Abstract—For the problem of identifying a target module that is embedded in a dynamic network with known interconnection structure, different sets of conditions are available for the set of node signals to be measured and the set of excitation signals to be applied at particular node locations. In previous work these conditions have typically been derived from either an indirect identification approach, considering external excitation signals as inputs, or from a direct identification approach, considering measured node signals as inputs. While both approaches lead to different sets of (sufficient) conditions, in this paper we extend the flexibility in the sufficient conditions for selection of excitation and measured node signals, by combining both direct and indirect approaches. As a result we will show the benefits of using both external excitation signals and node signals as predictor inputs. The provided conditions allow us to design sensor selection and actuation schemes with considerable freedom for consistent identification of a target module.

I. INTRODUCTION

In recent years increasing attention has been given to the identification of large-scale dynamically interconnected systems (modules), known as dynamic networks. Among the large amount of literature on this topic, there are three main research trends. The first one deals with the identification of the interconnection structure (topology) of systems in the dynamic network [1], [2], [3], [4]. The second deals with identification of the full network dynamics [5], [6], [7], while the third deals with identification of a target module in the dynamic network under the assumption of known topology (known as local module identification, see [8], [9], [10], [11], [12], [13]).

In this paper we focus on the local module identification problem. In [8], the classical direct-method [14] for closed-loop identification has been generalized to a dynamic network framework using a MISO identification setup. It introduces a method to achieve a consistent estimate of the target module when all the node signals in the MISO setup are measured. In [15], an extension has been made towards the situation where some of the node signals might be non-measurable. In [16] and [17], an approach has been introduced to consistently estimate the target module in the presence of confounding variables due to non-measured nodes and noise correlation. The direct method has been extended to a Bayesian setting in [9], where regularized kernel-based methods are used to reduce the mean-square error of the target module estimates. In [12] and [10], the situation has been addressed where the node measurements are affected by sensor noise.

An important condition in the works that use the direct method [8], [15], [16], [17] is that all parallel paths from the input of the target module to its output and all loops through the output node should pass through a measured node signal that is included as a predictor input. This requirement ensures that the identified module using the direct method is equal to the target module. However, in practical situations, there can be parallel paths and loops that might have all nodes non-measured. This creates a restriction for the selection of measured node signals.

In indirect method as in [11], [7], external excitation signals are used as predictor inputs for an open loop MIMO identification problem. These methods involve two steps: (1) First obtain consistent estimates of a transfer function from external signals to measured node signal; (2) Using these estimates obtain consistent estimates of the target module (we call this step as post-processing). In [7], the freedom in selection of measured node signal is exploited under the condition that all nodes are excited.

In this paper we extend the flexibility in the sufficient conditions for selection of excitation and measured node signals for consistent target module estimates and thereby generalizing the sensing and actuation schemes. We relax the above discussed condition on the parallel paths and loops around the output node. This relaxation in conditions are achieved by combining elements of both direct and indirect approaches. We use both the node signals and external excitation signals as predictor inputs, allow post-processing of module estimates, use MIMO identification setting and thereby mixing both direct and indirect methods. The provided conditions allow us to design sensor selection and actuation schemes with considerable freedom for consistent identification of a target module.

II. NETWORK AND IDENTIFICATION SETUP

A. Dynamic network setup

Following the basic setup of [8], a dynamic network is built up of $L$ scalar internal variables or nodes $w_j$, $j = 1, \ldots, L$, and $K$ external variables $r_k$, $k = 1, \ldots, K$. Each internal variable is described as:
where \( q^{-1} \) is the delay operator, i.e. \( q^{-1} w_j(t) = w_j(t - 1) \);
- \( G_{ji} \) is a proper rational transfer referred to as modules;
- \( u_j(t) \) is generated by the external variables \( r_k(t) \) that can directly be manipulated by the user and is given by \( u_j(t) = \sum_{k=1}^{K} R_{jk}(q)r_k(t) \) where \( R_{jk} \) are stable, proper rational transfer functions;
- \( v_j \) is process noise, where the vector process \( v = [v_1 \ldots v_L]^T \) is modelled as a stationary stochastic process with rational spectral density \( \Phi_v(\omega) \), such that there exists a white noise process \( e := [e_1 \ldots e_L]^T \), with covariance matrix \( \Lambda > 0 \) such that \( v(t) = H(q)e(t) \), where \( H \) is square, stable, monic and minimum-phase.

We will assume that the standard regularity conditions on the data are satisfied that are required for convergence results of prediction error identification method\(^1\). In this paper we consider the situation where \( u_j(t) = \sum_{k=1}^{K} R_{jk}(q)r_k(t) \), and \( R_{jk} = 1 \) if \( j = k \), \( R_{jk} = 0 \) if \( j \neq k \), and \( j = 1, \ldots, L \).

When combining the \( L \) node signals we arrive at the full network expression

\[
[w_1] = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1} & \cdots & G_{L \cdot L - 1} & 0 \\ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \\ \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_L \\ \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \\ \end{bmatrix}
\]

which results in the matrix equation:

\[
w = Gw + Rw + He.
\]

We will assume that the dynamic network is stable, i.e. \((I - G)^{-1}\) is stable, and well posed (see [18] for details).

The identification problem to be considered is the problem of identifying one particular module \( G_{ji}(q) \) on the basis of a selection of measured variables \( w \), and possibly \( r \).

**B. Direct method**

Let us define \( N_j^- \) as the set of node indices \( k \) such that \( G_{jk} \neq 0 \), i.e. the node indices in \( N_j^- \) are the \( w \)-in-neighbors of the node signal \( w_j \). The nodes corresponding to the set of node indices \( k \) such that \( G_{kj} \neq 0 \) are called the \( w \)-out-neighbors of \( w_j \) (i.e. \( N_j^+ \)). Let \( D_j \) denote the set of indices of the internal variables that are chosen as predictor inputs. Let \( \hat{D}_j \) denote the set of indices not in \( \{j\} \cup D_j \), i.e. \( \hat{D}_j = \{1, \ldots, L\} \setminus \{j\} \cup D_j \). Let \( w_0 \) denote the vector \( [w_{k_1} \cdots w_{k_n}]^T \), where \( \{k_1, \ldots, k_n\} = D_j \). Let \( w_0 \) denote the vector \( [w_{k_1} \cdots w_{k_n}]^T \), where \( \{k_1, \ldots, k_n\} = D_j \), and where the \( \ell \)th entry is zero if \( w_\ell \) is not present in the network. The vectors \( w_0, w_2, w_0, w_0 \) and \( w_2 \) are defined analogously. The ordering of the elements in \( w_0, w_2, \) and \( u_0 \) is not important, as long as it is the same for all vectors. The transfer function matrix between \( w_0 \) and \( w_j \) is denoted \( G_{jo} \). The other transfer function matrices are defined analogously.

By this notation, the network equation (2) is rewritten as:

\[
\begin{bmatrix} w_j \\ w_0 \\ w_2 \\ w_0 \\ w_0 \\ w_2 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 & G_{jo} & G_{jo} \\ G_{jo} & 0 & G_{jo} \\ G_{jo} & G_{jo} & 0 \\ G_{jo} & G_{jo} & G_{jo} \\ G_{jo} & G_{jo} & G_{jo} \\ G_{jo} & G_{jo} & G_{jo} \end{bmatrix} \begin{bmatrix} w_j \\ v_j \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} u_j \\ u_0 \\ u_2 \\ u_0 \end{bmatrix},
\]

where \( G_{jo} \) and \( G_{jo} \) have zeros on the diagonal.

Identification of module \( G_{ji} \) can be done by selecting \( D_j \) such that \( i \in D_j \), and subsequently estimating a multiple-input single output model for the transfer functions in \( G_{jo} \). This can be done by considering the one-step-ahead predictor\(^2\) \( \hat{w}_j(t|t-1) = \mathbb{E}[w_j(t) | w_j(t-1), w_1(t-1), \ldots, w_L(t-1)] \), and the resulting prediction error ((14)) \( \varepsilon_j(t, \theta) = w_j(t) - \hat{w}_j(t|t-1, \theta) \), given by

\[
\varepsilon_j(t, \theta) = H_j(\theta)^{-1} \left( w_j - \sum_{k \in D_j} G_{jk}(\theta) w_k - u_j \right)
\]

where arguments \( q \) and \( t \) have been dropped for notational clarity. The parameterized transfer functions \( G_{jk}(\theta), \theta \in D_j \) and \( H_j(\theta) \) are estimated by minimizing the sum of squared (prediction) errors: \( V_j(\theta) = \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon_j^2(t, \theta) \), where \( N \) is the length of the data set. We refer to this identification method as the direct method, [8]. Let \( \hat{\theta}_N \) denote the minimizing argument of \( V_j(\theta) \).

**C. Indirect method**

As an alternative approach, following the setup of [7], the network model (2) can be re-written as \( w = T_{wr} r + \bar{v} \) where \( T_{wr} = (I - G)^{-1} R \) and \( \bar{v} = (I - G)^{-1} H e \). Using the known external references \( r \) as predictor inputs and measured signals \( w \) as predicted outputs, it is well known that, under appropriate conditions, a consistent estimate \( \hat{T}_{wr} \) of \( T_{wr} \) can be obtained using open loop MIMO identification methods. On the basis of \( \hat{T}_{wr} \), an estimate of \( G \) can be obtained by solving \( (I - G) \hat{T}_{wr} = R \). By solving only a subset of these equations, a target module embedded in the dynamic network can be identified. We refer to this type of identification method that uses external signals as predictor inputs as the indirect method.

**III. BACKGROUND AND MOTIVATING EXAMPLE**

In this section we provide a summary of existing results related to the topic of this paper and subsequently highlight the motivation of the paper using a suitable example. In [15] it has been shown that we can identify the target module \( G_{ji} \) consistently provided that we choose the selection of predictor input signals to satisfy particular properties. One of the main conditions is formulated next.

**Property 1:** To identify a target module \( G_{ji} \), consider a set of internal variables \( w_k, k \in D_j \). Let \( D_j \) satisfy the following properties:

1. \( i \in D_j \) and \( j \notin D_j \);
2. Every path from \( w_i \) to \( w_j \) is blocked by a node \( w_k, k \in D_j \) (parallel path condition);
3. Every loop through \( w_j \) is blocked by a node \( w_k, k \in D_j \) (loop condition).

1See [14] page 249. This includes the property that \( e(t) \) has bounded moments of order higher than 4.

2\( \mathbb{E} \) refers to \( \lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \mathbb{E} \), and \( w_j^f \) and \( w_j^D \) refer to signal samples \( w_j(\tau) \) and \( w_k(\tau), k \in D_j \), respectively, for all \( \tau \leq t \).
When this property is satisfied, using a MISO identification setup with $w_j$ as predicted output and $w_{o_j}$ as predictor inputs, the direct method as discussed in section II-B provides consistent estimate of the target module, if in addition there are no confounding variables for the estimation problem $w_{o_j} \rightarrow w_j^3$. In [16] additional conditions have been formulated for the selection of $D_j$ so as to avoid the presence of confounding variables, typically by choosing additional predictor inputs defined by the set $B_j \subseteq \mathcal{L}\{D_j \cup \{j\}\}$. This situation has been analyzed for the case where all disturbance signals $v$ are mutually uncorrelated, i.e. its spectral density $\Phi_v$ being diagonal. In this situation it is still required that $D_j$ satisfies Property 1. This restrictive property is required for the target module in the dynamic network to be invariant in an immersed network where all non-measured signals are being removed [15].

![Fig. 1. Example network](image)

**Example 1:** Consider a dynamic network as represented in Figure 1 with all noises in $v$ uncorrelated with each other. For identifying the target module $G_{21}$ (in green box), we have $j = 2$, and in order to satisfy Property 1 we need $D_j = \{1, 3, 5\}$ where $w_3$ is included to block parallel path from $w_1$ to $w_2$, and $w_5$ is included to block the loop through $w_2$. Using this set of measured nodes, we arrive at an immersed network after removing the non-measured node as in Figure 2. We can observe that the module between $w_1$ and $w_2$ (the green box) is $G_{21}$ and remains invariant.

![Fig. 2. Immersed network of network in figure 1 [15] where the nonmeasured node $w_3$ has been removed (immersed), and where $v_2 = v_2 + G_{24}v_4$.](image)

If $w_3$ and $w_5$ are not selected in $D_j$, and so $D_j = \{1\}$, we arrive at an immersed network after removing all non-measured nodes, as depicted in Figure 3. We can now observe that the dynamic module between $w_1$ and $w_2$ (the green box in figure 3) is not equal to $G_{21}$. The terms $(1 - G_{25}G_{52})^{-1}$ and $G_{23}G_{31}$ are due to the fact that in this situation the loop and parallel path condition in property 1 are not satisfied, respectively. In this paper we are going to relax these restrictive conditions in property 1 and increase the freedom in the selection of measured node signals.

For the approach based on the indirect identification method, in [11] a method has been presented to identify a target module using external signals as predictor inputs, along the following reasoning. 

**Proposition 1 (from [11]):** In order to identify a target module $G_{ji}$, perform the following experiment:

1) Excite node $w_i$ and all its out-neighbors with sufficiently rich signals. Include these excitation signals as predictor inputs;

2) measure the out-neighbors of $w_i$. Include them as predicted outputs.

Under these conditions and using full order models for the elements of $T_{wr}$, consistent estimates $\hat{T}_{N^+N^+i}$, $\hat{T}_{N^+N^+i}$ of $T_{N^+N^+i}$ and $T_{N^+N^+i}$ can be obtained using an open loop MIMO identification method. Then a consistent estimate of $\hat{G}_{N^+i}$ (which includes the target module) is obtained by,

$$\hat{G}_{N^+i} = \left[\hat{T}_{N^+N^+i}\right]^{-1}\hat{T}_{N^+i} \quad \square$$

A dual of this proposition with in-neighbors of $w_j$ is also provided in [11]. It can be observed that a consistent estimate of the target module is obtained from consistent estimates of elements of $T_{wr}$. We will refer to this step (5) of manipulating identified objects, as post-processing. Considering the earlier Example 1, we can now consistently identify our target module using an open loop MIMO identification setup with $\{r_1, r_2, r_3\}$ as inputs and $\{w_2, w_3\}$ as outputs. However this requires restrictive conditions on the nodes to be excited and nodes to be measured, i.e. measured excitation signals $r_1, r_2, r_3$. Further relaxations of these restrictive conditions on excitation and measured node signals will be addressed in the sequel.

**Definition 1 (confounding variable):** Consider a dynamic network defined by

$$w = Gw + He + u \quad (6)$$
with \( \text{cov}(e) = I \) and \( H \) is not necessarily monic, and consider the graph related to this network, with node signals \( w \) and \( e \). Let \( w_1 \) and \( w_2 \) be two subsets of measured node signals in \( w \) and let \( w \) be the set of non-measured node signals in \( w \).

Then a noise component \( e_i \) in \( e \) is a **confounding variable** for the estimation problem \( w_k \rightarrow w_j \), if in the graph there exist simultaneous paths\(^4\) from \( e_i \) to node signals \( w_k, k \in \mathcal{A} \) and \( w_n, n \in \mathcal{Y} \), while these paths are either direct\(^5\) or only run through nodes that are in \( w_2 \).

**IV. Example to Illustrate the Introduced Method**

In this section we illustrate the introduced method in this paper with suitable examples. In this paper, we combine the ideas of both the direct and indirect method such that we introduce flexibility in the selection of excitation and measured node signals. We use both the measured node signals as well as the excitation signals as predictor inputs. In addition to that, we do not restrict to the situation of invariance of our target module after immersion as in the direct method, but use the mechanism of *post-processing* from the indirect method to consistently identify the target module.

**Example 2:** We now consider the same network as in Example 1 but with two constraints: (a) it is not possible to measure \( w_3 \) and \( w_5 \); (b) it is not possible to excite node \( w_1 \). It can be inferred that it is not possible to consistently estimate \( G_{ji} = G_{21} \) using the direct method due to constraint (a). Similarly due to constraint (b), it is not possible with the indirect method either.

As shown in Example 1, if we do not measure \( w_3 \) and \( w_5 \) our target module changes to \( (1- G_{23}G_{52})^{-1}(G_{21} + G_{23}G_{31}) \) in the immersed network. However, we can see that this module also contains the target module of interest \( G_{21} \). Therefore we might extract the target module from this term if we know (or) find the other contributions.

Consider that we excite node \( w_3 \), \( w_5 \) and measure node \( w_4 \). After immersing the non-measured nodes (see [15]) we end up in a dynamic network setup as in Figure 4. Now consider the identification problem \( \{w_1, w_4, r_2, r_3\} \rightarrow \{w_2, w_4\} \). We can infer the following from the figure:

1. Identifying the transfer from \( r_3 \) \( \rightarrow w_4 \) provides \( G_{43} \) and the transfer from \( w_1 \) \( \rightarrow w_4 \) provide \( G_{43}G_{31} \). Thus we can identify \( G_{31} \):
2. The transfer from \( r_3 \) \( \rightarrow w_2 \) provides \( (1 - G_{23}G_{52})^{-1}G_{23} \). The term \( (1 - G_{23}G_{52})^{-1} \) is due to the fact that in the original network there is a loop around \( w_2 \) which is not “blocked” by a measured node. This term given by the transfer from \( r_2 \) \( \rightarrow w_2 \). Now, we can obtain \( G_{23} \):
3. The term \( G_{23}G_{31} \) is due to the fact that in the original network there is a path from \( w_1 \) to \( w_2 \) through \( w_3 \) which

\[ A \text{ simultaneous path from } e_1 \text{ to node signal } w_1 \text{ and } w_2 \text{ implies that there exist a path from } e_1 \text{ to } w_1 \text{ as well as from } e_1 \text{ to } w_2. \]

\[ A \text{ direct path from } e_1 \text{ to node signal } w_1 \text{ implies that there exist a path from } e_1 \text{ to } w_3 \text{ which do not pass through nodes in } w. \]

is not “blocked” by a measured node. Knowing \( G_{23} \) and \( G_{31} \) from the above two steps, we obtain the term \( G_{23}G_{31} \). We also have \( (1 - G_{23}G_{52})^{-1} \). Eventually we obtain our target module of interest from the transfer \( w_1 \) \( \rightarrow w_2 \) (i.e. \( (1 - G_{23}G_{52})^{-1}(G_{21} + G_{23}G_{31}) \)).

This shows that we can consistently identify the target module \( G_{21} \) if we know or could consistently identify the transfer from \( \{w_1, w_4, r_2, r_3\} \rightarrow \{w_2, w_4\} \). Thus we move to a MIMO identification problem using the prediction error method with \( \{w_2, w_4\} \) as predicted outputs and \( \{w_1, w_4, r_2, r_3\} \) as predictor inputs.

**Remark 1:** Excitation at the output node \( r_2 \) is required as predictor input since the loops through \( r_1 \) are not block by a measured node. However we can still relax this under these conditions, which will be provided in this paper.

**Remark 2:** The consistency results may still require some excitation conditions, which will be specified later on.

From the discussed example, we can now conjecture the following generalization:

1. Violating the parallel path condition can be handled by exciting a node in the parallel path, including the excitation signal in the predictor input, and by measuring a
and Q\textsuperscript{w} that serve as predictor inputs. Next we decompose outputs (excluding the output node of target module); w\textsuperscript{predicted outputs} in order to avoid measurement of additional node signals. adding predictor inputs (see [16], [21]). In this paper we handle using measured nodes as predicted outputs\textsuperscript{6}.

Remark 3: In handling the parallel path condition it will appear that we actually have to add one additional constraint (see Property 2 later on). If the mentioned descendant node is the input of the target module, then this node needs to be excited with an external signal, which is included as predictor input.

Remark 4: If we consider again Example 1 in Figure 1, then the parallel path problem that occurs when w\textsubscript{3} can not be measured, can be compensated for by measuring a descendant from w\textsubscript{3}, which in this case could also be w\textsubscript{1}. Since w\textsubscript{1} is the input of the target module, the previous remark now leads to the situation that w\textsubscript{1} also needs to be excited, which is a situation that was excluded in Example 2.

In the sequel of this paper, we will derive the formal results that underly the above conjectured statements.

V. CONCEPTS AND NOTATION

We will denote u\textsubscript{y} as the node signals in u that serve as predicted outputs, and o\textsubscript{D} as the node signals in o that serve as predictor inputs, and r\textsubscript{P} as the external excitation signals in r that serve as predictor inputs. Next we decompose u\textsubscript{y} and o\textsubscript{D} in disjoint sets according to: \( \mathcal{Y} = \mathcal{Q} \cup \mathcal{O} \cup \{o\} \); \( \mathcal{D} = \mathcal{Q} \cup \mathcal{A} \) where w\textsubscript{Q} are the node signals that are common in u\textsubscript{y} and o\textsubscript{D}; w\textsubscript{A} is the set of node signals that are only predicted outputs (excluding the output node of target module); w\textsubscript{o} is the output w\textsubscript{j} of the target module; if j \( \in \mathcal{Q} \) then \{o\} is void. Additionally we denote \( w_{x} \) as the node signals in w that are neither predicted output nor predictor input, i.e., \( \mathcal{Z} = \mathcal{L} \setminus (\mathcal{D} \cup \mathcal{Y}) \), where \( \mathcal{L} = \{1, 2, \cdots, L\} \). Next we define the set related to r\textsubscript{P} as \( \mathcal{P} \subseteq \mathcal{L} \).

VI. MIMO IDENTIFICATION SETUP

The identification that we need to perform refers to the estimation problem \((w\textsubscript{P}, r\textsubscript{P}) \rightarrow u\textsubscript{y}\). In order to analyse this problem, based on system equation (6), we write the system equations for the output variables w\textsubscript{y}.

Proposition 2: The systems equations for the output variables in u\textsubscript{y} can always be written as,

\[ u\textsubscript{y} = Gw\textsubscript{Q} + H\xi\textsubscript{Q} + Rr\textsubscript{P}, \]

\textsuperscript{6}Confounding variables can also be handled in other ways, for example, adding predictor inputs (see [16], [21]). In this paper we handle using predicted outputs in order to avoid measurement of additional node signals. where \( \xi\textsubscript{Q} \) a white noise process with dimensions conforming to w\textsubscript{y}, with \( \text{cov}(\xi) = \tilde{A} \) and with \( H \) being monic, stable and stably invertible.

Proof: The proof is provided in the appendix. \( \square \)

As a result we can set up a predictor model based on a parametrized model set determined by \( \mathcal{M} := \{(G(\theta), \tilde{H}(\theta), \tilde{R}(\theta), \tilde{A}(\theta)) : \theta \in \Theta \} \), while the actual data generating system is represented by \( \mathcal{S} = (G(\theta_{0}), \tilde{H}(\theta_{0}), \tilde{R}(\theta_{0}), \tilde{A}(\theta_{0})) \). The corresponding identification problem is defined by considering the one-step-ahead prediction of w\textsubscript{y}, according to

\[ \hat{y}_{3}(t \mid t-1) := \mathbb{E}\{y_{3}(t) \mid w_{y}^{t-1}, u_{y}^{t}, r_{P}^{t}\} \]

where \( w_{y}^{t}, r_{P}^{t} \) denotes the past of w\textsubscript{y}, r\textsubscript{P} respectively, i.e., \{w\textsubscript{y}(k) and r\textsubscript{P}(k), k \leq t\}. The resulting prediction error becomes:

\[ e(t, \theta) := y_{3}(t) - \hat{y}_{3}(t \mid t-1; \theta) \]

\[ = \tilde{H}(q, \theta)^{-1} \left[ w_{y}(t) - \tilde{G}(q, \theta)w_{o}(t) - \tilde{R}(q, \theta)r_{P}(t) \right], \]

and the weighted least squares identification criterion

\[ \hat{\theta}_{N} = \arg \min_{\theta} \frac{1}{N} \sum_{t=0}^{N-1} e^{T}(t, \theta)W e(t, \theta), \]

with W any positive definite weighting matrix. This parameter estimate then leads to an estimated subnetwork \( \tilde{G}\textsubscript{yP}(q, \hat{\theta}_{N}) \), with the estimated module \( \tilde{G}\textsubscript{y}(q, \hat{\theta}_{N}) \) as one of its scalar entries.

VII. MAIN RESULTS

In this section we first present the consistency results for the above considered identification problem.

Theorem 1: Consider a (MIMO) network identification setup with predictor inputs \((w\textsubscript{P}, r\textsubscript{P})\) and predicted outputs w\textsubscript{y} as in (7). Then a prediction error identification method according to (8)-(9), applied to a parametrized model set \( \mathcal{M} \) will provide a consistent estimate of \( \hat{G} \) and \( \tilde{R} \), if the following conditions on the sets of nodes are satisfied:

a) there are no confounding variables for the estimation problem \( w_{A} \rightarrow w_{y} \); b) Every measured node signal that has an unmeasured path to a node signal in w\textsubscript{y} is included in w\textsubscript{o}; and additionally:

1) \( \mathcal{M} \) is chosen to satisfy \( \mathcal{S} \in \mathcal{M} \); 2) \( \Phi_{q}(\omega) > 0 \) for a sufficiently high number of frequencies, where \( \kappa(t) := [w_{y}^{T} \xi_{Q}^{T} \xi_{Q}^{T} r_{P}^{T}]^{T}; \)
3) All \( r_{k}, k \in \mathcal{P} \) are uncorrelated to all \( \xi_{l}, l \in \mathcal{Y} \cup \mathcal{A}; \)
4) All the elements in \( G_{\mathcal{Q}A}, G_{\mathcal{Q}Q}, G_{\mathcal{A}Q}, G_{\mathcal{A}A} \) are strictly proper (or) all existing paths/loops from \( w_{Q}, w_{P}, w_{0} \) to \( w_{Q}, w_{0} \) have at least a delay.

Proof: The proof is provided in the appendix. \( \square \)

The estimate \( \hat{G} \) contains the estimate of \( \hat{G}_{\mathcal{A}} \) as one of its elements since \( w_{i} \subseteq w_{Q} \). However our final goal is to estimate our target module \( \hat{G}_{\mathcal{A}} \) which will be present in \( \hat{G} \).
but will need to be extracted from this matrix through post-processing. For this post-processing step we will require two additional sets:

- A set $Z_r \subseteq \mathcal{Z} \cap \mathcal{P}$ which represents externally excited nodes in unmeasured paths\(^7\) from $w_i$ to $w_j$ and in loops around $w_j$; and
- A set $\mathcal{T} \subseteq \mathcal{Y}\setminus\{j\}$ which, for each of the nodes in $Z_r$, represents a measured descendant node that has an unmeasured path from $w_{x_r}$, while $w_j$ is excluded from $\mathcal{T}$. Note that for each node $w_k, k \in Z_r$, the corresponding element in $\mathcal{T}$ is a measured node, and therefore cannot be in the corresponding unmeasured path from $w_i$ to $w_j$ or loops around $w_j$, that passes through $w_k$. Therefore the descendant in $\mathcal{T}$ typically breaks out of these unmeasured parallel paths/loops, as illustrated in Figure 5.

These two sets will play a major role in extracting the target module estimate from the identification result $\hat{G}_i, \hat{R}_i$. The properties that $Z_r$ and $\mathcal{T}$ need to satisfy to realize this post-processing are formulated next.

**Property 2 (Properties of $Z_r$ and $\mathcal{T}$):** Let $Z_r$ and $\mathcal{T}$ satisfy the following properties:

1) All unmeasured paths from $w_i$ to $w_j$ pass through a node $w_k, k \in Z_r$ that has an unmeasured path to a node $w_l, l \in \mathcal{T}$;
2) All unmeasured paths from $w_i$ to $w_r$ pass through a node $w_k, k \in Z_r$ and $G_{r_t} = 0$;
3) If $i \in \mathcal{T}$:
   a) $w_i$ is excited by an external excitation signal $r_i$;
   b) If there exist unmeasured loops through $w_j$ and $w_j$ is not excited by an external excitation signal $r_j$, then:
      a) The unmeasured loops through $w_j$ pass through a node $w_k, k \in Z_r$ that has an unmeasured path to a node $w_n, n \in \mathcal{T}$;
      b) All unmeasured paths from $w_i$ to $w_r$ pass through a node $w_k, k \in Z_r$ and $G_{r_j} = 0$;
4) Every $w_k, k \in Z_r$ is excited by an external excitation signal $r_k$;
5) Every $w_k, k \in Z_r$ is excited by an external excitation signal $r_k$;
6) For every subset of $Z_r$ (i.e. $S_r \subseteq Z_r$) with cardinality $C$, there are unmeasured paths to at least $C$ nodes in $w_r$ with each node in $w_{S_r}$, having at least one path.

**Theorem 2:** Consider the situation of Theorem 1. Let $i \in \mathcal{D}$ and let the sets $Z_r$ and $\mathcal{T}$ satisfy Property 2. Then a consistent estimate of target module $\hat{G}_{ji}$ is obtained as

$$G_{ji}(\theta_N) = \hat{R}_{ji}^{-1}(\theta_N) (\hat{G}_{ji}(\theta_N) - \hat{R}_{ji},(\theta)\hat{R}_{jz},(\theta_N)G_{rz}(\theta_N))$$

where\(^8\)

1) $\hat{R}_{ji} = \hat{R}_{ij}$ if $w_j$ is excited by an external signal $u_j$;
2) $\hat{R}_{ji} = \left(1 - (1 + \hat{R}_{jz},(\theta)\hat{R}_{jz},G_{rz})^{-1}\hat{R}_{jz},(\theta)\hat{R}_{jz},G_{rz}\right)^{-1}$ if $w_j$ is not excited by an external signal $u_j$;
3) $\hat{G}_{r_i} = \hat{G}_{r_i}$ if $i \notin \mathcal{T}$;
4) $\hat{G}_{r_i} = (\hat{G}_{r_i} + \hat{R}_{i_i})$ if $i \in \mathcal{T}$, where $\hat{R}_{i_i}$ is a column matrix with every element as zero except the element corresponding to node $w_i$ which is $\hat{R}_{i_i}(1 - \hat{R}_{i_i}^{-1})$

**Proof:** The proof is provided in the appendix.

Here $[\ldots]$ correspond to the left inverse of the matrix. The left inverse exists if set $Z_r$ and $\mathcal{T}$ has Property 2.

We interpret Property 2 using the network in Figure 5. We have one unmeasured parallel path from $w_i$ to $w_j$ and one unmeasured loop through $w_j$. Considering the parallel path, the excited node $w_2$ and its measured descendant $w_1$ ensures that Property 2a) and 2b) are satisfied with $w_2$ in $w_{x_r}$ and $w_1$ in $w_r$. Similarly, considering the unmeasured loop through $w_j$, the excited node $w_5$ and its measured descendant $w_1$ ensures that Property 2d) is satisfied with $w_5$ in $w_{x_r}$. Property 2e) is satisfied with both $w_2$ and $w_5$ being excited by external signals. However, Property 2f) is not satisfied if $w_{x_r} = \{w_2, w_5\}$ and $w_r = w_1$. For the subset $S_r = Z_r$ with cardinality equal to 2, there are unmeasured paths to only 1 node in $w_r$ (i.e. $w_1$). Hence we choose $w_5$ in $w_r$, which is a descendant of $w_5$ and ensure that Property 2f) is satisfied. Property 2c) is redundant for this case since $i \notin \mathcal{T}$. It is important to note that $w_r$ can be any node in the network that satisfies the Property 2 and thus relaxes the sensor placement scheme.

**VIII. ALGORITHM FOR SIGNAL SELECTION**

In this section, we provide the algorithm for signal selection that provides the required identification setup for the introduced identification method in this paper.

**Algorithm A**

1) Select target module $G_{ji}$;
2) Include $j$ in the index set $\mathcal{Y}$ of node variables that are to be predicted.
3) Let $\mathcal{D}_j$ be a set of nodes that includes $w_i$ and (some)nodes that block the parallel path from $w_i$ to $w_j$ and loops through $w_j$. Then
   - include $\mathcal{D}_j$ in $\mathcal{D}$;
4) Select a set $Z_r, \mathcal{T}$ that satisfies Property 2 and include it in $\mathcal{P}$ and $\mathcal{Y}$ respectively;

\(^7\)An unmeasured path is a path that runs through nodes in $w_x$ only. Analogously, we can define an unmeasured loop through a node $w_j$.

\(^8\)notation $(\theta_N)$ is dropped in the following expressions.
5) Determine $Z = \mathcal{L}\{D \cup Y\}$;

6) For every $k \in D$:
   a) if there exist a confounding variable $e_k$ in $e$ for the estimation problem $w_k \rightarrow w_j$, then include $k$ in $Y$;

7) If $Y$ has changed, return to step 6;

8) For every $k \in Y \setminus D$:
   a) if there exist an unmeasured path from $w_k$ to a node in $w_j$, include $k$ in $D$;

9) Determine $Q$ as the intersection of $Y$ and $D$;

10) If $j \notin Q$ then set $w_0 = w_j$, else $w_0$ is void;

11) Determine $A = D \setminus Q$ and $O = Y \setminus (Q \cup \{o\})$;

12) For every $k \in \mathcal{L} \setminus A$:
   a) if there exist a direct path from $r_k$ to $w_k$, then include $k$ in $P$;

When this algorithm finishes, we acquire sets $Y, D, P, T, Z_r$ such that the conditions in Proposition 2 are satisfied and $Z_r, T$ has Property 2.

IX. CONCLUSIONS

A new local module identification method has been introduced that consistently identifies the target module under known topology, with a generalized scheme for selection of measured node signals and excitation signals of nodes. We provide flexibility in the sufficient conditions to identify a target module with consideration freedom in sensor selection and actuation schemes. This is achieved by combining elements of the direct and indirect identification approaches. We use both node signals and external excitation signals as predictor inputs, allow post-processing of module estimates, and use MIMO identification setting, thereby mixing both the approaches. With this step we remove restrictive conditions on measured node signals and excitation signals that are present in the currently available methods, e.g. concerning parallel paths and loops around the output.

APPENDIX I

PROOF OF PROPOSITION 2

On the basis of the decomposition of node signals as defined in Section V we are going to rewrite the system’s equations (6) in the following structured form:

\[
\begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix}
= \begin{bmatrix}
G_{QQ} & G_{Qo} & G_{Q0} & G_{QA} & G_{Qz} \\
G_{Qo} & G_{oo} & G_{Qo} & G_{oa} & G_{oz} \\
G_{Q0} & G_{Qo} & G_{oo} & G_{AA} & G_{Az} \\
G_{QA} & G_{Ao} & G_{Ao} & G_{AA} & G_{Az} \\
G_{Qz} & G_{zo} & G_{zo} & G_{Az} & G_{zz}
\end{bmatrix}
\begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix}
\]

where we make the notation agreement that the matrix $H$ is not necessarily monic, and the scaling of the white noise process $e$ is such that $\text{cov}(e) = I$. Using $Z = Z_r \cup Z_u$, we write $w_z = [w_{z1}^T \ w_{z2}^T]^T$. $G_{z*} = [G_{z*}^T \ G_{z*}^T]^T$, $G_{xz} = [G_{xz} \ G_{xz}]$, $G_{zz} = [G_{zz} \ G_{zz}]$, and $R_{zz} = \begin{bmatrix} R_{z1z1} & 0 \\
0 & R_{z2z2} \end{bmatrix}$ where $* = Q, o, O, A, Z_r, Z_u$.

Next we will eliminate the non-measured node signals $w_z$ in the equation (10), by abstracting these nodes.

**Proposition 3:** The system equations for the measured variables $w_{0-o}$ can be written as,

\[
\begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix}
= \begin{bmatrix}
G_{Qo} & G_{Q0} & G_{QA} & G_{Qz} \\
G_{Qo} & G_{oo} & G_{oa} & G_{oz} \\
G_{Q0} & G_{Qo} & G_{oa} & G_{Az} \\
G_{QA} & G_{Ao} & G_{AA} & G_{Az} \\
G_{Qz} & G_{zo} & G_{Az} & G_{zz}
\end{bmatrix}
\begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix}
+ \begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix}
+ \begin{bmatrix}
\hat{w}_Q \\
\hat{w}_o \\
\hat{w}_0 \\
\hat{w}_A \\
\hat{w}_z
\end{bmatrix},
\]

with $\text{cov}(e) = I$, and where

\[
\begin{align*}
\tilde{G}_{z,r} &= G_{z,r} + G_{z,z}(I - G_{z,z})^{-1}G_{z,u}r, \\
\tilde{G}_{z,e} &= G_{z,e} + G_{z,u}(I - G_{z,z})^{-1}G_{z,u}r, \\
\tilde{G}_{z,u} &= G_{z,u} + G_{z,z}(I - G_{z,z})^{-1}G_{z,u}r. \\
\end{align*}
\]

**Proof:** From (10), the fifth (block) row provides the equation for $w_z$ and using $w_z = [w_{z1}^T \ w_{z2}^T]^T$, we have $w_z = (I - G_{z,z})^{-1} \left[ \sum_{i \in G_z} \tilde{G}_{z,i} w_i + \sum_{e \in \epsilon} H_{z,e} e + \sum_{r \in R_u} R_{z,r} \right]$ where $\epsilon = \{Q, o, O, A, Z_r, Z_u \}$. Using the equation of $w_z$, we can obtain the expression for $G_{Az}, G_{Az}, G_{Az}, G_{Az}, G_{Az}, G_{Az}, G_{Az}, G_{Az}$. Substituting the above expressions in the respective block rows of (10), we obtain a network representation without
following the similar steps as above with the new network representation, we can now write the expression for \( w_{2x} \). Using this we can obtain the expression for \( \hat{G}_{ax}, w_{2x} \).

Substituting these expressions in the respective block rows of the new network representation, we obtain the result of the proposition 3.

The spectral density of \( \tilde{v} \) is given by \( \Phi_{\hat{v}} = \tilde{H} \hat{H}^* \).

Applying a spectral factorization [22] to \( \Phi_{\hat{v}} \) will deliver \( \Phi_{\hat{v}} = \tilde{H} \hat{H}^* \) with \( \tilde{H} \) a monic, stable and minimum phase rational matrix, and \( \hat{H} \) a positive definite (constant) matrix. Then there exists a white noise process \( \tilde{\xi} \) defined by \( \tilde{H} \tilde{\xi} = \tilde{v} \), with \( \text{cov}(\tilde{\xi}) = \hat{\Lambda} \), where \( \tilde{H} \tilde{\xi} = \tilde{v} \) is of the form

\[
\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} & \tilde{H}_{14} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} & \tilde{H}_{24} \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & \tilde{H}_{34} \\ \tilde{H}_{41} & \tilde{H}_{42} & \tilde{H}_{43} & \tilde{H}_{44} \end{bmatrix}
\]

and where the block dimensions are conformable to the dimensions of \( \tilde{v} \), \( \tilde{\xi} \), \( \tilde{u} \), and \( \hat{u} \) respectively. As a result, (11) can be rewritten as

\[
\begin{bmatrix} \tilde{u}_Q \\ \tilde{u}_o \\ \tilde{u}_A \end{bmatrix} = \begin{bmatrix} \hat{G}_{Q0} & \hat{G}_{O0} & \hat{G}_{A0} \\ \hat{G}_{Qo} & \hat{G}_{Oo} & \hat{G}_{Ao} \\ \hat{G}_{QA} & \hat{G}_{OA} & \hat{G}_{AA} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\xi} \\ \tilde{u} \end{bmatrix} + \begin{bmatrix} \hat{H}_{14} \\ \hat{H}_{24} \\ \hat{H}_{34} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \end{bmatrix} + \tilde{u}.
\]

By denoting

\[
\begin{bmatrix} \hat{H}_{14} \\ \hat{H}_{24} \\ \hat{H}_{34} \end{bmatrix} := \begin{bmatrix} \hat{H}_{14} & \hat{H}_{12} & \hat{H}_{13} \\ \hat{H}_{21} & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{31} & \hat{H}_{32} & \hat{H}_{33} \end{bmatrix}
\]

and premultiplying (20) with

\[
\begin{bmatrix} I & 0 & 0 & -\hat{H}_{14} \\ 0 & I & 0 & -\hat{H}_{24} \\ 0 & 0 & I & -\hat{H}_{34} \\ 0 & 0 & 0 & I \end{bmatrix}
\]

while only keeping the identity terms on the left hand side, we obtain an equivalent network equation:

\[
\begin{bmatrix} \tilde{u}_Q \\ \tilde{u}_o \\ \tilde{u}_A \end{bmatrix} = \begin{bmatrix} \hat{G}_{Q0} & \hat{G}_{O0} & \hat{G}_{A0} \\ \hat{G}_{Qo} & \hat{G}_{Oo} & \hat{G}_{Ao} \\ \hat{G}_{QA} & \hat{G}_{OA} & \hat{G}_{AA} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\xi} \\ \tilde{u} \end{bmatrix} + \begin{bmatrix} \hat{H}_{14} & \hat{H}_{12} & \hat{H}_{13} \\ \hat{H}_{21} & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{31} & \hat{H}_{32} & \hat{H}_{33} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \end{bmatrix} + \begin{bmatrix} \tilde{u}_Q \\ \tilde{u}_o \\ \tilde{u}_A \end{bmatrix} \begin{bmatrix} r_Q \\ r_o \\ r_A \end{bmatrix}
\]

with

\[
\tilde{C}_{oa} = \hat{G}_{oa} - \hat{H}_{14} \hat{G}_{AA} + \hat{H}_{14}, \quad \tilde{C}_{oa} = \hat{G}_{oa} - \hat{H}_{24} \hat{G}_{A}, + \hat{H}_{24}, \quad \tilde{C}_{oa} = \hat{G}_{oa} - \hat{H}_{34} \hat{G}_{A}, + \hat{H}_{34}, \quad \tilde{C}_{oa} = \hat{G}_{oa} - \hat{H}_{44} \hat{G}_{A}, + \hat{H}_{44},
\]

so \( \Phi_{\hat{v}} = \tilde{H} \hat{H}^* \), and where \( \tilde{H} \tilde{\xi} = \tilde{v} \) is of the form

\[
\tilde{H}_{14} = \hat{H}_{14} + \hat{H}_{12} \hat{H}_{24} + \hat{H}_{13} \hat{H}_{34}
\]

\[
\tilde{H}_{24} = \hat{H}_{24} + \hat{H}_{22} \hat{H}_{24} + \hat{H}_{23} \hat{H}_{34}
\]

\[
\tilde{H}_{34} = \hat{H}_{34} + \hat{H}_{32} \hat{H}_{34} + \hat{H}_{33} \hat{H}_{34}
\]

where \( \ast \in \{ Q \cup \{ o \} \cup O \}, n \in \{ 1, 2, 3 \} \) and \( o \in \{ A \cup Z_r \cup Z_a \} \).

The next step is now to show that the block elements \( \tilde{G}_{oo}, \tilde{G}_{oo}, \tilde{G}_{oo}, \tilde{G}_{oo}, \tilde{G}_{oo}, \tilde{G}_{oo} \) in \( G \) can be made 0. This can be done by variable substitution as follows:

The second row in (37) is replaced by an explicit expression for \( w_o \) according to

\[
w_o = (1 - \tilde{G}_{oo})^{-1}[\tilde{G}_{oo} \tilde{v} + \tilde{G}_{oo} \tilde{\xi} + \hat{H}_{21} \tilde{\xi} + \hat{H}_{22} \tilde{\xi} + \hat{H}_{23} \tilde{\xi} + \hat{R}_{oo} r_o + \hat{R}_{oa} r_A + \hat{R}_{oz} r_z + \hat{R}_{oz} r_z] \]

Additionally, this expression for \( w_o \) is substituted into the first and third block row of (37), to remove the \( w_o \)-dependent term on the right hand side, leading to

\[
w_o \begin{bmatrix} \tilde{u}_Q \\ \tilde{u}_o \\ \tilde{u}_A \end{bmatrix} = \begin{bmatrix} \hat{G}_{o0} & \hat{G}_{Oo} & \hat{G}_{Ao} \\ \hat{G}_{Qo} & \hat{G}_{Qo} & \hat{G}_{Qo} \\ \hat{G}_{QA} & \hat{G}_{QA} & \hat{G}_{QA} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\xi} \\ \tilde{u} \end{bmatrix} + \begin{bmatrix} \hat{H}_{14} & \hat{H}_{12} & \hat{H}_{13} \\ \hat{H}_{21} & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{31} & \hat{H}_{32} & \hat{H}_{33} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \end{bmatrix} + \begin{bmatrix} \tilde{u}_Q \\ \tilde{u}_o \\ \tilde{u}_A \end{bmatrix} \begin{bmatrix} r_Q \\ r_o \\ r_A \end{bmatrix}
\]

with

\[
\begin{align*}
\tilde{G}_{oa} &= \hat{G}_{oa} - \hat{H}_{14} \hat{G}_{AA} + \hat{H}_{14} \\
\tilde{G}_{oa} &= \hat{G}_{oa} - \hat{H}_{24} \hat{G}_{A} + \hat{H}_{24} \\
\tilde{G}_{oa} &= \hat{G}_{oa} - \hat{H}_{34} \hat{G}_{A} + \hat{H}_{34} \\
\tilde{G}_{oa} &= \hat{G}_{oa} - \hat{H}_{44} \hat{G}_{A} + \hat{H}_{44}
\end{align*}
\]
\[
G_{a*} = \hat{G}_{a*} + \tilde{C}^m_{o} \hat{G}_{o*}, \quad (52)
\]
\[
\hat{H}^m_{a*} = \hat{H}^m_{a*} + \tilde{C}^m_{o} \hat{H}^m_{o*}, \quad (53)
\]
\[
\hat{R}^m_{a*} = \hat{R}^m_{a*} + \tilde{C}^m_{o} \hat{R}^m_{o*}, \quad (54)
\]
\[
\hat{R}_{a*} = \hat{R}_{a*} + \tilde{C}^m_{o} \hat{R}_{o*}. \quad (55)
\]

Since these operations, the matrix \( \hat{G}_{o*} \) might not be hollow, we move any diagonal terms of this matrix to the left hand side of the equation, and premultiply the first (block) equation by the diagonal matrix \((I - \text{diag}(\hat{G}_{o*}))^{-1}\), to obtain the expression

\[
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix} =
\begin{bmatrix}
\hat{G}_{o*} & 0 & 0 & \tilde{C}^m_{o} \\
G_{o*} & 0 & 0 & \tilde{C}^m_{o} \\
G_{o*} & 0 & 0 & \tilde{C}^m_{o} \\
\hat{G}_{o*} & \hat{G}_{a*} & \hat{G}_{o*} & \hat{G}_{a*}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix} +
\begin{bmatrix}
\hat{H}^m_{11} & \hat{H}^m_{12} & \hat{H}^m_{13} & 0 \\
\hat{H}^m_{21} & \hat{H}^m_{22} & \hat{H}^m_{23} & 0 \\
\hat{H}^m_{31} & \hat{H}^m_{32} & \hat{H}^m_{33} & 0 \\
\hat{H}^m_{41} & \hat{H}^m_{42} & \hat{H}^m_{43} & \hat{H}^m_{44}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix}.
\]

Therefore we can write (56) as,

\[
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix} = \hat{G}
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix} + \begin{bmatrix}
H_{11} & H_{12} & H_{13} & 0 \\
H_{21} & H_{22} & H_{23} & 0 \\
H_{31} & H_{32} & H_{33} & 0 \\
H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_Q \\
\tilde{w}_o \\
\tilde{w}_p \\
\tilde{w}_A
\end{bmatrix} + \tilde{u} \quad (62)
\]

This completes the proof for obtaining (7). □

**APPENDIX II**

**PROOF OF THEOREM 1**

In the first part of the proof we derive the elements of (7) when the first two conditions of Theorem 1 are satisfied. Starting from the expression (11) the spectral density \( \Phi_{\hat{v}} \) can be written as \( \hat{H} \hat{H}^{*} \) while it is denoted as

\[
\Phi_{\hat{v}} = \begin{bmatrix}
\Phi_{\hat{v},\hat{v}} & \Phi_{\hat{v},\hat{e}} & \Phi_{\hat{v},\hat{a}} & \Phi_{\hat{v},\hat{a}} \nonumber \\
\Phi_{\hat{v},\hat{e}} & \Phi_{\hat{e},\hat{e}} & \Phi_{\hat{e},\hat{a}} & \Phi_{\hat{e},\hat{a}} \\
\Phi_{\hat{v},\hat{a}} & \Phi_{\hat{e},\hat{a}} & \Phi_{\hat{a},\hat{a}} & \Phi_{\hat{a},\hat{a}} \\
\Phi_{\hat{v},\hat{a}} & \Phi_{\hat{e},\hat{a}} & \Phi_{\hat{a},\hat{a}} & \Phi_{\hat{a},\hat{a}}
\end{bmatrix}. \quad (63)
\]

In this structure we are particularly going to analyse the elements \( \Phi_{\hat{v},\hat{v}} \), \( \Phi_{\hat{e},\hat{e}} \), \( \Phi_{\hat{a},\hat{a}} \), where \( \Phi_{\hat{v},\hat{v}} = \hat{H}_{11} \hat{H}_{22}^{*} + \hat{H}_{22} \hat{H}_{11}^{*} + \hat{H}_{13} \hat{H}_{31}^{*} + \hat{H}_{23} \hat{H}_{32}^{*} \). Similarly we have \( \Phi_{\hat{e},\hat{e}} \) and \( \Phi_{\hat{a},\hat{a}} \). We have \( \hat{H}_{13} = \hat{H}_{31}^{*} \) and \( \hat{H}_{23} = \hat{H}_{32}^{*} \) and \( \hat{H}_{12} = \hat{H}_{21}^{*} \). Similarly we can write \( \hat{H}_{11} \), \( \hat{H}_{22} \), \( \hat{H}_{33} \). We write the equation of \( \Phi_{\hat{v},\hat{v}} \) as \( \Phi_{\hat{v},\hat{v}} = \sum_{k} H_{11}^{*} H_{11} + \sum_{k} H_{22}^{*} H_{22} + \sum_{k} H_{33}^{*} H_{33} \). Similarly we can write \( \Phi_{\hat{e},\hat{e}} \), \( \Phi_{\hat{a},\hat{a}} \). The term \( \sum_{k} H_{11}^{*} H_{11} \) in \( \Phi_{\hat{v},\hat{v}} \) correspond to the confounding variable in \( \epsilon \) that has direct paths and the other terms refer to the cases with unmeasured paths. If **condition 1 in Proposition 2** is satisfied, we ensure that there are no confounding variables in \( \epsilon \) for the estimation problem \( w_{\hat{v}} \rightarrow w_{\hat{e}} \). This implies that \( H_{11}^{*} H_{11} = H_{22}^{*} H_{22} = H_{33}^{*} H_{33} = 0 \) for \( * = \hat{v}, \hat{e}, \hat{a} \). Therefore \( \Phi_{\hat{v},\hat{v}} = \Phi_{\hat{e},\hat{e}} = \Phi_{\hat{a},\hat{a}} = 0 \). Thus we get a block diagonal structure in equation (63). Then the spectral density \( \Phi_{\hat{v}} \) has the unique spectral factorization \( \Phi_{\hat{v}} = \begin{bmatrix} H_{11}^{*} H_{11} & 0 \\
0 & H_{22}^{*} H_{22}
\end{bmatrix} = \hat{H} \hat{H}^{*} \). Where \( \hat{H} \) and \( \hat{H} \) are monic, stable, minimum phase and \( \hat{H} \) is of the form given in proposition 2.

Now we substitute \( \hat{H} \) in (11). If **condition 2 in Proposition 2** is satisfied, this ensures that \( G_{o*} = G_{o0} = 0 \) where \( * = \hat{v}, \hat{e}, \hat{a} \). We substitute it in (11). Now we have the (1,1) block-elements of the resulting matrix of \( \hat{G} \) to have non-zero elements in the diagonal. By multiplying the first block row with a diagonal matrix \((I - \text{diag}(\hat{G}_{o*}))^{-1}\) and writing only the block rows for
where $\xi_w$, $\xi_o$, and $\xi_s$ white noise processes with dimensions conforming to $w_q$, $w_o$, and $w_q$, respectively, with $\text{cov}(\xi_w') = \Lambda$ and with $H$ being monic, stable and stably invertible and

$$
\begin{align*}
\mathcal{G}_{\omega} &= (I - \text{diag}(\mathcal{G}_{\omega}))^{-1} (\mathcal{G}_{\omega} - \text{diag}(\mathcal{G}_{\omega})) \\
\mathcal{G}_{s} &= \mathcal{G}_{s} = 0 \\
\mathcal{G}_{s} &= (I - \text{diag}(\mathcal{G}_{s}))^{-1} \mathcal{G}_{s} \\
\bar{R}_{\infty} &= (I - \text{diag}(\mathcal{G}_{s}))^{-1} \bar{R}_{\infty}
\end{align*}
$$

(65) (66) (67) (68)

where $* = Q, o, \Omega$, $n = Q, A$ and $\phi = Q, o, \Omega, A, Z, z, u$ where $\mathcal{G}_{\omega}$ has diagonal elements as zero. This is represented in equation (7).

The second part of the proof is to prove consistency results. This can be shown by following a similar reasoning as in Theorem 1 of [21].

### APPENDIX III

#### PROOF OF THEOREM 2

The target module that is the objective of our identification is given by $G_{ji}$, with $w_j \in (w_q, w_o)$ and $w_i \in (w_i, w_q)$. From (65), (67) and since $j \neq i$, we have $\mathcal{G}_{ji} = (I - \mathcal{G}_{ji})^{-1} \mathcal{G}_{ji} = (I - \mathcal{G}_{ji})^{-1} (G_{ji} + \mathcal{G}_{ji} (I - \mathcal{G}_{z_i x_j}))^{-1} \mathcal{G}_{zi} + \mathcal{G}_{ji} (I - \mathcal{G}_{z_i x_j})^{-1} \mathcal{G}_{zi}$. **Condition 1 in Property 2** ensures the second term is zero. Now in the sequel we find expressions for $(I - \mathcal{G}_{ji})^{-1}$ and $\mathcal{G}_{ji} (I - \mathcal{G}_{z_i x_j})^{-1} \mathcal{G}_{zi}$ using elements of (7) in order to extract $G_{ji}$ from $G_{ji}$.

If there are unmeasured loops through $w_j$, then $\mathcal{G}_{ji} \neq 0$. From (68), $\bar{R}_{ji} = (I - \mathcal{G}_{ji})^{-1} \bar{R}_{ji} = (I - \mathcal{G}_{ji})^{-1} \bar{R}_{ji}$. Therefore if node $j$ is excited by an external excitation signal (i.e. $\bar{R}_{ji} = 1$) then $\bar{R}_{ji} = (I - \mathcal{G}_{ji})^{-1}$. Also from (68) we have $\bar{R}_{\eta x_j} = (I - \mathcal{G}_{ji})^{-1} \bar{R}_{\eta x_j} = (I - \mathcal{G}_{ji})^{-1} G_{ji} (I - \mathcal{G}_{z_i x_j})^{-1} \mathcal{G}_{zi}$.

**Condition 6 in Property 2** ensures that the left inverse of $\bar{R}_{\eta x_j}$ exists. Then $G_{z_i x_j} = \bar{R}_{\eta x_j} G_{\eta x_j}$. From (14), we have $\bar{G}_{ji} = \bar{G}_{ji} (I - \mathcal{G}_{z_i x_j})^{-1} \mathcal{G}_{zi} + \mathcal{G}_{ji} (I - \mathcal{G}_{z_i x_j})^{-1} \mathcal{G}_{zi}$.

### REFERENCES


