the case $n_g = 1$ (i.e G has one pole in $z = -a_1$, shows that the resulting transformed system has a pole in $D - a_1C(I + a_1A)^{-1}B$, which is equal to $G_b(-a_1^{-1})$. Since G_{ort} can be written as the product of (the transform of) such first order systems, it follows that all poles can be written in this way. □

6 Asymptotic bias and variance errors

In this section we will review some of the main results of the least squares (linear regression) identification method as sketched in section 4. For the proofs of the results, the reader is referred to [14, 13].

Some important properties of the asymptotic parameter estimate hinge on the characteristics of the matrix $R(n)$, that occurs in (5).

Proposition 6.1 *Consider the matrix $R(n)$ as in (5).*

(a) $R(n)$ is block-Toeplitz, being the covariance matrix related to the spectral density function $\tilde{\Phi}_u(\omega)$.

(b) For all n, the eigenvalues $\lambda_j(R(n))$ of $R(n)$ are bounded by

$$ess \inf_\omega \Phi_u(\omega) \leq \lambda_j(R(n)) \leq ess \sup_\omega \Phi_u(\omega);$$

(c) $\lim_{n\to\infty} \max_j \lambda_j(R(n)) = ess \sup_\omega \Phi_u(\omega)$.

These results on $R(n)$ can be employed in the derivation of upper bounds for the asymptotic bias errors both in estimated parameters and in the resulting transfer function estimate. Combining equations (5) and (6), it follows that $\theta^* - \theta_0 = R(n)^{-1}\bar{E}[\psi(t)\psi_e^T(t)\theta_e]$. Consequently

$$\|\theta^* - \theta_0\|_2 \leq \|R(n)^{-1}\|_2 \cdot \|\bar{E}[\psi(t)\psi_e^T(t)]\|_2 \cdot \|\theta_e\|_2, \quad (11)$$

where for a (matrix) operator T, $\|T\|_2$ refers to the induced operator 2-norm. For simplicity of notation we have skipped the dependence of θ^* (and θ_0) on n.

The asymptotic bias errors can now be bounded as follows.

Proposition 6.2 *Consider the identification set-up as discussed in section 4. Then*

(a) $\|\theta^* - \theta_0\|_2 \leq \dfrac{ess \sup_\omega \Phi_u(\omega)}{ess \inf_\omega \Phi_u(\omega)} \cdot \|\theta_e\|_2$

(b) For all $\omega_1 \in I\!R$, $|G(e^{i\omega_1}, \theta^*) - G_0(e^{i\omega_1})| \leq$

$$\leq \|V_0(e^{i\omega_1})\|_\infty \{\sqrt{n_b n} \frac{ess \sup_\omega \Phi_u(\omega)}{ess \inf_\omega \Phi_u(\omega)} \|\theta_e\|_2 + \|\theta_e\|_1\}, \quad (12)$$

where $\|V_0(e^{i\omega_1})\|_\infty$ is the ℓ_∞-induced operator norm of the matrix $V_0(e^{i\omega_1}) \in \mathbb{C}^{n_b\times 1}$, i.e. the maximum absolute value over the elements in $V_0(e^{i\omega_1})$.

(c) $\|G(z,\theta^*) - G_0(z)\|_{\mathcal{H}_2} \leq \{1 + \dfrac{ess \sup_\omega \Phi_u(\omega)}{ess \inf_\omega \Phi_u(\omega)}\}\|\theta_e\|_2,$

where $\|\theta_e\|_2 = \sqrt{\sum_{k=n}^\infty L_k^{(0)} L_k^{(0)^T}}$. □

In this result an upper bound for $\|R(n)^{-1}\|_2$ is employed as provided by Proposition 6.1. In many situations the input signal and its statistical properties will be known, and $\|R(n)^{-1}\|_2$ can be exactly calculated. In that case we can replace $(ess \inf_\omega \Phi_u(\omega))^{-1}$ in the expressions above by $\|R(n)^{-1}\|_2$.

Note that these asymptotic bias errors are dependent on the basis functions chosen. The factor $\|\theta_e\|_2^2$ is determined by the convergence rate of the series expansion of G_0 in the generalized basis.

Proposition 6.3 ([5]) *Let $G_0(z)$ have eigenvalues μ_i, $i = 1, \cdots, n_g$, and let $G_b(z)$ have eigenvalues ρ_j, $j = 1, \cdots n_b$. Denote*

$$\lambda := \max_i \prod_{j=1}^{n_b} \left| \frac{\mu_i - \rho_j}{1 - \mu_i \rho_j} \right|. \quad (13)$$

Then there exists a constant $c \in I\!R$ such that for all $\eta > \lambda$

$$\|\theta_e\|_2 \leq c \cdot \frac{\eta^{n+1}}{\sqrt{1-\eta^2}}. \quad (14)$$

When the two sets of eigenvalues converge to each other, λ will tend to 0, the upper bound on $\|\theta_e\|_2$ will decrease drastically, and the bias error will reduce accordingly.

The results of Proposition 6.2 (a), (c), show that we achieve consistency of the parameter and transfer function estimates as $n \to \infty$ provided that the input spectrum is bounded away from 0 and $\|\theta_e\|_2 \to 0$ for $n \to \infty$. The latter condition is guaranteed if $G_0 \in \mathcal{H}_2$.

As in the FIR-case, corresponding with $G_b(z) = z^{-1}$, we can arrive at consistent parameter estimates for finite n under specific experimental conditions.

Corollary 6.4 *Consider the identification setup as discussed in section 4. If $\tilde{H}_u$ is an inner transfer function, then for all $n \geq 1$ it follows that $\theta^* = \theta_0$.*

Note that a special case of the situation of an inner $\tilde{H}_u$ is obtained if the input signal u is uncorrelated (white noise). In that situation $H_u = 1$ and consequently $\tilde{H}_u = I_{n_b}$, being inner.

For the asymptotic variance of the estimate the following generalization of the (classical) FIR result is obtained.

Theorem 6.5 *Assume the spectral density $\Phi_u(\omega)$ to be bounded away from zero and sufficiently smooth. Then for $N, n \to \infty$, $n^2/N \to 0$:*

$$\sqrt{\frac{N}{n}} cov(G(e^{i\omega_1}, \hat{\theta}_N), G(e^{i\omega_2}, \hat{\theta}_N)) \to$$
$$\begin{cases} 0 & \text{for } G_b(e^{i\omega_1}) \neq G_b(e^{i\omega_2}), \\ V_0^T(e^{i\omega_1})V_0(e^{-i\omega_1}) \cdot \dfrac{\Phi_v(\omega_1)}{\Phi_u(\omega_1)} & \text{for } \omega_1 = \omega_2. \end{cases}$$

Theorem 6.5 gives a closed form expression for the asymptotic covariance. It implies that the variance of the transfer function estimate for a specific ω_1 is given by the noise to input signal ratio weighted with an additional weighting factor that is determined by the basis functions. This additional weighting, which is not present in the case of FIR estimation, again generalizes the weighting that is also present in the case of Laguerre [15] and Kautz functions [16]. Since the frequency function $V_0(e^{i\omega})$ has a low pass character, it ensures that the variance will have a roll-off at high frequencies. This is unlike the case of FIR estimation, where the absolute variance generally increases with increasing frequency.

The role of V_0 in this variance expression clearly shows that there is a design variable involved that can be chosen also from a point of view of variance reduction. In that case V_0 has to be chosen in such a way that it reduces the effect of the noise ($\Phi_v(\omega)$) in those frequency regions where the noise is dominating.

7 Simulation example

In order to illustrate the identification method considered in this paper, we will show the results of an example where an identification is performed on the basis of simulation data.

The simulated system is determined by $G_0(z) =$

$$\frac{0.2530z^{-1}-0.9724z^{-2}+1.4283z^{-3}-0.9493z^{-4}+0.2410z^{-5}}{1-4.15z^{-1}+6.8831z^{-2}-5.6871z^{-3}+2.3333z^{-4}-0.3787z^{-5}} \quad (15)$$

having poles: $0.95 \pm 0.2i$, $0.85 \pm 0.09i$ and 0.55. The static gain of the system is 0.9966.

An output noise is added to the simulated output, coloured by a second order noise filter. As input signal is chosen a zero mean unit variance white noise signal, leading to a signal-to-noise ratio at the output of 10dB, being equivalent to around 30% noise disturbance in amplitude.

Orthogonal basis functions have been chosen generated by a fourth order inner function, having poles: $0.9 \pm 0.3i$ and $0.7 \pm 0.2i$. We have used a data set of input and output signals with length $N = 1200$, and have estimated 5 coefficients of the series expansion.

In figure 3 the Bode amplitude plot of G_0 is sketched together with the amplitude plots of each of the four components of V_0, i.e. the first four basis functions. Note that all other basis functions will show the same Bode amplitude plot, as they only differ in postmultiplication by an inner function, which does not change its amplitude. We have used 5 different realizations of 1200 data points to estimate 5 different models. Their Bode amplitude plots are given in figure 1 and the corresponding step responses in figure 4.

To illustrate the power of the identification method, we have made a comparison with the identification of 5th order (least squares) output error models. In figures 2 and 5 the results of the estimated 5th order output error models are sketched. Here also five different realizations of the input/output data are used. It can be observed that the models based on the generalized orthogonal basis functions have a good ability to identify the resonant behaviour of the system in the frequency range from 0.15 to 0.5 rad/sec, while the output error models clearly have less performance here. The variance of both types of identification methods seem to be comparable. Note that the OE algorithm requires a nonlinear optimization whereas the expansion procedure is a convex optimization problem.

Discussion

We have discussed and illustrated the use of a general class of orthonormal basis functions in system identification. These basis functions generalize the well known pulse, Laguerre and Kautz basis functions. When chosen properly, these basis functions can provide a linear model parametrization that can provide accurate estimates by only identifying a few parameters, and utilizing simple linear regression schemes. Expressions for asymptotic bias and variance are available.

The identification method discussed, points to the use of iterative procedures, where the basis functions are updated iteratively by previously estimated models. Such iterative methods already have been applied succesfully in practical experiments, see e.g. De Callafon *et al.*(1993).

Apart from the identification of nominal models, the basis functions introduced here, have also been applied succesfully in the identification of model uncertainty bounds, see De Vries (1994) and Hakvoort (1994).

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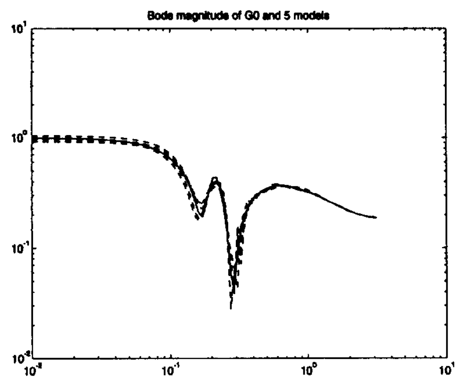

Fig. 1: Bode amplitude plot of simulated system G_0 (solid) and of five estimated finite expansion models with $n = 5$, $N = 1200$ (dashed).

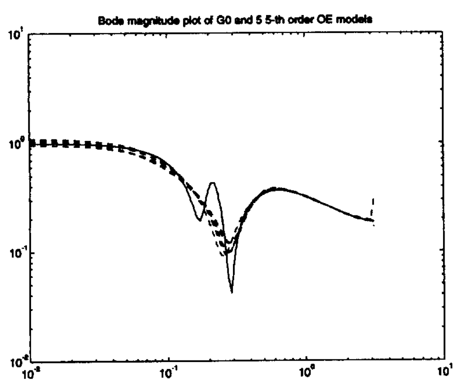

Fig. 2: Bode amplitude plot of G_0 (solid) and five 5-th order OE models (dashed).

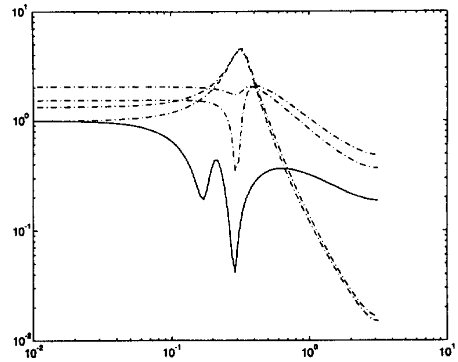

Fig. 3: Bode amplitude plot of simulated system G_0 (solid) and of basis functions V_0 (four-dimensional) (dashed).

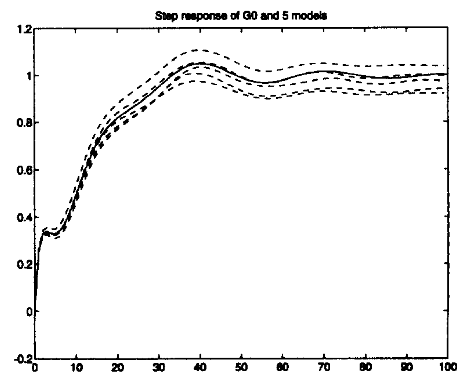

Fig. 4: Step response of simulated system G_0 (solid) and of five estimated finite expansion models with $n = 5$, $N = 1200$ (dashed).

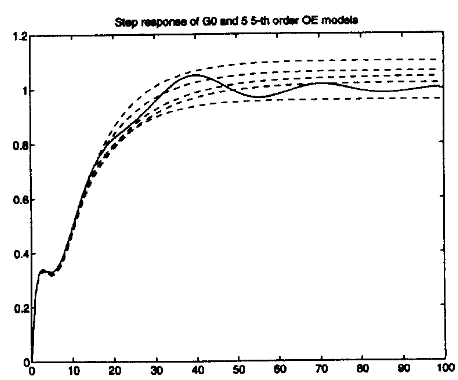

Fig. 5: Step response of G_0 (solid) and five 5-th order OE models (dashed).