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Generalized sensing and actuation schemes for local module identification in dynamic networks

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***Abstract*— For the problem of identifying a target module that is embedded in a dynamic network with known interconnection structure, different sets of conditions are available for the set of node signals to be measured and the set of excitation signals to be applied at particular node locations. In previous work these conditions have typically been derived from either an indirect identification approach, considering external excitation signals as inputs, or from a direct identification approach, considering measured node signals as inputs. While both approaches lead to different sets of (sufficient) conditions, in this paper we extend the flexibility in the sufficient conditions for selection of excitation and measured node signals, by combining both direct and indirect approaches. As a result we will show the benefits of using both external excitation signals and node signals as predictor inputs. The provided conditions allow us to design sensor selection and actuation schemes with considerable freedom for consistent identification of a target module.**

I. INTRODUCTION

In recent years increasing attention has been given to the identification of large-scale dynamically interconnected systems (modules), known as dynamic networks. Among the large amount of literature on this topic, there are three main research trends. The first one deals with the identification of the interconnection structure (topology) of systems in the dynamic network [1], [2], [3], [4]. The second deals with identification of the full network dynamics [5], [6], [7], while the third deals with identification of a target module in the dynamic network under the assumption of known topology (known as local module identification, see [8], [9], [10], [11], [12], [13]).

In this paper we focus on the local module identification problem. In [8], the classical *direct-method* [14] for closed-loop identification has been generalized to a dynamic network framework using a MISO identification setup. It introduces a method to achieve a consistent estimate of the target module when all the node signals in the MISO setup are measured. In [15], an extension has been made towards the situation where some of the node signals might be non-measurable. In [16] and [17], an approach has been introduced to consistently estimate the target module in the presence of confounding variables due to non-measured nodes and noise correlation. The direct method has been extended to a Bayesian setting in [9], where regularized kernel-based methods are used to reduce the mean-square error of the target module estimates. In [12] and [10], the situation has been addressed where the node measurements are affected by sensor noise.

An important condition in the works that use the direct method [8], [15], [16], [17] is that all parallel paths from the input of the target module to its output and all loops through the output node should pass through a measured node signal that is included as a predictor input. This requirement ensures that the identified module using the direct method is equal to the target module. However, in practical situations, there can be parallel paths and loops that might have all nodes non-measured. This creates a restriction for the selection of measured node signals.

In *indirect method* as in [11], [7], external excitation signals are used as predictor inputs for an open loop MIMO identification problem. These methods involve two steps: (1) First obtain consistent estimates of a transfer function from external signals to measured node signal; (2) Using these estimates obtain consistent estimates of the target module (we call this step as *post-processing*). In [7], the freedom in selection of measured node signal is exploited under the condition that all nodes are excited.

In this paper we extend the flexibility in the sufficient conditions for selection of excitation and measured node signals for consistent target module estimates and thereby generalizing the sensing and actuation schemes. We relax the above discussed condition on the parallel paths and loops around the output node. This relaxation in conditions are achieved by combining elements of both direct and indirect approaches. We use both the node signals and external excitation signals as predictor inputs, allow post-processing of module estimates, use MIMO identification setting and thereby mixing both direct and indirect methods. The provided conditions allow us to design sensor selection and actuation schemes with considerable freedom for consistent identification of a target module.

II. NETWORK AND IDENTIFICATION SETUP

A. Dynamic network setup

Following the basic setup of [8], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1,\dots,L$, and K *external variables* r_k, $k = 1,\dots K$. Each internal variable is described as:

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$$w_j(t) = \sum_{\substack{l=1\\ l\neq j}}^{L} G_{jl}(q)w_l(t) + u_j(t) + v_j(t) \qquad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl} is a proper rational transfer referred to as *modules*;
- $u_j(t)$ is generated by the *external variables* $r_k(t)$ that can directly be manipulated by the user and is given by $u_j(t) = \sum_{k=1}^{K} R_{jk}(q)r_k(t)$ where R_{jk} are stable, proper rational transfer functions;
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase.

We will assume that the standard regularity conditions on the data are satisfied that are required for convergence results of prediction error identification method[1]. In this paper we consider the situation where $u_j(t) = \sum_{k=1}^{K} R_{jk}(q)r_k(t)$, and $R_{jk} = 1$ if $j = k$, $R_{jk} = 0$ if $j \neq k$, and $j = 1, \ldots, L$.

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1\ L} \\ G_{L1} & \cdots & G_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_L \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + Rr + He. \qquad (2)$$

We will assume that the dynamic network is stable, i.e. $(I - G)^{-1}$ is stable, and well posed (see [18] for details).

The identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of a selection of measured variables w, and possibly r.

B. Direct method

Let us define $\mathcal{N}_j^-$ as the set of node indices k such that $G_{jk} \neq 0$, i.e. the node signals in $\mathcal{N}_j^-$ are the w-in-neighbors of the node signal w_j. The nodes corresponding to the set of node indices k such that $G_{kj} \neq 0$ are called the w-out-neighbors of w_j (i.e. $\mathcal{N}_j^+$). Let $\mathcal{D}_j$ denote the set of indices of the internal variables that are chosen as predictor inputs. Let $\mathcal{Z}_j$ denote the set of indices not in $\{j\} \cup \mathcal{D}_j$, i.e. $\mathcal{Z}_j = \{1, \ldots, L\} \backslash \{\{j\} \cup \mathcal{D}_j\}$. Let $w_\mathcal{D}$ denote the vector $[w_{k_1} \ \cdots \ w_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{D}_j$. Let $u_\mathcal{D}$ denote the vector $[u_{k_1} \ \cdots \ u_{k_n}]^T$, where $\{k_1, \ldots, k_n\} = \mathcal{D}_j$, and where the ℓth entry is zero if u_ℓ is not present in the network. The vectors $w_\mathcal{Z}$, $v_\mathcal{D}$, $v_\mathcal{Z}$ and $u_\mathcal{Z}$ are defined analogously. The ordering of the elements in $w_\mathcal{D}$, $v_\mathcal{D}$, and $u_\mathcal{D}$ is not important, as long as it is the same for all vectors. The transfer function matrix between $w_\mathcal{D}$ and w_j is denoted $G_{j\mathcal{D}}$. The other transfer function matrices are defined analogously.

[1]See [14] page 249. This includes the property that $e(t)$ has bounded moments of order higher than 4.

By this notation, the network equation (2) is rewritten as:

$$\begin{bmatrix} w_j \\ w_\mathcal{D} \\ w_\mathcal{Z} \end{bmatrix} = \begin{bmatrix} 0 & G_{j\mathcal{D}} & G_{j\mathcal{Z}} \\ G_{\mathcal{D}j} & G_{\mathcal{D}\mathcal{D}} & G_{\mathcal{D}\mathcal{Z}} \\ G_{\mathcal{Z}j} & G_{\mathcal{Z}\mathcal{D}} & G_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_j \\ w_\mathcal{D} \\ w_\mathcal{Z} \end{bmatrix} + \begin{bmatrix} v_j \\ v_\mathcal{D} \\ v_\mathcal{Z} \end{bmatrix} + \begin{bmatrix} u_j \\ u_\mathcal{D} \\ u_\mathcal{Z} \end{bmatrix}, \qquad (3)$$

where $G_{\mathcal{D}\mathcal{D}}$ and $G_{\mathcal{Z}\mathcal{Z}}$ have zeros on the diagonal.

Identification of module G_{ji} can be done by selecting $\mathcal{D}_j$ such that $i \in \mathcal{D}_j$, and subsequently estimating a multiple-input single output model for the transfer functions in $G_{j\mathcal{D}}$. This can be done by considering the one-step-ahead predictor[2] $\hat{w}_j(t|t-1) := \bar{\mathbb{E}}\{w_j(t) \mid w_j^{t-1}, w_{\mathcal{D}_j}^t\}$, and the resulting prediction error ([14]) $\varepsilon_j(t,\theta) = w_j(t) - \hat{w}_j(t|t-1,\theta)$, given by

$$\varepsilon_j(t,\theta) = H_j(\theta)^{-1}\Big(w_j - \sum_{k\in\mathcal{D}_j} G_{jk}(\theta)w_k - u_j\Big) \qquad (4)$$

where arguments q and t have been dropped for notational clarity. The parameterized transfer functions $G_{jk}(\theta)$, $k \in \mathcal{D}_j$ and $H_j(\theta)$ are estimated by minimizing the sum of squared (prediction) errors: $V_j(\theta) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j^2(t,\theta)$, where N is the length of the data set. We refer to this identification method as the *direct method*, [8]. Let $\hat{\theta}_N$ denote the minimizing argument of $V_j(\theta)$.

C. Indirect method

As an alternative approach, following the setup of [7], the network model (2) can be re-written as $w = T_{wr}r + \bar{v}$ where $T_{wr} = (I-G)^{-1}R$ and $\bar{v} = (I-G)^{-1}He$. Using the known external references r as predictor inputs and measured signals w as predicted outputs, it is well known that, under appropriate conditions, a consistent estimate $\hat{T}_{wr}$ of T_{wr} can be obtained using open loop MIMO identification methods. On the basis of $\hat{T}_{wr}$, an estimate of $\hat{G}$ can be obtained by solving $(I-\hat{G})\hat{T}_{wr} = R$. By solving only a subset of these equations, a target module embedded in the dynamic network can be identified. We refer to this type of identification method that uses external signals as predictor inputs as the *indirect method*.

III. Background and Motivating example

In this section we provide a summary of existing results related to the topic of this paper and subsequently highlight the motivation of the paper using a suitable example. In [15] it has been shown that we can identify the target module G_{ji} consistently provided that we choose the selection of predictor input signals to satisfy particular properties. One of the main conditions is formulated next.

Property 1: To identify a target module G_{ji}, consider a set of internal variables $w_k, k \in \mathcal{D}_j$. Let $\mathcal{D}_j$ satisfy the following properties:

1) $i \in \mathcal{D}_j$ and $j \notin \mathcal{D}_j$;
2) Every path from w_i to w_j is blocked by a node $w_k, k \in \mathcal{D}_j$ (*parallel path condition*);
3) Every loop through w_j is blocked by a node $w_k, k \in \mathcal{D}_j$ (*loop condition*).

[2]$\bar{\mathbb{E}}$ refers to $\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}$, and w_j^ℓ and $w_{\mathcal{D}_j}^\ell$ refer to signal samples $w_j(\tau)$ and $w_k(\tau)$, $k \in \mathcal{D}_j$, respectively, for all $\tau \leqslant \ell$.

When this property is satisfied, using a MISO identification setup with w_j as predicted output and $w_{\mathcal{D}_j}$ as predictor inputs, the *direct method* as discussed in section II-B provides consistent estimate of the target module, if in addition there are no confounding variables for the estimation problem $w_{\mathcal{D}_j} \rightarrow w_j$[3]. In [16] additional conditions have been formulated for the selection of $\mathcal{D}_j$ so as to avoid the presence of confounding variables, typically by choosing additional predictor inputs defined by the set $\mathcal{B}_j \subset \mathcal{L}\backslash\{\mathcal{D}_j \cup \{j\}\}$. This situation has been analyzed for the case where all disturbance signals v are mutually uncorrelated, i.e. its spectral density Φ_v being diagonal. In this situation it is still required that $\mathcal{D}_j$ satisfies Property 1. This restrictive property is required for the target module in the dynamic network to be invariant in an immersed network where all non-measured signals are being removed [15].

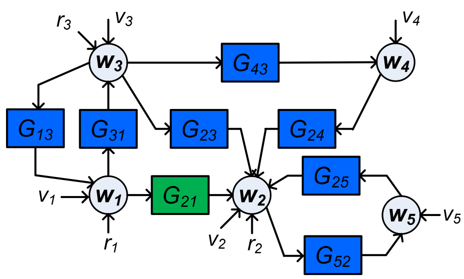

Fig. 1. Example network

Example 1: Consider a dynamic network as represented in Figure 1 with all noises in v uncorrelated with each other. For identifying the target module G_{21} (in green box), we have $j = 2$, and in order to satisfy Property 1 we need $\mathcal{D}_j = \{1, 3, 5\}$ where w_3 is included to block parallel path from w_1 to w_2, and w_5 is included to block the loop through w_2. Using this set of measured nodes, we arrive at an immersed network after removing the non-measured node as in Figure 2. We can observe that the module between w_1 and w_2 (the green box) is G_{21} and remains invariant.

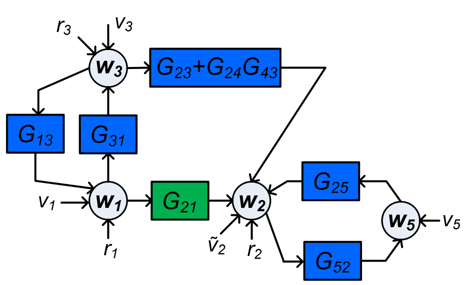

Fig. 2. Immersed network of network in figure 1 [15] where the nonmeasured node w_4 has been removed (immersed), and where $\tilde{v}_2 = v_2 + G_{24}v_4$.

If w_3 and w_5 are not selected in $\mathcal{D}_j$, and so $\mathcal{D}_j = \{1\}$, we arrive at an immersed network after removing all non-measured nodes, as depicted in Figure 3. We can now observe

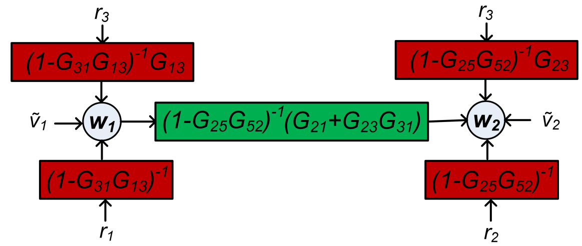

Fig. 3. Immersed network of network in figure 1[15] where the non-measured nodes w_3, w_4, w_5 have been removed (immersed), and where $\tilde{v}_1 = (1 - G_{31}G_{13})^{-1}(v_1 + G_{13}v_3)$ and $\tilde{v}_2 = (1 - G_{25}G_{52})^{-1}(v_2 + (G_{23} + G_{24}G_{43})v_3 + G_{24}v_4 + G_{25}v_5)$.

that the dynamic module between w_1 and w_2 (the green box in figure 3) is not equal to G_{21}. The terms $(1 - G_{25}G_{52})^{-1}$ and $G_{23}G_{31}$ are due to the fact that in this situation the loop and parallel path condition in property 1 are not satisfied, respectively. In this paper we are going to relax these restrictive conditions in property 1 and increase the freedom in the selection of measured node signals.

For the approach based on the indirect identification method, in [11] a method has been presented to identify a target module using external signals as predictor inputs, along the following reasoning.

Proposition 1 (from [11]): In order to identify a target module G_{ji}, perform the following experiment:

1) Excite node w_i and all its out-neighbors with sufficiently rich signals. Include these excitation signals as predictor inputs;
2) measure the out-neighbors of w_i. Include them as predicted outputs.

Under these conditions and using full order models for the elements of T_{wr}, consistent estimates $\hat{T}_{\mathcal{N}_i^+\mathcal{N}_i^+}$, $\hat{T}_{\mathcal{N}_i^+i}$ of $T_{\mathcal{N}_i^+\mathcal{N}_i^+}$ and $T_{\mathcal{N}_i^+i}$ can be obtained using an open loop MIMO identification method. Then a consistent estimate of $\hat{G}_{\mathcal{N}_i^+i}$ (which includes the target module) is obtained by,

$$\hat{G}_{\mathcal{N}_i^+i} = [\hat{T}_{\mathcal{N}_i^+\mathcal{N}_i^+}]^{-1}\hat{T}_{\mathcal{N}_i^+i} \qquad \square \tag{5}$$

A dual of this proposition with in-neighbors of w_j is also provided in [11]. It can be observed that a consistent estimate of the target module is obtained from consistent estimates of elements of T_{wr}. We will refer to this step (5) of manipulating identified objects, as *post-processing*. Considering the earlier Example 1, we can now consistently identify our target module using an open loop MIMO identification setup with $\{r_1, r_2, r_3\}$ as inputs and $\{w_2, w_3\}$ as outputs. However this requires restrictive conditions on the nodes to be excited and nodes to be measured, i.e. measured excitation signals r_1, r_2, r_3. Further relaxations of these restrictive conditions on excitation and measured node signals will be addressed in the sequel.

Definition 1 (confounding variable): Consider a dynamic network defined by

$$w = Gw + He + u \tag{6}$$

[3] A confounding variable is an unmeasured variable that induces correlation between the input and output signal of an estimation problem [19].

with $cov(e) = I$ and H is not necessarily monic, and consider the graph related to this network, with node signals w and e. Let $w_{\mathcal{X}}$ and $w_{\mathcal{Y}}$ be two subsets of measured node signals in w and let $w_{\mathcal{Z}}$ be the set of non-measured node signals in w.

Then a noise component e_ℓ in e is a *confounding variable for the estimation problem* $w_{\mathcal{X}} \rightarrow w_{\mathcal{Y}}$, if in the graph there exist simultaneous paths[4] from e_ℓ to node signals $w_k, k \in \mathcal{A}$ and $w_n, n \in \mathcal{Y}$, while these paths are either direct[5] or only run through nodes that are in $w_{\mathcal{Z}}$. $\square$

IV. Example to Illustrate the Introduced Method

In this section we illustrate the introduced method in this paper with suitable examples. In this paper, we combine the ideas of both the direct and indirect method such that we introduce flexibility in the selection of excitation and measured node signals. We use both the measured node signals as well as the excitation signals as predictor inputs. In addition to that, we do not restrict to the situation of invariance of our target module after immersion as in the direct method, but use the mechanism of *post-processing* from the indirect method to consistently identify the target module.

Example 2: We now consider the same network as in Example 1 but with two constraints: (a) it is not possible to measure w_3 and w_5; (b) it is not possible to excite node w_1. It can be inferred that it is not possible to consistently estimate $G_{ji} = G_{21}$ using the direct method due to constraint (a). Similarly due to constraint (b), it is not possible with the indirect method either.

As shown in Example 1, if we do not measure w_3 and w_5 our target module changes to $(1 - G_{25}G_{52})^{-1}(G_{21} + G_{23}G_{31})$ in the immersed network. However, we can see that this module also contains the target module of interest G_{21}. Therefore we might extract the target module from this term if we know (or) find the other contributions.

Consider that we excite node w_3, w_5 and measure node w_4. After immersing the non-measured nodes (see [15]) we end up in a dynamic network setup as in Figure 4. Now consider the identification problem $\{w_1, w_4, r_2, r_3\} \rightarrow \{w_2, w_4\}$. We can infer the following from the figure:

1) Identifying the transfer from $r_3 \rightarrow w_4$ provides G_{43} and the transfer from $w_1 \rightarrow w_4$ provide $G_{43}G_{31}$. Thus we can identify G_{31};
2) The transfer from $r_3 \rightarrow w_2$ provides $(1 - G_{25}G_{52})^{-1}G_{23}$. The term $(1-G_{25}G_{52})^{-1}$ is due to the fact that in the original network there is a loop around w_2 which is not "blocked" by a measured node. This term given by the transfer from $r_2 \rightarrow w_2$. Now, we can obtain G_{23}.
3) The term $G_{23}G_{31}$ is due to the fact that in the original network there is a path from w_1 to w_2 through w_3 which is not "blocked" by a measured node. Knowing G_{23} and G_{31} from the above two steps, we obtain the term $G_{23}G_{31}$. We also have $(1 - G_{25}G_{52})^{-1}$. Eventually we obtain our target module of interest from the transfer $w_1 \rightarrow w_2$ (i.e. $(1 - G_{25}G_{52})^{-1}(G_{21} + G_{23}G_{31})$).

This shows that we can consistently identify the target module G_{21} if we know or could consistently identify the transfer from $\{w_1, w_4, r_2, r_3\} \rightarrow \{w_2, w_4\}$. Thus we move to a MIMO identification problem using the prediction error method with $\{w_2, w_4\}$ as predicted outputs and $\{w_1, w_4, r_2, r_3\}$ as predictor inputs.

Remark 1: Excitation at the output node r_2 is required as predictor input since the loops through w_j are not blocked by a measured node. However we can still relax this under these conditions, which will be provided in this paper.

Remark 2: The consistency results may still require some excitation conditions, which will be specified later on.

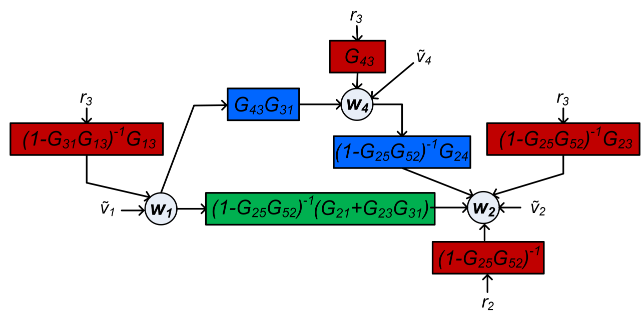

Fig. 4. Immersed network of network in Figure 1 where the nonmeasured nodes w_3, w_5 have been removed (immersed), and where $\tilde{v}_1 = (1 - G_{31}G_{13})^{-1}(v_1 + G_{13}v_3)$, $\tilde{v}_2 = (1 - G_{25}G_{52})^{-1}(v_2 + G_{23}v_3 + G_{25}v_5)$ and $\tilde{v}_4 = v_4 + G_{43}v_3$.

A. Handling confounding variables

We can observe from Figure 4 that the noise at predictor input w_1 and at predicted outputs w_2, w_4 are correlated due to v_3. This is due to the fact that in the original network, v_3 (in turn e_3) has simultaneous paths to w_1 and w_2 (also w_1 and w_4), while these paths run through the unmeasured node w_3. Therefore e_3, which is a confounding variable, creates noise correlation between predictor inputs and predicted outputs. When using the prediction error framework with the MIMO setup as explained above (i.e. with $\{w_2, w_4\}$ as predicted outputs), we only model the noise from $\{e_2, e_4\} \rightarrow \{w_2, w_4\}$ but not from the confounding variable e_3. This leads to a lack of consistency property of identified modules [16]. If we also predict w_2 (include it also as predicted output), we now model the noise from e_3 as well. This leads to consistent estimates. This has been studied in [20] for a two-node example network. Therefore for the Example 2, we need the MIMO identification setup $\{w_1, w_4, r_2, r_3\} \rightarrow \{w_1, w_2, w_4\}$.

From the discussed example, we can now conjecture the following generalization:

1) Violating the parallel path condition can be handled by exciting a node in the parallel path, including the excitation signal in the predictor input, and by measuring a

[4]A simultaneous path from e_1 to node signal w_1 and w_2 implies that there exist a path from e_1 to w_1 as well as from e_1 to w_2.

[5]A direct path from e_1 to node signal w_1 implies that there exist a path from e_1 to w_1 which do not pass through nodes in w.

descendant node from the excited node, different from the output, and by including this descendant node in the predicted output;

2) Violating the loop condition can be handled by either
   - exciting the output node and including the excitation signal in the predictor input; or
   - exciting a node in the loop, including the excitation signal in the predictor input, and by measuring a descendant node from the excited node, different from the output, and by including this descendant node in the predicted output;

3) Confounding variables can be handled by including measured nodes as predicted outputs[6].

Remark 3: In handling the parallel path condition it will appear that we actually have to add one additional constraint (see Property 2 later on). If the mentioned descendant node is the input of the target module, then this node needs to be excited with an external signal, which is included as predictor input.

Remark 4: If we consider again Example 1 in Figure 1, then the parallel path problem that occurs when w_3 can not be measured, can be compensated for by measuring a descendant from w_3, which in this case could also be w_1. Since w_1 is the input of the target module, the previous remark now leads to the situation that w_1 also needs to be excited, which is a situation that was excluded in Example 2.

In the sequal of this paper, we will derive the formal results that underly the above conjectured statements.

V. Concepts and notation

We will denote $w_{\mathcal{Y}}$ as the node signals in w that serve as predicted outputs, and $w_{\mathcal{D}}$ as the node signals in w that serve as predictor inputs, and $r_{\mathcal{P}}$ as the external excitation signals in r that serve as predictor inputs. Next we decompose $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ in disjoint sets according to: $\mathcal{Y} = \mathcal{Q} \cup \mathcal{O} \cup \{o\}$; $\mathcal{D} = \mathcal{Q} \cup \mathcal{A}$ where $w_{\mathcal{Q}}$ are the node signals that are common in $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$; $w_{\mathcal{O}}$ is the set of node signals that are only predicted outputs (excluding the output node of target module); w_o is the output w_j of the target module; if $j \in \mathcal{Q}$ then $\{o\}$ is void. Additionally we denote $w_{\mathcal{Z}}$ as the node signals in w that are neither predicted output nor predictor input, i.e. $\mathcal{Z} = \mathcal{L} \backslash \{\mathcal{D} \cup \mathcal{Y}\}$, where $\mathcal{L} = \{1, 2, \cdots L\}$. Next we define the set related to $r_{\mathcal{P}}$ as $\mathcal{P} \subseteq \mathcal{L}$.

VI. MIMO identification setup

The identification that we need to perform refers to the estimation problem $(w_{\mathcal{D}}, r_{\mathcal{P}}) \rightarrow w_{\mathcal{Y}}$. In order to analyse this problem, based on system equation (6), we write the system equations for the output variables $w_{\mathcal{Y}}$.

Proposition 2: The systems equations for the output variables in $w_{\mathcal{Y}}$ can always be written as,

$$w_{\mathcal{Y}} = \bar{G} w_{\mathcal{D}} + \bar{H} \xi_{\mathcal{Y}} + \bar{R} r_{\mathcal{P}}, \tag{7}$$

where $\xi_{\mathcal{Y}}$ a white noise process with dimensions conforming to $w_{\mathcal{Y}}$, with $cov(\xi_{\mathcal{Y}}) = \bar{\Lambda}$ and with $\bar{H}$ being monic, stable and stably invertible.

Proof: The proof is provided in the appendix. □

As a result we can set up a predictor model based on a parametrized model set determined by

$$\mathcal{M} := \left\{ (\bar{G}(\theta), \bar{H}(\theta), \bar{R}(\theta), \bar{\Lambda}(\theta)), \theta \in \Theta \right\},$$

while the actual data generating system is represented by $\mathcal{S} = (\bar{G}(\theta_o), \bar{H}(\theta_o), \bar{R}(\theta_0), \bar{\Lambda}(\theta_0))$. The corresponding identification problem is defined by considering the one-step-ahead prediction of $w_{\mathcal{Y}}$, according to

$$\hat{w}_{\mathcal{Y}}(t|t-1) := \mathbb{E}\{w_{\mathcal{Y}}(t) \mid w_{\mathcal{Y}}^{t-1}, w_{\mathcal{D}}^{t}, r_{\mathcal{P}}^{t}\}$$

where $w_{\mathcal{D}}^{t}, r_{\mathcal{P}}^{t}$ denotes the past of $w_{\mathcal{D}}, r_{\mathcal{P}}$ respectively, i.e. $\{w_{\mathcal{D}}(k)$ and $r_{\mathcal{P}}(k), k \leqslant t\}$. The resulting prediction error becomes:

$$\begin{aligned} \varepsilon(t,\theta) &:= w_{\mathcal{Y}}(t) - \hat{w}_{\mathcal{Y}}(t|t-1;\theta) \qquad (8)\\ &= \bar{H}(q,\theta)^{-1} \left[w_{\mathcal{Y}}(t) - \bar{G}(q,\theta) w_{\mathcal{D}}(t) - \bar{R}(q,\theta) r_{\mathcal{P}}(t) \right], \end{aligned}$$

and the weighted least squares identification criterion

$$\hat{\theta}_N = \arg\min_{\theta} \frac{1}{N} \sum_{t=0}^{N-1} \varepsilon^T(t,\theta) W \varepsilon(t,\theta), \tag{9}$$

with W any positive definite weighting matrix. This parameter estimate then leads to an estimated subnetwork $\bar{G}_{\mathcal{YD}}(q, \hat{\theta}_N)$, with the estimated module $\bar{G}_{ji}(q, \hat{\theta}_N)$ as one of its scalar entries.

VII. Main Results

In this section we first present the consistency results for the above considered identification problem.

Theorem 1: Consider a (MIMO) network identification setup with predictor inputs $(w_{\mathcal{D}}, r_{\mathcal{P}})$ and predicted outputs $w_{\mathcal{Y}}$ as in (7). Then a prediction error identification method according to (8)-(9), applied to a parametrized model set $\mathcal{M}$ will provide a consistent estimate of $\bar{G}$ and $\bar{R}$, if the following conditions on the sets of nodes are satisfied:

a) there are no confounding variables for the estimation problem $w_{\mathcal{A}} \rightarrow w_{\mathcal{Y}}$;
b) Every measured node signal that has an unmeasured path to a node signal in $w_{\mathcal{Y}}$ is included in $w_{\mathcal{D}}$;

and additionally:

1) $\mathcal{M}$ is chosen to satisfy $\mathcal{S} \in \mathcal{M}$;
2) $\Phi_{\kappa}(\omega) > 0$ for a sufficiently high number of frequencies, where $\kappa(t) := \begin{bmatrix} w_{\mathcal{Y}}^{\top} & \xi_{\mathcal{Q}}^{\top} & w_{\mathcal{A}}^{\top} & r_{\mathcal{P}}^{\top} \end{bmatrix}^{\top}$;
3) All $r_k, k \in \mathcal{P}$ are uncorrelated to all $\xi_{\ell}, \ell \in \mathcal{Y} \cup \mathcal{A}$;
4) All the elements in $G_{\mathcal{QQ}}, G_{\mathcal{QA}}, G_{o\mathcal{Q}}, G_{o\mathcal{A}}$ are strictly proper (or) all existing paths/loops from $w_{\mathcal{Q}}, w_o, w_{\mathcal{O}}$ to $w_{\mathcal{Q}}$, w_o and $w_{\mathcal{O}}$ have at least a delay. □

Proof: The proof is provided in the appendix. □

The estimate $\bar{G}$ contains the estimate of $\bar{G}_{ji}$ as one of its elements since $w_i \in w_{\mathcal{D}}$. However our final goal is to estimate our target module G_{ji} which will be present in $\bar{G}$

[6] Confounding variables can also be handled in other ways, for example, adding predictor inputs(see [16], [21]). In this paper we handle using predicted outputs in order to avoid measurement of additional node signals.

but will need to be extracted from this matrix through *post-processing*. For this post-processing step we will require two additional sets:

- A set $\mathcal{Z}_r \subseteq \mathcal{Z} \cap \mathcal{P}$ which represents externally excited nodes in unmeasured paths[7] from w_i to w_j and in loops around w_j; and
- A set $\mathcal{T} \subseteq \mathcal{Y} \backslash \{j\}$ which, for each of the nodes in $\mathcal{Z}_r$ represents a measured descendant node that has an unmeasured path from $w_{\mathcal{Z}_r}$, while w_j is excluded from $\mathcal{T}$. Note that for each node $w_k, k \in \mathcal{Z}_r$, the corresponding element in $\mathcal{T}$ is a measured node, and therefore cannot be in the corresponding unmeasured path from w_i to w_j or loops around w_j, that passes through w_k. Therefore the descendant in $\mathcal{T}$ typically breaks out of these unmeasured parallel paths/loops, as illustrated in Figure 5.

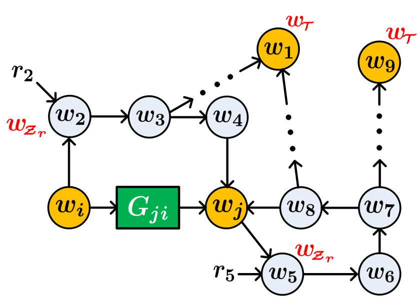

Fig. 5. Example network with all measured nodes in yellow. Modules and noise are not shown for convenience purpose. Arrows with dots indicate unmeasured path.

These two sets wil play a major role in extracting the target module estimate from the identification result $\bar{G}, \bar{R}$. The properties that $\mathcal{Z}_r$ and $\mathcal{T}$ need to satisfy to realize this post-processing are formulated next.

Property 2 (Properties of $\mathcal{Z}_r$ and $\mathcal{T}$): Let $\mathcal{Z}_r$ and $\mathcal{T}$ satisfy the following properties:

1) All unmeasured paths from w_i to w_j pass through a node $w_k, k \in \mathcal{Z}_r$ that has an unmeasured path to a node $w_\ell, \ell \in \mathcal{T}$;
2) All unmeasured paths from w_i to $w_{\mathcal{T}}$ pass through a node $w_k, k \in \mathcal{Z}_r$ and $G_{\mathcal{T}i} = 0$;
3) If $i \in \mathcal{T}$:
   a) w_i is excited by an external excitation signal r_i;
4) If there exist unmeasured loops through w_j and w_j is not excited by an external excitation signal r_j, then:
   a) The unmeasured loops through w_j pass through a node $w_k, k \in \mathcal{Z}_r$ that has an unmeasured path to a node $w_n, n \in \mathcal{T}$;
   b) All unmeasured paths from w_j to $w_{\mathcal{T}}$ pass through a node $w_k, k \in \mathcal{Z}_r$ and $G_{\mathcal{T}j} = 0$;
5) Every $w_k, k \in \mathcal{Z}_r$ is excited by an external excitation signal r_k;
6) For every subset of $\mathcal{Z}_r$ (i.e. $\mathcal{S}_r \subseteq \mathcal{Z}_r$) with cardinality C, there are unmeasured paths to at least C nodes in $w_{\mathcal{T}}$ with each node in $w_{\mathcal{S}_r}$ having at least one path.

Theorem 2: Consider the situation of Theorem 1. Let $i \in \mathcal{D}$ and let the sets $\mathcal{Z}_r$ and $\mathcal{T}$ satisfy Property 2. Then a consistent estimate of target module G_{ji} is obtained as

$$G_{ji}(\hat{\theta}_N) = \check{R}_{jj}^{-1}(\hat{\theta}_N)\Big(\bar{G}_{ji}(\hat{\theta}_N) - \bar{R}_{j\mathcal{Z}_r}(\theta) R^{\dagger}_{\mathcal{T}\mathcal{Z}_r}(\hat{\theta}_N)\check{G}_{\mathcal{T}i}(\hat{\theta}_N)\Big)$$

where[8]

1) $\check{R}_{jj} = \bar{R}_{jj}$ if w_j is excited by an external signal u_j;
2) $\check{R}_{jj} = \Big(1 - (1 + \bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j})^{-1}\bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j}\Big)^{-1}$ if w_j is not excited by an external signal u_j;
3) $\check{G}_{\mathcal{T}i} = \bar{G}_{\mathcal{T}i}$ if $i \notin \mathcal{T}$;
4) $\check{G}_{\mathcal{T}i} = (\bar{G}_{\mathcal{T}i} + \check{R}_{ii})$ if $i \in \mathcal{T}$, where $\check{R}_{ii}$ is a column matrix with every element as zero except the element corresponding to node w_i which is $\bar{R}_{ii}(1 - \bar{R}_{ii}^{-1})$ □

Proof: The proof is provided in the appendix. □

Here $[\ .\ ]^{\dagger}$ correspond to the left inverse of the matrix. The left inverse exists if set $\mathcal{Z}_r$ and $\mathcal{T}$ has Property 2.

We interpret Property 2 using the network in Figure 5. We have one unmeasured parallel path from w_i to w_j and one unmeasured loop through w_j. Considering the parallel path, the excited node w_2 and its measured descendant w_1 ensures that Property 2a) and 2b) are satisfied with w_2 in $w_{\mathcal{Z}_r}$ and w_1 in $w_{\mathcal{T}}$. Similarly, considering the unmeasured loop through w_j, the excited node w_5 and its measured descendant w_1 ensures that Property 2d) is satisfied with w_5 in $w_{\mathcal{Z}_r}$. Property 2e) is satisfied with both w_2 and w_5 being excited by external signals. However, Property 2f) is not satisfied if $w_{\mathcal{Z}_r} = \{w_2, w_5\}$ and $w_{\mathcal{T}} = w_1$. For the subset $\mathcal{S}_r = \mathcal{Z}_r$ with cardinality equal to 2, there are unmeasured paths to only 1 node in $w_{\mathcal{T}}$ (i.e. w_1). Hence we choose w_9 in $w_{\mathcal{T}}$, which is a descendant of w_5 and ensure that Property 2f) is satisfied. Property 2c) is redundant for this case since $i \notin \mathcal{T}$. It is important to note that $w_{\mathcal{T}}$ can be any node in the network that satisfies the Property 2 and thus relaxes the sensor placement scheme.

VIII. Algorithm for Signal Selection

In this section, we provide the algorithm for signal selection that provides the required identification setup for the introduced identification method in this paper.

Algorithm A

1) Select target module G_{ji};
2) Include j in the index set $\mathcal{Y}$ of node variables that are to be predicted.
3) Let $\mathcal{D}_j$ be a set of nodes that includes w_i and (some)nodes that block the parallel path from w_i to w_j and loops through w_j. Then
   - include $\mathcal{D}_j$ in $\mathcal{D}$;
4) Select a set $\mathcal{Z}_r$, $\mathcal{T}$ that satisfies Property 2 and include it in $\mathcal{P}$ and $\mathcal{Y}$ respectively;

[7] An unmeasured path is a path that runs through nodes in $w_{\mathcal{Z}}$ only. Analogously, we can define an unmeasured loop through a node w_j.

[8] notation $(\hat{\theta}_N)$ is dropped in the following expressions.

5) Determine $\mathcal{Z} = \mathcal{L}\backslash\{\mathcal{D} \cup \mathcal{Y}\}$;
6) For every $k \in \mathcal{D}$:
   a) if there exist a confounding variable e_ℓ in e for the estimation problem $w_k \rightarrow w_{\mathcal{Y}}$, then include k in $\mathcal{Y}$;
7) If $\mathcal{Y}$ has changed, return to step 6;
8) For every $k \in \mathcal{Y}\backslash\mathcal{D}$:
   a) if there exist an unmeasured path from w_k to a node in $w_{\mathcal{Y}}$, include k in $\mathcal{D}$;
9) Determine $\mathcal{Q}$ as the intersection of $\mathcal{Y}$ and $\mathcal{D}$;
10) If $j \notin \mathcal{Q}$ then set $w_o = w_j$, else w_o is void;
11) Determine $\mathcal{A} = \mathcal{D}\backslash\mathcal{Q}$ and $\mathcal{O} = \mathcal{Y}\backslash(\mathcal{Q} \cup \{o\})$;
12) For every $k \in \mathcal{L}\backslash\mathcal{A}$:
   a) if there exist a direct path from r_k to w_k, then include k in $\mathcal{P}$;

When this algorithm finishes, we acquire sets $\mathcal{Y}, \mathcal{D}, \mathcal{P}, \mathcal{T}, \mathcal{Z}_r$ such that the conditions in Proposition 2 are satisfied and $\mathcal{Z}_r, \mathcal{T}$ has Property 2.

IX. Conclusions

A new local module identification method has been introduced that consistently identifies the target module under known topology, with a generalized scheme for selection of measured node signals and excitation of nodes. We provide flexibility in the sufficient conditions to identify a target module which creates considerable freedom in sensor selection and actuation schemes. This is achieved by combining elements of the direct and indirect identification approaches. We use both node signals and external excitation signals as predictor inputs, allow post-processing of module estimates, and use MIMO identification setting, thereby mixing both the approaches. With this step we remove restrictive conditions on measured node signals and excitation signals that are present in the currently available methods, e.g. concerning parallel paths and loops around the output.

Appendix I
Proof of Proposition 2

On the basis of the decomposition of node signals as defined in Section V we are going to rewrite the system's equations (6) in the following structured form:

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{QQ}} & G_{\mathcal{Q}o} & G_{\mathcal{QO}} & G_{\mathcal{QA}} & G_{\mathcal{QZ}} \\ G_{o\mathcal{Q}} & G_{oo} & G_{o\mathcal{O}} & G_{o\mathcal{A}} & G_{o\mathcal{Z}} \\ G_{\mathcal{OQ}} & G_{\mathcal{O}o} & G_{\mathcal{OO}} & G_{\mathcal{OA}} & G_{\mathcal{OZ}} \\ G_{\mathcal{AQ}} & G_{\mathcal{A}o} & G_{\mathcal{AO}} & G_{\mathcal{AA}} & G_{\mathcal{AZ}} \\ G_{\mathcal{ZQ}} & G_{\mathcal{Z}o} & G_{\mathcal{ZO}} & G_{\mathcal{ZA}} & G_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} H_{\mathcal{QQ}} & H_{\mathcal{Q}o} & H_{\mathcal{QO}} & H_{\mathcal{QA}} & H_{\mathcal{QZ}} \\ H_{o\mathcal{Q}} & H_{oo} & H_{o\mathcal{O}} & H_{o\mathcal{A}} & H_{o\mathcal{Z}} \\ H_{\mathcal{OQ}} & H_{\mathcal{O}o} & H_{\mathcal{OO}} & H_{\mathcal{OA}} & H_{\mathcal{OZ}} \\ H_{\mathcal{AQ}} & H_{\mathcal{A}o} & H_{\mathcal{AO}} & H_{\mathcal{AA}} & H_{\mathcal{AZ}} \\ H_{\mathcal{ZQ}} & H_{\mathcal{Z}o} & H_{\mathcal{ZO}} & H_{\mathcal{ZA}} & H_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{O}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} R_{\mathcal{QQ}} & 0 & 0 & 0 & 0 \\ 0 & R_{oo} & 0 & 0 & 0 \\ 0 & 0 & R_{\mathcal{OO}} & 0 & 0 \\ 0 & 0 & 0 & R_{\mathcal{AA}} & 0 \\ 0 & 0 & 0 & 0 & R_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}} \end{bmatrix} \tag{10}$$

where we make the notation agreement that the matrix H is not necessarily monic, and the scaling of the white noise process e is such that $cov(e) = I$. Using $\mathcal{Z} = \mathcal{Z}_r \cup \mathcal{Z}_u$, we write $w_{\mathcal{Z}} = \begin{bmatrix} w_{\mathcal{Z}_r}^\top & w_{\mathcal{Z}_u}^\top \end{bmatrix}^\top$, $G_{\mathcal{Z}\star} = \begin{bmatrix} G_{\mathcal{Z}_r\star}^\top & G_{\mathcal{Z}_u\star}^\top \end{bmatrix}^\top$, $G_{\star\mathcal{Z}} = \begin{bmatrix} G_{\star\mathcal{Z}_r} & G_{\star\mathcal{Z}_u} \end{bmatrix}$, $G_{\mathcal{ZZ}} = \begin{bmatrix} G_{\mathcal{ZZ}_r} & G_{\mathcal{ZZ}_u} \end{bmatrix}$, and $R_{\mathcal{ZZ}} = \begin{bmatrix} R_{\mathcal{Z}_r\mathcal{Z}_r} & 0 \\ 0 & R_{\mathcal{Z}_u\mathcal{Z}_u} \end{bmatrix}$ where $\star = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}, \mathcal{Z}_r, \mathcal{Z}_u$. We can define $H_{\mathcal{Z}\star}$, $H_{\star\mathcal{Z}}$ and $H_{\mathcal{ZZ}}$ analogously.

Next we will eliminate the non-measured node signals $w_{\mathcal{Z}}$ in the equation (10), by abstracting these nodes.

Proposition 3: The system equations for the measured variables $w_{\mathcal{D}\cup\mathcal{Y}}$ can be written as,

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \breve{G}_{\mathcal{QQ}} & \breve{G}_{\mathcal{Q}o} & \breve{G}_{\mathcal{QO}} & \breve{G}_{\mathcal{QA}} \\ \breve{G}_{o\mathcal{Q}} & \breve{G}_{oo} & \breve{G}_{o\mathcal{O}} & \breve{G}_{o\mathcal{A}} \\ \breve{G}_{\mathcal{OQ}} & \breve{G}_{\mathcal{O}o} & \breve{G}_{\mathcal{OO}} & \breve{G}_{\mathcal{OA}} \\ \breve{G}_{\mathcal{AQ}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{AO}} & \breve{G}_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \underbrace{\breve{v}}_{\breve{H}e} + \underbrace{\breve{u}}_{\breve{R}r},$$

$$\breve{v} = \begin{bmatrix} \breve{H}_{\mathcal{QQ}} & \breve{H}_{\mathcal{Q}o} & \breve{H}_{\mathcal{QO}} & \breve{H}_{\mathcal{QA}} & \breve{H}_{\mathcal{QZ}_r} & \breve{H}_{\mathcal{QZ}_u} \\ \breve{H}_{o\mathcal{Q}} & \breve{H}_{oo} & \breve{H}_{o\mathcal{O}} & \breve{H}_{o\mathcal{A}} & \breve{H}_{o\mathcal{Z}_r} & \breve{H}_{o\mathcal{Z}_u} \\ \breve{H}_{\mathcal{OQ}} & \breve{H}_{\mathcal{O}o} & \breve{H}_{\mathcal{OO}} & \breve{H}_{\mathcal{OA}} & \breve{H}_{\mathcal{OZ}_r} & \breve{H}_{\mathcal{OZ}_u} \\ \breve{H}_{\mathcal{AQ}} & \breve{H}_{\mathcal{A}o} & \breve{H}_{\mathcal{AO}} & \breve{H}_{\mathcal{AA}} & \breve{H}_{\mathcal{AZ}_r} & \breve{H}_{\mathcal{AZ}_u} \end{bmatrix} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{O}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}_r} \\ e_{\mathcal{Z}_u} \end{bmatrix}$$

$$\breve{u} = \begin{bmatrix} \breve{R}_{\mathcal{QQ}} & 0 & 0 & 0 & \breve{R}_{\mathcal{QZ}_r} & \breve{R}_{\mathcal{QZ}_u} \\ 0 & \breve{R}_{oo} & 0 & 0 & \breve{R}_{o\mathcal{Z}_r} & \breve{R}_{o\mathcal{Z}_u} \\ 0 & 0 & \breve{R}_{\mathcal{OO}} & 0 & \breve{R}_{\mathcal{OZ}_r} & \breve{R}_{\mathcal{OZ}_u} \\ 0 & 0 & 0 & \breve{R}_{\mathcal{AA}} & \breve{R}_{\mathcal{AZ}_r} & \breve{R}_{\mathcal{AZ}_u} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}_r} \\ r_{\mathcal{Z}_u} \end{bmatrix} \tag{11}$$

with $cov(e) = I$, and where

$$\breve{G}_{\mathcal{Z}_r\mathcal{Z}_r} = G_{\mathcal{Z}_r\mathcal{Z}_r} + G_{\mathcal{Z}_r\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u\mathcal{Z}_r} \tag{12}$$

$$\breve{G}_{\star\mathcal{Z}_r} = G_{\star\mathcal{Z}_r} + G_{\star\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u\mathcal{Z}_r} \tag{13}$$

$$\breve{G}_{\star\square} = G_{\star\square} + G_{\star\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u\square} + \breve{G}_{\star\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r\square}, \tag{14}$$

$$\breve{H}_{\star\diamond} = H_{\star\diamond} + G_{\star\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}H_{\mathcal{Z}_u\diamond} + \breve{G}_{\star\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}H_{\mathcal{Z}_r\diamond}, \tag{15}$$

$$\breve{R}_{\star\square} = R_{\star\square}, \tag{16}$$

$$\breve{R}_{\star\mathcal{Z}_u} = \left(\breve{G}_{\star\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r\mathcal{Z}_u} + G_{\star\mathcal{Z}_u}\right) \times (I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}R_{\mathcal{Z}_u\mathcal{Z}_u}, \tag{17}$$

$$\breve{R}_{\star\mathcal{Z}_r} = \breve{G}_{\star\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}R_{\mathcal{Z}_r\mathcal{Z}_r} \tag{18}$$

where $\star, \square = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}$ and $\diamond = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}, \mathcal{Z}_r, \mathcal{Z}_u$. $\square$

Proof: From (10), the fifth (block) row provides the equation for $w_{\mathcal{Z}}$ and using $w_{\mathcal{Z}} = \begin{bmatrix} w_{\mathcal{Z}_r}^\top & w_{\mathcal{Z}_u}^\top \end{bmatrix}^\top$, we have $w_{\mathcal{Z}_u} = (I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}[\sum_{k\in\star} G_{\mathcal{Z}_u\star}w_\star + \sum_{\ell\in\square} H_{\mathcal{Z}_u\square}e_\square + \sum_{k\in\diamond} R_{\mathcal{Z}_u\diamond}r_\diamond]$ where $\star = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}, \mathcal{Z}_r$ and $\square = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}, \mathcal{Z}_r, \mathcal{Z}_u$. Using the equation of $w_{\mathcal{Z}_u}$, we can obtain the expression for $G_{\mathcal{AZ}_u}w_{\mathcal{Z}_u}$, $G_{\mathcal{QZ}_u}w_{\mathcal{Z}_u}$, $G_{\mathcal{OZ}_u}w_{\mathcal{Z}_u}$, $G_{o\mathcal{Z}_u}w_{\mathcal{Z}_u}$ and $G_{\mathcal{Z}_r\mathcal{Z}_u}w_{\mathcal{Z}_u}$. Substituting the above expressions in the respective block rows of (10), we obtain a network representation without

$w_{\mathcal{Z}_u}$ in w. Following the similar steps as above with the new network representation, we can now write the expression for $w_{\mathcal{Z}_r}$. Using this we can obtain the expression for $\breve{G}_{\star\mathcal{Z}_r} w_{\mathcal{Z}_r}$. Substituting these expressions in the respective block rows of the new network representation, we obtain the result of the proposition 3 $\square$

The spectral density of $\breve{v}$ is given by $\Phi_{\breve{v}} = \breve{H}\breve{H}^*$. Applying a spectral factorization [22] to $\Phi_{\breve{v}}$ will deliver $\Phi_{\breve{v}} = \tilde{H}\tilde{\Lambda}\tilde{H}^*$ with $\tilde{H}$ a monic, stable and minimum phase rational matrix, and $\tilde{\Lambda}$ a positive definite (constant) matrix. Then there exists a white noise process $\tilde{\xi}$ defined by $\tilde{\xi} := \tilde{H}^{-1}\breve{H}e$ such that $\tilde{H}\tilde{\xi} = \breve{v}$, with cov($\tilde{\xi}$) = $\tilde{\Lambda}$, while $\tilde{H}$ is of the form

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} & \tilde{H}_{14} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} & \tilde{H}_{24} \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & \tilde{H}_{34} \\ \tilde{H}_{41} & \tilde{H}_{42} & \tilde{H}_{43} & \tilde{H}_{44} \end{bmatrix} \tag{19}$$

and where the block dimensions are conformable to the dimensions of $w_{\mathcal{Q}}$, w_o, $w_{\mathcal{U}}$ and $w_{\mathcal{A}}$ respectively. As a result, (11) can be rewritten as

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \breve{G}_{\mathcal{Q}\mathcal{Q}} & \breve{G}_{\mathcal{Q}o} & \breve{G}_{\mathcal{Q}\mathcal{O}} & \breve{G}_{\mathcal{Q}\mathcal{A}} \\ \breve{G}_{o\mathcal{Q}} & \breve{G}_{oo} & \breve{G}_{o\mathcal{O}} & \breve{G}_{o\mathcal{A}} \\ \breve{G}_{\mathcal{O}\mathcal{Q}} & \breve{G}_{\mathcal{O}o} & \breve{G}_{\mathcal{O}\mathcal{O}} & \breve{G}_{\mathcal{O}\mathcal{A}} \\ \breve{G}_{\mathcal{A}\mathcal{Q}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{A}\mathcal{O}} & \breve{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \tilde{H} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{O}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix} + \breve{u}. \tag{20}$$

By denoting

$$\begin{bmatrix} \breve{H}_{14} \\ \breve{H}_{24} \\ \breve{H}_{34} \end{bmatrix} := \begin{bmatrix} \tilde{H}_{14}\tilde{H}_{44}^{-1} \\ \tilde{H}_{24}\tilde{H}_{44}^{-1} \\ \tilde{H}_{34}\tilde{H}_{44}^{-1} \end{bmatrix} \tag{21}$$

and premultiplying (20) with

$$\begin{bmatrix} I & 0 & 0 & -\breve{H}_{14} \\ 0 & I & 0 & -\breve{H}_{24} \\ 0 & 0 & I & -\breve{H}_{34} \\ 0 & 0 & 0 & I \end{bmatrix} \tag{22}$$

while only keeping the identity terms on the left hand side, we obtain an equivalent network equation:

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \breve{G}'_{\mathcal{Q}\mathcal{Q}} & \breve{G}'_{\mathcal{Q}o} & \breve{G}'_{\mathcal{Q}\mathcal{O}} & \breve{G}'_{\mathcal{Q}\mathcal{A}} \\ \breve{G}'_{o\mathcal{Q}} & \breve{G}'_{oo} & \breve{G}'_{o\mathcal{O}} & \breve{G}'_{o\mathcal{A}} \\ \breve{G}'_{\mathcal{O}\mathcal{Q}} & \breve{G}'_{\mathcal{O}o} & \breve{G}'_{\mathcal{O}\mathcal{O}} & \breve{G}'_{\mathcal{O}\mathcal{A}} \\ \breve{G}_{\mathcal{A}\mathcal{Q}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{A}\mathcal{O}} & \breve{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \tilde{H}'_{11} & \tilde{H}'_{12} & \tilde{H}'_{13} & 0 \\ \tilde{H}'_{21} & \tilde{H}'_{22} & \tilde{H}'_{23} & 0 \\ \tilde{H}'_{31} & \tilde{H}'_{32} & \tilde{H}'_{33} & 0 \\ \tilde{H}_{41} & \tilde{H}_{42} & \tilde{H}_{43} & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{O}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix}$$
$$+ \begin{bmatrix} \breve{R}'_{\mathcal{Q}\mathcal{Q}} & 0 & 0 & \breve{R}'_{\mathcal{Q}\mathcal{A}} & \breve{R}'_{\mathcal{Q}\mathcal{Z}_r} & \breve{R}'_{\mathcal{Q}\mathcal{Z}_u} \\ 0 & \breve{R}'_{oo} & 0 & \breve{R}'_{o\mathcal{A}} & \breve{R}'_{o\mathcal{Z}_r} & \breve{R}'_{o\mathcal{Z}_u} \\ 0 & 0 & \breve{R}'_{\mathcal{O}\mathcal{O}} & \breve{R}'_{\mathcal{O}\mathcal{A}} & \breve{R}'_{\mathcal{O}\mathcal{Z}_r} & \breve{R}'_{\mathcal{O}\mathcal{Z}_u} \\ 0 & 0 & 0 & \breve{R}_{\mathcal{A}\mathcal{A}} & \breve{R}_{\mathcal{A}\mathcal{Z}_r} & \breve{R}_{\mathcal{A}\mathcal{Z}_u} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}_r} \\ r_{\mathcal{Z}_u} \end{bmatrix} \tag{23}$$

with

$$\begin{aligned}
\breve{G}'_{\mathcal{Q}\mathcal{A}} &= \breve{G}_{\mathcal{Q}\mathcal{A}} - \breve{H}_{14}\breve{G}_{\mathcal{A}\mathcal{A}} + \breve{H}_{14} && (24)\\
\breve{G}'_{\mathcal{Q}\star} &= \breve{G}_{\mathcal{Q}\star} - \breve{H}_{14}\breve{G}_{\mathcal{A}\star} && (25)\\
\breve{G}'_{o\star} &= \breve{G}_{o\star} - \breve{H}_{24}\breve{G}_{\mathcal{A}\star} && (26)\\
\breve{G}'_{\mathcal{O}\star} &= \breve{G}_{\mathcal{O}\star} - \breve{H}_{34}\breve{G}_{\mathcal{A}\star} && (27)\\
\breve{G}'_{o\mathcal{A}} &= \breve{G}_{o\mathcal{A}} - \breve{H}_{24}\breve{G}_{\mathcal{A}\mathcal{A}} + \breve{H}_{24} && (28)\\
\breve{G}'_{\mathcal{O}\mathcal{A}} &= \breve{G}_{\mathcal{O}\mathcal{A}} - \breve{H}_{34}\breve{G}_{\mathcal{A}\mathcal{A}} + \breve{H}_{34} && (29)\\
\tilde{H}'_{1\square} &= \tilde{H}_{1\square} - \breve{H}_{14}\tilde{H}_{4\square} && (30)\\
\tilde{H}'_{2\square} &= \tilde{H}_{2\square} - \breve{H}_{24}\tilde{H}_{4\square} && (31)\\
\tilde{H}'_{3\square} &= \tilde{H}_{3\square} - \breve{H}_{34}\tilde{H}_{4\square} && (32)\\
\breve{R}'_{\mathcal{Q}\diamond} &= \breve{R}_{\mathcal{Q}\diamond} - \breve{H}_{14}\breve{R}_{\mathcal{A}\diamond} && (33)\\
\breve{R}'_{o\diamond} &= \breve{R}_{o\diamond} - \breve{H}_{24}\breve{R}_{\mathcal{A}\diamond} && (34)\\
\breve{R}'_{\mathcal{O}\diamond} &= \breve{R}_{\mathcal{O}\diamond} - \breve{H}_{34}\breve{R}_{\mathcal{A}\diamond} && (35)
\end{aligned}$$

where $\star \in \{\mathcal{Q} \cup \{o\} \cup \mathcal{O}\}$, $\square \in \{1, 2, 3\}$ and $\diamond \in \{\mathcal{A} \cup \mathcal{Z}_r \cup \mathcal{Z}_u\}$.

The next step is now to show that that the block elements $\breve{G}'_{\mathcal{Q}o}$, $\breve{G}'_{oo}$, $\breve{G}'_{\mathcal{O}o}$, $\breve{G}'_{\mathcal{Q}\mathcal{O}}$, $\breve{G}'_{o\mathcal{O}}$, $\breve{G}'_{\mathcal{O}\mathcal{O}}$ in G can be made 0. This can be done by variable substitution as follows:

The second row in (37) is replaced by an explicit expression for w_o according to

$$\begin{aligned} w_o &= (1 - \breve{G}'_{oo})^{-1}[\breve{G}'_{o\mathcal{Q}} w_{\mathcal{Q}} + \breve{G}'_{o\mathcal{O}} w_{\mathcal{O}} + \breve{G}'_{o\mathcal{A}} w_{\mathcal{A}} + \tilde{H}'_{21}\tilde{\xi}_{\mathcal{Q}} \\ &+ \tilde{H}'_{22}\tilde{\xi}_o + \tilde{H}'_{23}\tilde{\xi}_{\mathcal{O}} + \breve{R}'_{oo} r_o + \breve{R}'_{o\mathcal{A}} r_{\mathcal{A}} + \breve{R}'_{o\mathcal{Z}_r} r_{\mathcal{Z}_r} + \breve{R}'_{o\mathcal{Z}_u} r_{\mathcal{Z}_u}] \end{aligned} \tag{36}$$

Additionally, this expression for w_o is substituted into the first and third block row of (37), to remove the w_o-dependent term on the right hand side, leading to

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \breve{G}''_{\mathcal{Q}\mathcal{Q}} & 0 & \breve{G}''_{\mathcal{Q}\mathcal{O}} & \breve{G}''_{\mathcal{Q}\mathcal{A}} \\ \breve{G}''_{o\mathcal{Q}} & 0 & \breve{G}''_{o\mathcal{O}} & \breve{G}''_{o\mathcal{A}} \\ \breve{G}''_{\mathcal{O}\mathcal{Q}} & 0 & \breve{G}''_{\mathcal{O}\mathcal{O}} & \breve{G}''_{\mathcal{O}\mathcal{A}} \\ \breve{G}_{\mathcal{A}\mathcal{Q}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{A}\mathcal{O}} & \breve{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \tilde{H}''_{11} & \tilde{H}''_{12} & \tilde{H}''_{13} & 0 \\ \tilde{H}''_{21} & \tilde{H}''_{22} & \tilde{H}''_{23} & 0 \\ \tilde{H}''_{31} & \tilde{H}''_{32} & \tilde{H}''_{33} & 0 \\ \tilde{H}_{41} & \tilde{H}_{42} & \tilde{H}_{43} & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{O}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix}$$
$$+ \begin{bmatrix} \breve{R}''_{\mathcal{Q}\mathcal{Q}} & \breve{R}''_{\mathcal{Q}o} & 0 & \breve{R}''_{\mathcal{Q}\mathcal{A}} & \breve{R}''_{\mathcal{Q}\mathcal{Z}_r} & \breve{R}''_{\mathcal{Q}\mathcal{Z}_u} \\ 0 & \breve{R}''_{oo} & 0 & \breve{R}''_{o\mathcal{A}} & \breve{R}''_{o\mathcal{Z}_r} & \breve{R}''_{o\mathcal{Z}_u} \\ 0 & \breve{R}''_{\mathcal{O}o} & \breve{R}''_{\mathcal{O}\mathcal{O}} & \breve{R}''_{\mathcal{O}\mathcal{A}} & \breve{R}''_{\mathcal{O}\mathcal{Z}_r} & \breve{R}''_{\mathcal{O}\mathcal{Z}_u} \\ 0 & 0 & 0 & \breve{R}_{\mathcal{A}\mathcal{A}} & \breve{R}_{\mathcal{A}\mathcal{Z}_r} & \breve{R}_{\mathcal{A}\mathcal{Z}_u} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}_r} \\ r_{\mathcal{Z}_u} \end{bmatrix} \tag{37}$$

with

$$\begin{aligned}
\breve{G}''_{o\star} &= (I - \breve{G}'_{oo})^{-1}\breve{G}'_{o\star} && (38)\\
\tilde{H}''_{2\star} &= (I - \breve{G}'_{oo})^{-1}\tilde{H}'_{2\star} && (39)\\
\breve{R}''_{o\star} &= (I - \breve{G}'_{oo})^{-1}\breve{R}'_{o\star} && (40)\\
\breve{G}''_{\mathcal{Q}\star} &= \breve{G}'_{\mathcal{Q}\star} + \breve{G}'_{\mathcal{Q}o}\breve{G}''_{o\star} && (41)\\
\tilde{H}''_{1\star} &= \tilde{H}'_{1\star} + \breve{G}'_{\mathcal{Q}o}\tilde{H}''_{2\star} && (42)\\
\breve{G}''_{\mathcal{O}\star} &= \breve{G}'_{\mathcal{O}\star} + \breve{G}'_{\mathcal{O}o}\breve{G}''_{o\star} && (43)\\
\tilde{H}''_{3\star} &= \tilde{H}'_{3\star} + \breve{G}'_{\mathcal{O}o}\tilde{H}''_{2\star} && (44)\\
\breve{R}''_{\mathcal{Q}\star} &= \breve{R}'_{\mathcal{Q}\star} + \breve{G}'_{\mathcal{Q}o}\breve{R}''_{o\star} && (45)\\
\breve{R}''_{\mathcal{O}\star} &= \breve{R}'_{\mathcal{O}\star} + \breve{G}'_{\mathcal{O}o}\breve{R}''_{o\star}. && (46)
\end{aligned}$$

In a similar way we can make $\breve{G}'_{\mathcal{Q}\mathcal{O}}$, $\breve{G}'_{o\mathcal{O}}$, $\breve{G}'_{\mathcal{O}\mathcal{O}}$ in G as 0. Therefore,

$$\begin{aligned}
\bar{G}_{\mathcal{O}\star} &= (I - \breve{G}''_{\mathcal{O}\mathcal{O}})^{-1}\breve{G}''_{\mathcal{O}\star} && (47)\\
\tilde{H}'''_{3\star} &= (I - \breve{G}''_{\mathcal{O}\mathcal{O}})^{-1}\tilde{H}''_{3\star} && (48)\\
\bar{R}_{\mathcal{O}\star} &= (I - \breve{G}''_{\mathcal{O}\mathcal{O}})^{-1}\breve{R}''_{\mathcal{O}\star} && (49)\\
\breve{G}'''_{\mathcal{Q}\star} &= \breve{G}''_{\mathcal{Q}\star} + \breve{G}''_{\mathcal{Q}\mathcal{O}}\breve{G}'''_{\mathcal{O}\star} && (50)\\
\tilde{H}'''_{1\star} &= \tilde{H}''_{1\star} + \breve{G}'_{\mathcal{Q}\mathcal{O}}\tilde{H}'''_{3\star} && (51)
\end{aligned}$$

$$
\begin{aligned}
\bar{G}_{o\star} &= \breve{G}''_{o\star} + \breve{G}''_{o\mathcal{O}}\breve{G}'''_{\mathcal{O}\star} && (52)\\
\tilde{H}'''_{2\star} &= \tilde{H}''_{2\star} + \breve{G}''_{o\mathcal{O}}\tilde{H}'''_{3\star} && (53)\\
\breve{R}'''_{\mathcal{Q}\star} &= \breve{R}''_{\mathcal{Q}\star} + \breve{G}''_{\mathcal{Q}\mathcal{O}}\breve{R}'''_{\mathcal{O}\star} && (54)\\
\bar{R}_{o\star} &= \breve{R}''_{o\star} + \breve{G}''_{o\mathcal{O}}\breve{R}'''_{\mathcal{O}\star}. && (55)
\end{aligned}
$$

Since because of these operations, the matrix $\breve{G}'''_{\mathcal{Q}\mathcal{Q}}$ might not be hollow, we move any diagonal terms of this matrix to the left hand side of the equation, and premultiply the first (block) equation by the diagonal matrix $(I - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}}))^{-1}$, to obtain the expression

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{G}_{\mathcal{Q}\mathcal{Q}} & 0 & 0 & \bar{G}_{\mathcal{Q}\mathcal{A}} \\ \bar{G}_{o\mathcal{Q}} & 0 & 0 & \bar{G}_{o\mathcal{A}} \\ \bar{G}_{\mathcal{O}\mathcal{Q}} & 0 & 0 & \bar{G}_{\mathcal{O}\mathcal{A}} \\ \breve{G}_{\mathcal{A}\mathcal{Q}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{A}\mathcal{O}} & \breve{G}_{\mathcal{A}\mathcal{A}} \end{bmatrix}}_{\bar{G}} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \tilde{H}''''_{11} & \tilde{H}''''_{12} & \tilde{H}''''_{13} & 0 \\ \tilde{H}'''_{21} & \tilde{H}'''_{22} & \tilde{H}'''_{23} & 0 \\ \tilde{H}'''_{31} & \tilde{H}'''_{32} & \tilde{H}'''_{33} & 0 \\ \tilde{H}_{41} & \tilde{H}_{42} & \tilde{H}_{43} & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{O}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix}
$$

$$
+ \underbrace{\begin{bmatrix} \bar{R}_{\mathcal{Q}\mathcal{Q}} & \bar{R}_{\mathcal{Q}o} & \bar{R}_{\mathcal{Q}\mathcal{O}} & \bar{R}_{\mathcal{Q}\mathcal{A}} & \bar{R}_{\mathcal{Q}\mathcal{Z}_r} & \bar{R}_{\mathcal{Q}\mathcal{Z}_u} \\ 0 & \bar{R}_{oo} & \bar{R}_{o\mathcal{Q}} & \bar{R}_{o\mathcal{A}} & \bar{R}_{o\mathcal{Z}_r} & \bar{R}_{o\mathcal{Z}_u} \\ 0 & \bar{R}_{\mathcal{O}o} & \bar{R}_{\mathcal{O}\mathcal{O}} & \bar{R}_{\mathcal{O}\mathcal{A}} & \bar{R}_{\mathcal{O}\mathcal{Z}_r} & \bar{R}_{\mathcal{O}\mathcal{Z}_u} \\ 0 & 0 & 0 & \breve{R}_{\mathcal{A}\mathcal{A}} & \breve{R}_{\mathcal{A}\mathcal{Z}_r} & \breve{R}_{\mathcal{A}\mathcal{Z}_u} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}_r} \\ r_{\mathcal{Z}_u} \end{bmatrix}}_{\bar{u}} \tag{56}
$$

with

$$
\begin{aligned}
\bar{G}_{\mathcal{Q}\mathcal{Q}} &= (I - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}}))^{-1}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}} - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}})), && (57)\\
\bar{G}_{\mathcal{Q}\mathcal{A}} &= (I - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}}))^{-1}\breve{G}'''_{\mathcal{Q}\mathcal{A}} && (58)\\
\tilde{H}''''_{1\star} &= (I - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}}))^{-1}\tilde{H}'''_{1\star} && (59)\\
\bar{R}_{\mathcal{Q}\star} &= (I - \mathrm{diag}(\breve{G}'''_{\mathcal{Q}\mathcal{Q}}))^{-1}\breve{R}'''_{\mathcal{Q}\star} && (60)
\end{aligned}
$$

As final step, we need the matrix $\tilde{H}_r := \begin{bmatrix} \tilde{H}''''_{11} & \tilde{H}''''_{12} & \tilde{H}''''_{13} \\ \tilde{H}'''_{21} & \tilde{H}'''_{22} & \tilde{H}'''_{23} \\ \tilde{H}'''_{31} & \tilde{H}'''_{32} & \tilde{H}'''_{33} \end{bmatrix}$ to be monic, stable and minimum phase to obtain the representation as in (7). To that end, we consider the stochastic process $\tilde{v}_{\mathcal{Y}} := \tilde{H}_r \tilde{\xi}_{\mathcal{Y}}$ with $\tilde{\xi}_{\mathcal{Y}} := \begin{bmatrix} \xi_{\mathcal{Q}}^\top & \tilde{\xi}_o^\top & \xi_{\mathcal{O}}^\top \end{bmatrix}^\top$. The spectral density of $\tilde{v}_{\mathcal{Y}}$ is then given by $\Phi_{\tilde{v}_{\mathcal{Y}}} = \tilde{H}_r \tilde{\Lambda}_{\mathcal{Y}} \tilde{H}_r^\star$ with $\tilde{\Lambda}_{\mathcal{Y}}$ the covariance matrix of $\tilde{\xi}_{\mathcal{Y}}$, that can be decomposed as $\tilde{\Lambda}_{\mathcal{Y}} = \tilde{\Gamma}_r \tilde{\Gamma}_r^T$. From spectral factorization [22] it follows that the spectral factor $\tilde{H}_r \tilde{\Gamma}_r$ of $\Phi_{\tilde{v}_{\mathcal{Y}}}$ satisfies

$$
\tilde{H}_r \tilde{\Gamma}_r = \bar{H}_s D \tag{61}
$$

with $\bar{H}_s$ a stable and minimum phase rational matrix, and D an "all pass" stable rational matrix satisfying $DD^\star = I$. The signal $\tilde{v}_{\mathcal{Y}}$ can then be written as

$$
\tilde{v}_{\mathcal{Y}} = \tilde{H}_r \tilde{\xi}_{\mathcal{Y}} = \bar{H}_s D \tilde{\Gamma}_r^{-1} \tilde{\xi}_{\mathcal{Y}}.
$$

By defining $\bar{H}_s^\infty := \lim_{z\to\infty} \bar{H}_s$, this can be rewritten as

$$
\tilde{v}_{\mathcal{Y}} = \tilde{H}_r \tilde{\xi}_{\mathcal{Y}} = \underbrace{\bar{H}_s (\bar{H}_s^\infty)^{-1}}_{\bar{H}} \underbrace{\bar{H}_s^\infty D \tilde{\Gamma}_r^{-1} \tilde{\xi}_{\mathcal{Y}}}_{\xi_{\mathcal{Y}}}.
$$

As a result, $\bar{H}$ is a monic stable and stably invertible rational matrix, and $\xi_{\mathcal{Y}}$ is a white noise process with spectral density given by $\bar{H}_s^\infty D \tilde{\Gamma}_r^{-1} \Phi_{\tilde{\xi}_{\mathcal{Y}}} \tilde{\Gamma}_r^{-T} D^\star (\bar{H}_s^\infty)^T = \bar{H}_s^\infty (\bar{H}_s^\infty)^T$. Therefore we can write (56) as,

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} = \bar{G} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} & \bar{H}_{13} & 0 \\ \bar{H}_{21} & \bar{H}_{22} & \bar{H}_{23} & 0 \\ \bar{H}_{31} & \bar{H}_{32} & \bar{H}_{33} & 0 \\ \bar{H}_{41} & \bar{H}_{42} & \bar{H}_{43} & \tilde{H}_{445} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{O}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix} + \bar{u} \tag{62}
$$

where $\begin{bmatrix} \bar{H}_{31} \bar{H}_{32} \bar{H}_{33} \end{bmatrix} = \begin{bmatrix} \tilde{H}_{31} \tilde{H}_{32} \tilde{H}_{33} \end{bmatrix} \tilde{\Gamma}_r D^{-1} (\bar{H}_s^\infty)^{-1}$. This completes the proof for obtaining (7). $\square$

Appendix II
Proof of Theorem 1

In the first part of the proof we derive the elements of (7) when the first two conditions of Theorem 1 are satisfied. Starting from the expression (11) the spectral density $\Phi_{\breve{v}}$ can be written as $\breve{H}\breve{H}^*$ while it is denoted as

$$
\Phi_{\breve{v}} = \begin{bmatrix} \Phi_{\breve{v}_{\mathcal{Q}}} & \Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_o} & \Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{O}}} & \Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}} \\ \Phi_{\breve{v}_o\breve{v}_{\mathcal{Q}}} & \Phi_{\breve{v}_o} & \Phi_{\breve{v}_o\breve{v}_{\mathcal{O}}} & \Phi_{\breve{v}_o\breve{v}_{\mathcal{A}}} \\ \Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{Q}}} & \Phi_{\breve{v}_{\mathcal{O}}\breve{v}_o} & \Phi_{\breve{v}_{\mathcal{O}}} & \Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{A}}} \\ \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_{\mathcal{Q}}} & \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_o} & \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_{\mathcal{O}}} & \Phi_{\breve{v}_{\mathcal{A}}} \end{bmatrix}. \tag{63}
$$

In this structure we are particularly going to analyse the elements $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}}, \Phi_{\breve{v}_o\breve{v}_{\mathcal{A}}}, \Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{A}}}$ where $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}} = \breve{H}_{\mathcal{Q}\mathcal{Q}}\breve{H}^*_{\mathcal{A}\mathcal{Q}} + \breve{H}_{\mathcal{Q}o}\breve{H}^*_{\mathcal{A}o} + \breve{H}_{\mathcal{Q}\mathcal{O}}\breve{H}^*_{\mathcal{A}\mathcal{O}} + \breve{H}_{\mathcal{Q}\mathcal{A}}\breve{H}^*_{\mathcal{A}\mathcal{A}} + \breve{H}_{\mathcal{Q}\mathcal{Z}_r}\breve{H}^*_{\mathcal{A}\mathcal{Z}_r} + \breve{H}_{\mathcal{Q}\mathcal{Z}_u}\breve{H}^*_{\mathcal{A}\mathcal{Z}_u}$. Similarly we have $\Phi_{\breve{v}_o\breve{v}_{\mathcal{A}}}$ and $\Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{A}}}$. We have $\breve{H}_{\mathcal{A}\star} = H_{\mathcal{A}\star} + G_{\mathcal{A}\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}H_{\mathcal{Z}\star} + +\breve{G}_{\mathcal{A}\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}H_{\mathcal{Z}_r\star} = H_{\mathcal{A}\star} + H^{(i)}_{\mathcal{A}\star}$. Here $H^{(i)}_{\mathcal{A}\star}$ includes the transfer in the path from $e_\star$ to $w_{\mathcal{A}}$ through nodes in $w_{\mathcal{Z}}$ only. Similarly we can write $\breve{H}_{\mathcal{Q}\star}, \breve{H}_{o\star}, \breve{H}_{\mathcal{O}\star}$. We rewrite the equation of $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}}$ as $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}} = \sum_{k\in\star} H_{\mathcal{Q}\star}H^*_{\mathcal{A}\star} + \sum_{k\in\star} H_{\mathcal{Q}\star}H^{(i)*}_{\mathcal{A}\star} + \sum_{k\in\star} H^{(i)*}_{\mathcal{Q}\star}H^*_{\mathcal{A}\star} + \sum_{k\in\star} H^{(i)*}_{\mathcal{Q}\star}H^{(i)*}_{\mathcal{A}\star}$ where $\star = \mathcal{Q}, o, \mathcal{A}, \mathcal{O}, \mathcal{Z}_r, \mathcal{Z}_u$. Similarly we can write $\Phi_{\breve{v}_o\breve{v}_{\mathcal{A}}}$ and $\Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{A}}}$. The term $\sum_{k\in\star} H_{\mathcal{Q}\star}H^*_{\mathcal{A}\star}$ in $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}}$ correspond to the confounding variable in e that has direct paths and the other terms refer to the cases with unmeasured paths. If **condition 1 in Proposition 2** is satisfied, we ensure that there are no confounding variables in e for the estimation problem $w_{\mathcal{A}} \to w_{\mathcal{Y}}$. This implies that $H_{\mathcal{Q}\star}H^*_{\mathcal{A}\star} = H_{\mathcal{Q}\star}H^{(i)*}_{\mathcal{A}\star} = H^{(i)*}_{\mathcal{Q}\star}H_{\mathcal{A}\star} = H^{(i)*}_{\mathcal{Q}\star}H^{(i)*}_{\mathcal{A}\star} = 0$ for $\star = \mathcal{Q}, o, \mathcal{A}, \mathcal{O}, \mathcal{Z}_r, \mathcal{Z}_u$. Therefore $\Phi_{\breve{v}_{\mathcal{Q}}\breve{v}_{\mathcal{A}}} = \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_{\mathcal{Q}}} = 0$ and similarly $\Phi_{\breve{v}_o\breve{v}_{\mathcal{A}}} = \Phi_{\breve{v}_{\mathcal{O}}\breve{v}_{\mathcal{A}}} = \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_o} = \Phi_{\breve{v}_{\mathcal{A}}\breve{v}_{\mathcal{O}}} = 0$. Thus we get a block diagonal structure in equation (63). Then the spectral density $\Phi_{\breve{v}}$ has the unique spectral factorization $\Phi_{\breve{v}} = \begin{bmatrix} \bar{H}\Lambda_{11}\bar{H}^* & 0 \\ 0 & \breve{H}\Lambda_{22}\breve{H}^* \end{bmatrix} = \tilde{H}\Lambda\tilde{H}^*$ where $\tilde{H}, \bar{H}$ and $\breve{H}$ are monic, stable, minimum phase and $\bar{H}$ is of the form given in proposition 2.

Now we substitute $\tilde{H}\xi_{\mathcal{Y}}$ for $\breve{H}e$ in equation (11). If **condition 2 in Proposition 2** is satisfied, this ensures that $\breve{G}_{\star o} = \breve{G}_{\star\mathcal{O}} = 0$ where $\star = \mathcal{Q}, o, \mathcal{O}$ and we substitute it in (11). Now we have the (1,1) block-elements of the resulting matrix of $\breve{G}$ to have non-zero elements in the diagonal. By multiplying the first block row with a diagonal matrix $(I - \mathrm{diag}(\breve{G}_{\mathcal{Q}\mathcal{Q}}))^{-1}$ and writing only the block rows for

$w_{\mathcal{Q}}, w_o, w_{\mathcal{O}}$ (first 3 (block) rows), we lead to expression

$$\underbrace{\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{O}} \end{bmatrix}}_{w_{\mathcal{Y}}} = \underbrace{\begin{bmatrix} \bar{G}_{\mathcal{Q}\mathcal{Q}} & \bar{G}_{\mathcal{Q}\mathcal{A}} \\ \bar{G}_{o\mathcal{Q}} & \bar{G}_{o\mathcal{A}} \\ \bar{G}_{\mathcal{O}\mathcal{Q}} & \bar{G}_{\mathcal{O}\mathcal{A}} \end{bmatrix}}_{\bar{G}} \underbrace{\begin{bmatrix} w_{\mathcal{Q}} \\ w_{\mathcal{A}} \end{bmatrix}}_{w_{\mathcal{D}}} + \underbrace{\begin{bmatrix} \bar{H}_{\mathcal{Q}\mathcal{Q}} & \bar{H}_{\mathcal{Q}o} & \bar{H}_{\mathcal{Q}\mathcal{O}} \\ \bar{H}_{o\mathcal{Q}} & \bar{H}_{oo} & \bar{H}_{o\mathcal{O}} \\ \bar{H}_{\mathcal{O}\mathcal{Q}} & \bar{H}_{\mathcal{O}o} & \bar{H}_{\mathcal{O}\mathcal{O}} \end{bmatrix}}_{\bar{H}} \underbrace{\begin{bmatrix} \xi_{\mathcal{Q}} \\ \xi_o \\ \xi_{\mathcal{O}} \end{bmatrix}}_{\xi_{\mathcal{Y}}}$$

$$+ \underbrace{\begin{bmatrix} \bar{R}_{\mathcal{Q}\mathcal{Q}} & 0 & 0 & \bar{R}_{\mathcal{Q}\mathcal{Z}_r} & \bar{R}_{\mathcal{Q}\mathcal{Z}_u} \\ 0 & \bar{R}_{oo} & 0 & \bar{R}_{o\mathcal{Z}_r} & \bar{R}_{o\mathcal{Z}_u} \\ 0 & 0 & \bar{R}_{\mathcal{O}\mathcal{O}} & \bar{R}_{\mathcal{O}\mathcal{Z}_r} & \bar{R}_{\mathcal{O}\mathcal{Z}_u} \end{bmatrix}}_{\bar{R}} \underbrace{\begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{O}} \\ r_{\mathcal{Z}_r} \\ r_{\mathcal{Z}_u} \end{bmatrix}}_{r_{\mathcal{P}}} \quad (64)$$

with $\xi_{\mathcal{Q}}$, ξ_o and $\xi_{\mathcal{O}}$ white noise processes with dimensions conforming to $w_{\mathcal{Q}}$, w_o and $w_{\mathcal{O}}$, respectively, with $cov(\xi_{\mathcal{Y}}) = \bar{\Lambda}$ and with $\bar{H}$ being monic, stable and stably invertible and

$$\bar{G}_{\mathcal{Q}\mathcal{Q}} = (I - \mathrm{diag}(\breve{G}_{\mathcal{Q}\mathcal{Q}}))^{-1}(\breve{G}_{\mathcal{Q}\mathcal{Q}} - \mathrm{diag}(\breve{G}_{\mathcal{Q}\mathcal{Q}})) \quad (65)$$

$$\breve{G}_{\star o} = \breve{G}_{\star\mathcal{O}} = 0 \quad (66)$$

$$\bar{G}_{\star\square} = (I - \mathrm{diag}(\breve{G}_{\star\star}))^{-1}\breve{G}_{\star\square} \quad (67)$$

$$\bar{R}_{\star\diamond} = (I - \mathrm{diag}(\breve{G}_{\star\star}))^{-1}\breve{R}_{\star\diamond} \quad (68)$$

where $\star = \mathcal{Q}, o, \mathcal{O}$, $\square = \mathcal{Q}, \mathcal{A}$ and $\diamond = \mathcal{Q}, o, \mathcal{O}, \mathcal{A}, \mathcal{Z}_r, \mathcal{Z}_u$ where $\bar{G}_{\mathcal{Q}\mathcal{Q}}$ has diagonal elements as zero. This is represented in equation (7).

The second part of the proof is to prove consistency results. This can be shown by following a similar reasoning as in Theorem 1 of [21]. □

APPENDIX III
PROOF OF THEOREM 2

The target module that is the objective of our identification is given by G_{ji}, with $w_j \in (w_{\mathcal{Q}}, w_o)$ and $w_i \in (w_{\mathcal{A}}, w_{\mathcal{Q}})$. From (65), (67) and since $j \neq i$, we have $\bar{G}_{ji} = (I - \breve{G}_{jj})^{-1}\breve{G}_{ji} = (I - \breve{G}_{jj})^{-1}(G_{ji} + G_{j\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u i} + \breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r i})$. **Condition 1 in Property 2** ensures that the second term is zero. Now in the sequel we find expressions for $(I - \breve{G}_{jj})^{-1}$ and $\breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r i}$ using elements of (7) in order to extract G_{ji} from $\bar{G}_{ji}$.

If there are unmeasured loops through w_j, then $\breve{G}_{jj} \neq 0$. From (68), $\bar{R}_{jj} = (I - \breve{G}_{jj})^{-1}\breve{R}_{jj} = (I - \breve{G}_{jj})^{-1}R_{jj}$. Therefore if node j is excited by an external excitation signal (i.e. $R_{jj} = 1$) then $\bar{R}_{jj} = (I - \breve{G}_{jj})^{-1}$. Also from (68) we have $\bar{R}_{j\mathcal{Z}_r} = (I - \breve{G}_{jj})^{-1}\breve{R}_{j\mathcal{Z}_r} = (I - \breve{G}_{jj})^{-1}\breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}R_{\mathcal{Z}_r\mathcal{Z}_r}$. If **condition 5 in Property 2** (i.e $R_{\mathcal{Z}_r\mathcal{Z}_r} = 1$), then $\bar{R}_{j\mathcal{Z}_r} = (I - \breve{G}_{jj})^{-1}\breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}$. Similarly we will have $\bar{R}_{\mathcal{T}\mathcal{Z}_r} = (I - \mathrm{diag}(\breve{G}_{\mathcal{T}\mathcal{T}}))^{-1}\breve{G}_{\mathcal{T}\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}$. Now from (67), $\bar{G}_{\mathcal{T}j} = (I - \mathrm{diag}(\breve{G}_{\mathcal{T}\mathcal{T}}))^{-1}\breve{G}_{\mathcal{T}j}$. From (14), we have $\breve{G}_{\mathcal{T}j} = G_{\mathcal{T}j} + G_{\mathcal{T}\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u j} + \breve{G}_{\mathcal{T}\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r j}$. The sum has three terms and the **condition 4b in Property 2** ensures that the first two terms are zero. Therefore, $\bar{G}_{\mathcal{T}j} = (I - \mathrm{diag}(\breve{G}_{\mathcal{T}\mathcal{T}}))^{-1}\breve{G}_{\mathcal{T}\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r j} = \bar{R}_{\mathcal{T}\mathcal{Z}_r}G_{\mathcal{Z}_r j}$. **Condition 6 in Property 2** ensures that a left inverse of $\bar{R}_{\mathcal{T}\mathcal{Z}_r}$ exists. Then $G_{\mathcal{Z}_r j} = \bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j}$. From (14), we have $\breve{G}_{jj} = G_{jj} + G_{j\mathcal{Z}_u}(I - G_{\mathcal{Z}_u\mathcal{Z}_u})^{-1}G_{\mathcal{Z}_u j} + \breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r j}$. $G_{jj} = 0$ and the by **condition 4a of Property 2** the second term is also zero. Therefore $\breve{G}_{jj} = \breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r j} = (1 - \breve{G}_{jj})\bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j}$. Thus we get $(1 - \breve{G}_{jj})^{-1} = \left(1 - (1 + \bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j})^{-1}\bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}j}\right)^{-1}$.

We now look into the term $\breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r i} = \breve{R}^{-1}_{jj}\bar{R}_{j\mathcal{Z}_r}G_{\mathcal{Z}_r i}$. Now we differentiate two different cases to get $G_{\mathcal{Z}_r i}$: when $i \notin \mathcal{T}$ and when $i \in \mathcal{T}$. When $i \notin \mathcal{T}$, following the similar reasoning for $G_{\mathcal{Z}_r j}$, we have $G_{\mathcal{Z}_r i} = \bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}\bar{G}_{\mathcal{T}i}$ provided **condition 2, 5 and 6 in Property 2** are satisfied. When $i \in \mathcal{T}$, then $i \in \mathcal{Q}$. Now in $\bar{G}_{\mathcal{T}i}$ column matrix we have an element $\bar{G}_{ii} = (I - \breve{G}_{ii})^{-1}(\breve{G}_{ii} - \mathrm{diag}(\breve{G}_{ii}))$. When **condition 3 in Property 2** is satisfied, from (68) we have $\bar{R}_{ii} = (1 - \breve{G}_{ii})^{-1}$. Therefore $(I - \breve{G}_{ii})^{-1}\mathrm{diag}(\breve{G}_{ii}) = \bar{R}_{ii}(1 - \bar{R}^{-1}_{ii})$. Let C_{ii} be a column matrix with every element as zero except the element corresponding to node w_i which is $\bar{R}_{ii}(1 - \bar{R}^{-1}_{ii})$. Therefore, $\bar{G}_{\mathcal{T}i} = (I - \mathrm{diag}(\breve{G}_{\mathcal{T}\mathcal{T}}))^{-1}\breve{G}_{\mathcal{T}\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r i} - C_{ii} = \bar{R}_{\mathcal{T}\mathcal{Z}_r}G_{\mathcal{Z}_r i} - C_{ii}$. This gives us $\breve{G}_{j\mathcal{Z}_r}(I - \breve{G}_{\mathcal{Z}_r\mathcal{Z}_r})^{-1}G_{\mathcal{Z}_r i} = \breve{R}^{-1}_{jj}\bar{R}_{j\mathcal{Z}_r}G_{\mathcal{Z}_r i} = \breve{R}^{-1}_{jj}\bar{R}_{j\mathcal{Z}_r}\bar{R}^{\dagger}_{\mathcal{T}\mathcal{Z}_r}(\bar{G}_{\mathcal{T}i} + C_{ii})$ □

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