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Excitation allocation for generic identifiability of a single module in dynamic networks: A graphic approach ⋆

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Abstract: For identifiability of a single module in a dynamic network, excitation signals need to be allocated at particular nodes in the network. Current techniques provide analysis tools for verifying identifiability in a given situation, but hardly address the synthesis question: where to allocate the excitation signals in case of a particular parameterized network model set. Starting from the graph topology of the considered network model set, a new analytic result for generic identifiability of a single module is derived, based on the concept of *disconnecting sets*. For the situation that there is no restriction on measurability of node signals, the vertices in a particular disconnecting set provide the potential locations to allocate the excitation signals. Synthesis approaches are then developed to allocate excitation signals to guarantee generic identifiability.

Keywords: System identification, identifiability, dynamic networks, graph theory

1. INTRODUCTION

Due to the increasing complexity of current technological systems, the study of large-scale interconnected dynamic systems receives considerable attention recently. As a modeling framework for dynamic networks, we consider the network of transfer functions introduced in (Van den Hof et al., 2013; Gonçalves and Warnick, 2008), where vertices represent the internal signals, and directed edges denote transfer functions which are called modules. Identification problems in this setup involve multiple aspects, including estimation of one local module (Van den Hof et al., 2013; Ramaswamy et al., 2018), estimation of the topology (Materassi and Innocenti, 2010; Shi et al., 2019), estimation of the full network model (Weerts et al., 2018b) and identifiability of the network models (Weerts et al., 2018a; Hendrickx et al., 2019; van Waarde et al., 2018).

The analysis of identifiability of dynamic networks aims for conditions under which network models in a parameterized model set can be distinguished based on data, which provides guidelines for users to choose an appropriate model set such that the identification problem has a unique solution. Two notions of identifiability have been addressed in the literature, including *global identifiability* (Weerts et al., 2018a; van Waarde et al., 2018), which requires models to be distinguishable from all other models in the model set, and generic identifiability (Hendrickx et al., 2019; Weerts et al., 2018c; Cheng et al., 2019), which means that models can be distinguished from *almost all* models in the set. In addition, two different settings are considered for identifiability study, in one setting, all node signals are measured and a subset of nodes is excited (Weerts et al., 2018a), while in the other setting, a subset of node signals is measured and all nodes are excited (Hendrickx et al., 2019; Bazanella et al., 2017). In this work, we focus on generic identifiability of a particular module in the network, while there are no restrictions on the measurability of node signals.

Analysis results for generic identifiability of a single module can be found in (Weerts et al., 2018c), where both algebraic and graphical conditions have been developed. However, the results do not provide a structured method to allocate excitation signals for generic identifiability. In this work, given a network model set and assuming users have the freedom to assign excitation signals, the research question is how to allocate the excitation signals such that one single parameterized module of a network becomes generically identifiable. A related synthesis question for generic identifiability of a full network is addressed in (Cheng et al., 2019).

In this work, a new analytic result for generic identifiability of a single module is developed using the concept of disconnecting sets, and the vertices in the disconnecting set provide the potential locations for allocating the excitation signals. Then synthesis approaches are developed to allocate excitation signals at vertices in the disconnecting such that generic identifiability of a single module is guaranteed.

This paper is organized as follows. The network model, generic identifiability and the existing analytic results are introduced in Section 2. A new analytic result based on disconnect sets is developed in Section 3, which leads to several synthesis approaches in Section 4. The paper is concluded in Section 5. All the proofs are collected in Appendix.

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2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Dynamic networks and graphical representation

A dynamic network encompasses the relationship among scalar-valued *internal signals* w_j, deterministic excitation signals r_k which can be manipulated by the user, and unmeasured disturbances e_l, where $j \in \{1, \cdots, L\}$, $k \in \{1, \cdots, K\}$, $l \in \{1, \cdots, p\}$ and $p \leqslant L$. Following Van den Hof et al. (2013), the interconnection among the signals is modeled using linear time-invariant systems, and for any j, the equation of w_j is written as

$$w_j(t) = \sum_{j \neq i} G_{ji}(q) w_i(t) + \sum_k R_{jk}(q) r_k(t) + \sum_l H_{jl}(q) e_l(t), \tag{1}$$

where q is the delay operator, i.e. $q^{-1} w_j(t) = w_j(t-1)$, and its matrix form is obtained as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t),$$

where $G(q)$ is hollow, $w(t) = [w_1(t), \cdots, w_L(t)]^\top$, $r(t) = [r_1(t), \cdots, r_K(t)]^\top$, and $e(t) = [e_1(t), \cdots, e_p(t)]^\top$. Furthermore, model (1) satisfies the following assumptions:

Assumption 1

- $G_{ji}(q)$ is a stable and proper rational transfer operator if nonzero;
- $R_{ji}(q)$ is a stable and proper rational transfer operator;
- $(I - G(q))^{-1}$ is stable and proper;
- $H(q)$ is monic, proper and minimum-phase when $p = L$; When $p < L$, i.e., rank-reduced noises, $H(q)$ is structured as $H(q) = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$, with H_a square, proper, monic, stable and minimum phase, see (Weerts et al., 2018a) for more details;
- $e(t)$ is a vector of white noises with the covariance matrix $\Lambda > 0$;
- $w(t)$ and $r(t)$ can be measured.

We further define $U(q) \triangleq [R(q) \quad H(q)]$ and call both excitation and noise signals *external signals*. We use *modules* to refer to the transfer functions in $G(q)$. Besides the algebraic representation of a dynamic network, we will denote its graphical representation as follows.

Definition 1. Let $\mathcal{U} = \{r_1, \cdots, r_K, e_1, \cdots, e_p\}$ and $\mathcal{W} = \{w_1, \cdots, w_L\}$. The graphical representation of model (1) is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{W} \cup \mathcal{U}$ is a set of vertices representing the internal signals and external signals, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes a set of directed edges such that $(w_i, w_j), (r_k, w_j), (e_l, w_j) \in \mathcal{E}$ if and only if $G_{ji} \neq 0$, $R_{kj} \neq 0$, $H_{jp} \neq 0$, respectively, for all w_i, w_j, r_k, e_l in $\mathcal{V}$.

For any directed edge $(w_i, w_j) \in \mathcal{E}$, w_i is called an in-neighbor of w_j, and w_j is called an out-neighbor of w_i. For any vertex in $\mathcal{V}$, e.g. w_i, $\mathcal{N}^+_{w_i}$ and $\mathcal{N}^-_{w_i}$ denote the set of all out-neighbors and the set of all in-neighbors of w_i, respectively. A (directed) *walk* in $\mathcal{G}$ from one vertex to another, e.g. from w_i to w_j, is a sequence of vertices and out-going edges starting from w_i to w_j, while a (directed) *path* from w_i to w_j is a walk without repeating any vertex. The *length* of a directed path is the number of edges in the path, and a single vertex is regarded as a directed path to itself with length zero. In addition, we use $w_i \to w_j$ to denote a directed path from w_i to w_j, and $\mathcal{V}_1 \to \mathcal{V}_2$ denotes paths from some vertices in $\mathcal{V}_1$ to some vertices in $\mathcal{V}_2$. We refer to *internal vertices* as the vertices in a path excluding the starting and the ending vertices, and note that a directed path with length zero or length one does not have any internal vertex.

Two directed paths are *internally vertex disjoint* if they do not share any internal vertex, while they are called *vertex disjoint* if they do not share any vertex, including the starting and ending vertices. If two paths share any common vertex, we say that they *intersect*. A vertex set is also called vertex disjoint with a path set if they do not share common vertices. Given two subsets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, $b_{\mathcal{V}_1 \to \mathcal{V}_2}$ denotes the maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$. Note that a set of maximum vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ may not be unique.

2.2 Generic identifiability of a single module

Generic identifiability is a property of a network model set which is built on the parameterization of model (1) and a parameter space.

Definition 2. A network model set for a network of L internal variables, K excitation signals and p noise process is defined as

$$\mathcal{M} = \{(G(q,\theta), R(q,\theta), H(q,\theta)), \Lambda(\theta) | \theta \in \Theta \subseteq \mathbb{R}^n\},$$

with all elements in $\mathcal{M}$ satisfying Assumption 1 and additionally the following assumptions:

- *A subset of entries in $G(q,\theta)$, $R(q,\theta)$ and $H(q,\theta)$ is known and thus nonparameterized, e.g. $G_{ji}(q) = 0$;*
- *Any parameterized entry in $\mathcal{M}$ is an analytic function of θ;*
- *Parameterized entries in $G(q,\theta)$ cover all possible strictly proper and stable transfer functions;*
- *All parameterized transfer functions are parameterized independently.*

Similar to the graphical representation of model (1), a graphical representation $\mathcal{G}$ of $\mathcal{M}$ can be defined with the same set of vertices, while a directed edge exists if and only if the corresponding entry in the transfer matrices of $\mathcal{M}$ is parametrized or known and non-zero. In this way a network model set also induces a graph $\mathcal{G}$.

In addition, we call a property of a function of θ to hold generically if it holds for all $\theta \in \Theta$ except a set of Lebesgue measure zero. For simplicity of notation, we sometimes use G to denote $G(q,\theta)$, similarly for other functions of θ and q. We denote $T \triangleq (I-G)^{-1}U$, $T_{wr} \triangleq (I-G)^{-1}R$ and $T_{we} \triangleq (I-G)^{-1}H$, while $\Phi_{\bar{v}}(w)$ is the power spectrum of $T_{we}(q)e(t)$.

Following Weerts et al. (2018c), generic identifiability of module G_{ji} is defined as follows.

Definition 3. Module G_{ji} of model set $\mathcal{M}$ is generically identifiable if it holds that

$$\begin{cases} T_{wr}(q,\theta_1) = T_{wr}(q,\theta_2) \\ \Phi_{\bar{v}}(\omega,\theta_1) = \Phi_{\bar{v}}(\omega,\theta_2), \forall w \end{cases} \implies G_{ji}(q,\theta_1) = G_{ji}(q,\theta_2)$$

for almost all θ_1 and θ_2 in Θ.

Since T_{wr} and $\Phi_{\bar{v}}$ can typically be identified from data, identifiability refers to the situation that a unique module can be found based on T_{wr} and $\Phi_{\bar{v}}$. Definition 3 also has a simplified implication:

Lemma 1. (Weerts et al. (2018c)). *Let $\mathcal{M}$ satisfy one of the following conditions:*

- *all modules in $G(q,\theta)$ are strictly proper, or*
- *there is no algebraic loop*[1] *in $G(q,\theta)$ and $H^\infty(\theta)\Lambda(\theta)H^\infty(\theta)^T$ is diagonal for all θ, with $H^\infty(\theta) := \lim_{z\to\infty} H(z,\theta)$;*

then module G_{ji} of model set $\mathcal{M}$ is generically identifiable if it holds that

$$T(q,\theta_1) = T(q,\theta_2) \implies G_{ji}(q,\theta_1) = G_{ji}(q,\theta_2),$$

for almost all θ_1 and θ_2 in Θ.

In this work, we consider model set $\mathcal{M}$ to satisfy both the assumptions in Definition 2 and the ones in Lemma 1. In this setting, generic identifiability of a single module is related to the generic rank of certain submatrices of $T(q,\theta)$ and the number of vertex disjoint paths from external signals to internal signals (Weerts et al., 2018a,c). The graphical conditions in (Weerts et al., 2018c) are collected here:

Theorem 2. Given a network model set $\mathcal{M}$ and its graph $\mathcal{G}$, let $\mathcal{U}_j$ denote the set of external signals which do not appear as inputs of w_j in (1) through parameterized transfer functions, and let $\mathcal{W}_j$ denote the set of internal signals which are inputs of w_j through parameterized modules, the following conditions are equivalent:

(1) module G_{ji} of $\mathcal{M}$ is generically identifiable;
(2) $b_{\mathcal{U}_j\to\mathcal{W}_j} > b_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$;
(3) there exists a set $\mathcal{P}$ of maximum vertex disjoint paths from $\mathcal{U}_j$ to $\mathcal{W}_j\setminus\{w_i\}$ and also a directed path from $\mathcal{U}_j$ to w_i, such that $\mathcal{P}$ and the directed path are vertex disjoint.

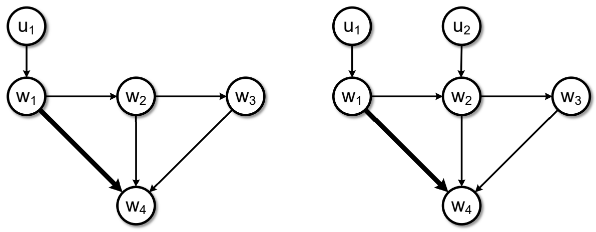
Note that $\mathcal{U}_j$ contains any external signal u_k that is no direct input to w_j with $R_{jk} = 0$ or $H_{jk} = 0$. Related graphic results can be found in (Hendrickx et al., 2019) for a different setting where all internal signals are excited and a subset of them is measured. For that same situation, a graphic simplification procedure is developed in (van Waarde et al., 2018) for testing global identifiability of a SIMO subsystem and a full network.

Theorem 2 is demonstrated in the following example.

Example 1. Given the network model set in Fig. 1(a), where the target module is G_{41} and all indicated transfer functions are parameterized, we have $\mathcal{W}_4 = \{w_1, w_2, w_3\}$ and $\mathcal{U}_4 = \{u_1\}$ such that $b_{\mathcal{U}_4\to\mathcal{W}_4} = 1$ consisting of one path $u_1 \to w_1$. However, since $b_{\mathcal{U}_4\to\mathcal{W}_4\setminus\{w_1\}}$ also consists of one path $u_1 \to w_2$ and thus $b_{\mathcal{U}_4\to\mathcal{W}_4\setminus\{w_1\}} = b_{\mathcal{U}_4\to\mathcal{W}_4} = 1$, G_{41} is not generically identifiable based on Theorem 2.

The above result allows to analyse generic identifiability for a given situation. However it does not address the synthesis problem: Given a network model set, how to allocate external signals such that module G_{ji} becomes generically identifiable? In this work, we assume that the user can allocate excitation signals such that a given model set can be modified. In the next sections, new analytic results are derived, which lead to several synthesis approaches.

[1] There exists an algebraic loop around node w_{n_1} if there exists a sequence of integers $n_1, \cdots n_k$ such that $G^\infty_{n_1n_2}G^\infty_{n_2n_3}\cdots G^\infty_{n_kn_1} \neq 0$, with $G^\infty_{n_1n_2} := \lim_{z\to\infty} G_{n_1n_2}(z)$.



(a) (b)

Fig. 1. Generic identifiability of G_{41} is considered (thick line). G_{41} is not generically identifiable in (a) but becomes generically identifiable in (b) if an extra signal u_2 is allocated at w_2.

Note that even if we consider the setting where all internal and excitation signals are available to be measured, generic identifiability of a single module does not require to measure all the signals. The necessary measured signals vary with the specific module of interest and the identification method.

3. DISCONNECTING SETS FOR GENERIC IDENTIFIABILITY

In this section, we derive a new analytic result based on the concept of disconnecting sets, which shows to be more suitable for synthesis. A vertex set $\mathcal{C}$ is a disconnecting set from a vertex set $\mathcal{V}_1$ to a set $\mathcal{V}_2$ if there is no directed path from $\mathcal{V}_1$ to $\mathcal{V}_2$ when the vertices in $\mathcal{C}$ are removed (Schrijver, 2003). In this case, we say that $\mathcal{V}_2$ is disconnected from $\mathcal{V}_1$ by $\mathcal{C}$. Note that a disconnecting set from $\mathcal{V}_1$ to $\mathcal{V}_2$ may also include vertices in $\mathcal{V}_1\cup\mathcal{V}_2$. We call a disconnecting set with the minimum cardinality a *minimum disconnecting set.* The duality between vertex disjoint paths and disconnecting sets is explained in Menger's theorem, which is also explored in (Hendrickx et al., 2019).

Theorem 3. (Menger's theorem (Schrijver, 2003)). *Let $\mathcal{V}_1$, $\mathcal{V}_2$ be two subsets of the vertices in a directed graph. The maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ equals the cardinality of a minimum disconnecting set from $\mathcal{V}_1$ to $\mathcal{V}_2$.*

We illustrate the relevance of the concept of disconnecting sets to our synthesis problem in the following example.

Example 2. Given the network model in Fig. 1(a), $\{w_2\}$ is a disconnecting set from $\{w_1\}$ to the other in-neighbors of w_4, i.e., $\{w_2, w_3\}$. Now, we allocate an extra excitation signal u_2 at w_2, as shown in Fig. 1(b), and we find that $b_{\mathcal{U}_4\to\mathcal{W}_4} > b_{\mathcal{U}_4\to\mathcal{W}_4\setminus\{w_1\}}$, which implies that G_{41} is generically identifiable according to Theorem 2. ∎

In Example 2, generic identifiability of G_{ji} is achieved when the vertices in a disconnecting set from $\{w_i\}$ to the other in-neighbors of w_j are excited. In the following, we prove that this result holds for the general case. Before proceeding, a graphical result is provided.

Lemma 4. *In a simple directed graph, given a set $\mathcal{P}$ of vertex disjoint paths from vertex set $\mathcal{V}_1$ to a vertex set $\mathcal{V}_2$, there exists a set $\mathcal{P}_{new}$ of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ such that $|\mathcal{P}_{new}| = |\mathcal{P}|$ and paths in $\mathcal{P}_{new}$ are internally vertex disjoint[2] with $\mathcal{V}$, where $\mathcal{V}$ can be $\mathcal{V}_1$, $\mathcal{V}_2$ or $\mathcal{V}_1 \cup \mathcal{V}_2$.*

The above graphical result will be used to derive the synthesis approach in Algorithm 1, as we will see later. Moreover, based on Lemma 4, a new analytic result for generic identifiability of a single module can be derived.

Theorem 5. *Given a network model set $\mathcal{M}$ and its graph $\mathcal{G}$, module G_{ji} is generically identifiable in $\mathcal{M}$ if and only if there exists a set of external signals $\bar{\mathcal{U}}_j \subseteq \mathcal{U}_j$ and a disconnecting set $\mathcal{W}_d$ from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ such that*

$$b_{\bar{\mathcal{U}}_j \to \mathcal{W}_d \cup \{w_i\}} = |\mathcal{W}_d| + 1. \qquad (2)$$

Note that a disconnecting set $\mathcal{W}_d$ from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ always exists, and one simple example is to let it be $\mathcal{W}_j \setminus \{w_i\}$. However, generic identifiability of G_{ji} requires the existence of a special disconnecting set which satisfies equation (2). In addition, note that $\mathcal{W}_d$ and $\mathcal{W}_j \setminus \{w_i\}$ are not necessarily disjoint, while $\{w_i\}$ and $\mathcal{W}_d$ have to be *disjoint* to satisfy equation (2).

Theorem 5 can be visualized in Fig. 2, where the paths from w_i and $\bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ intersect with $\mathcal{W}_d$. Due to condition (2), we can find a set of vertex disjoint paths consisting of a path $\bar{\mathcal{U}}_j \to w_i$ and multiple paths $\bar{\mathcal{U}}_j \to \mathcal{W}_d$. As shown in Fig. 2, we can further concatenate the paths $\bar{\mathcal{U}}_j \to \mathcal{W}_d$ and the paths $\mathcal{W}_d \to \mathcal{W}_j \setminus \{w_i\}$, which leads to a set of paths $\mathcal{P}$ from $\bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, that are vertex disjoint with the path $\bar{\mathcal{U}}_j \to w_i$. Then based on condition (3) of Theorem 2, G_{ji} is generically identifiable. Note that as shown in the proof of Theorem 5, condition (3) of Theorem 2 still holds if we replace $\mathcal{U}_j$ by $\bar{\mathcal{U}}_j$.

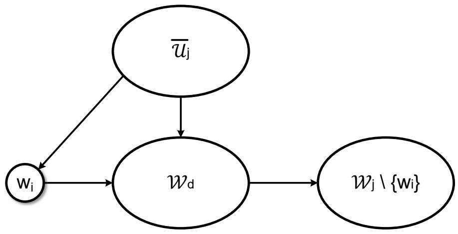


Fig. 2. A graphical visualization of Theorem 5.

To be consistent with Theorem 2 where $\mathcal{U}_j$ is used instead of $\bar{\mathcal{U}}_j$, the following result is immediate from Theorem 5.

Corollary 6. *Given a network model set $\mathcal{M}$ with its graph $\mathcal{G}$, module G_{ji} is generically identifiable in $\mathcal{M}$ if and only if there exists a disconnecting set $\mathcal{W}_d$ from $\{w_i\} \cup \mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ such that*

$$b_{\mathcal{U}_j \to \mathcal{W}_d \cup \{w_i\}} = |\mathcal{W}_d| + 1. \qquad (3)$$

Note that the disconnecting set in Corollary 6 can be different from the one in Theorem 5. Based on the above analytic results, we proceed to develop synthesis approaches.

[2] A path is internally vertex disjoint with a set of vertices $\mathcal{V}$, if the internal vertices of the path are not in $\mathcal{V}$.

4. SYNTHESIS APPROACH

Given a network model set with a set of initial external signals $\mathcal{U}_j^0$ for module G_{ji}, the synthesis problem aims to allocate a minimum number of additional excitation signals $\mathcal{U}_j^a$ such that generic identifiability of a module G_{ji} is guaranteed. We will consider the synthesis problem in two different settings, including one setting where $\mathcal{U}_j^0 = \emptyset$ and the other one where $\mathcal{U}_j^0 \neq \emptyset$. For synthesis, it is assumed that the transfer function R_{jk} of any additional excitation signal r_k for allocation is known and thus nonparameterized. This includes the case where r_k is allocated at another vertex rather than output w_j with $R_{jk} = 0$, and the case where r_k directly influences w_j with nonparameterized R_{jk}.

As shown in the last section, one of the key steps to solve the synthesis problem is the computation of a disconnecting set $\mathcal{W}_d$ as defined in Theorem 5. In the setting where $\mathcal{U}_j^0 = \emptyset$, one simple approach to compute $\mathcal{W}_d$ is based on the concepts of parallel paths and cycles around the output[3], introduced in (Dankers et al., 2015).

Proposition 7. *Consider a model set $\mathcal{M}$ with its graph $\mathcal{G}$, and let G_{ji} be the target module to identify. Construct a set $\mathcal{W}_d$ as follows:*

- *For any parallel path that enters w_j through a parametrized module, we select one internal vertex of the path into $\mathcal{W}_d$;*
- *For any cycle around w_j that enters w_j through a parametrized module, we select one internal vertex of the cycle into $\mathcal{W}_d$.*

Then $\mathcal{W}_d$ is a disconnecting set from $\{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$, and assigning distinct excitation signals to each vertex in $\mathcal{W}_d \cup \{w_i\}$ leads to generic identifiability of G_{ji} in $\mathcal{M}$.

In the above approach, equation (3) is satisfied with the disconnecting set $\mathcal{W}_d$ and the allocated signals $\mathcal{U}_j \triangleq \mathcal{U}_j^a$. The advantage of the approach is that the parallel paths and cycles around the output can be easily determined by inspection of the user, as demonstrated in (Dankers et al., 2015; Ramaswamy and Van den Hof, 2019) where the graphic tools are used to select predictors in a network for identification of a single module. However, this approach does not necessarily provide a minimum solution for the number of excitation signals.

To reduce the number of allocated excitation signals, a minimum disconnecting set should be used for synthesis. Recall that w_i should be excluded from $\mathcal{W}_d$ to satisfy equation (2), and thus a minimum disconnecting set $\mathcal{W}_d$ with the constraint that $w_i \notin \mathcal{W}_d$ needs to be found. As standard graphical algorithms for computing minimum disconnecting sets do not take into account any constraint, we redefine the disconnecting set so as to make standard algorithms applicable.

Proposition 8. *Given a network model set $\mathcal{M}$ and its graph $\mathcal{G}$, for any subset $\bar{\mathcal{U}}_j \subseteq \mathcal{U}_j$, a minimum disconnecting set from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ is a minimum disconnecting set $\mathcal{W}_d$ from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ subject to $w_i \notin \mathcal{W}_d$.*

[3] A parallel path is a directed path $w_i \to w_j$ which does not contain the edge (w_i, w_j), and a cycle around w_j is a directed walk starting and ending at w_j with distinct internal vertices.

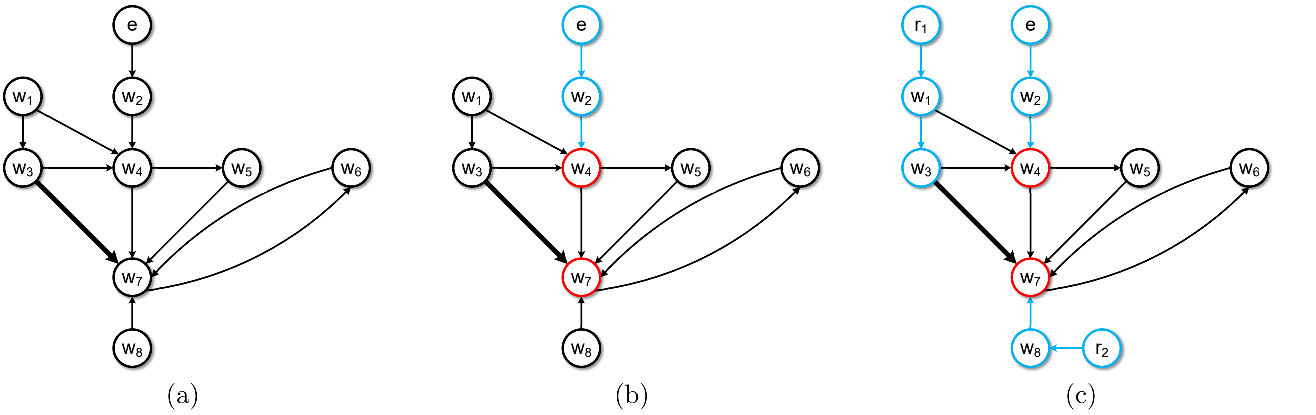


Fig. 3. An example of allocating signals for generic identifiability of G_{73} (thick line) using Algorithm 1. Starting from the network model in (a), a disconnecting set (red vertices) is computed in (b). Since there already exists an external signal e, which has a path to w_4, a vertex in the disconnecting set, we only need to add r_1 and r_2 as in (c), which achieves generic identifiability of G_{73}.

Based on the above proposition, a minimum disconnecting set $\mathcal{W}_d$ from $\mathcal{N}^+_{w_i} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ can be computed for the synthesis problem, which is an unconstrained problem and thus can be solved by standard graphic algorithms. Then to improve the approach in Proposition 7, the following result can be derived from Theorem 5.

Corollary 9. In a network model set $\mathcal{M}$ with its graph $\mathcal{G}$, given any minimum disconnecting set $\mathcal{W}_d$ from $\mathcal{N}^+_{w_i}$ to $\mathcal{W}_j \setminus \{w_i\}$, assigning distinct excitation signals to every vertex in $\mathcal{W}_d \cup \{w_i\}$ leads to generic identifiability of G_{ji} in $\mathcal{M}$.

Compared to Proposition 7, the method in Corollary 9 achieves the minimum number of excitation signals for the situation where $\mathcal{U}^0_j = \emptyset$, because a minimum disconnecting set is used.

The methods of allocating excitation signals in Corollary 9 and Proposition 7 are strictly local in the sense that all excitation signals are directly allocated at w_i and $\mathcal{W}_d$. If $\mathcal{U}^0_j \neq \emptyset$, the external signals that are initially present in $\mathcal{U}^0_j$ are not taken into account by the two methods, while $\mathcal{U}^0_j$ may affect generic identifiability of G_{ji} and reduce the number of additional excitation signals that are actually required. Therefore, for the setting $\mathcal{U}^0_j \neq \emptyset$, we present a more comprehensive synthesis approach in Algorithm 1 which makes use of the initially present external signals in $\mathcal{U}^0_j$ and explores the freedom of allocating additional excitation signals in the graph.

Given a network model set with a target module G_{ji} and a set of pre-existing signals $\mathcal{U}^0_j$, Algorithm 1 starts with a minimum disconnecting set $\mathcal{W}_d$ from $\{w_i\} \cup \mathcal{U}^0_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and then the target is to add a minimum number of excitation signals $\mathcal{U}^a_j$ such that equation (3) is satisfied with $\mathcal{U}_j \triangleq \mathcal{U}^0_j \cup \mathcal{U}^a_j$. Firstly, Algorithm 1 removes the elements in $\mathcal{W}_d \cup \{w_i\}$ which are already excited by the existing external signals through vertex disjoint paths. Then it allocates additional signals to excite the remaining elements in $\mathcal{W}_d \cup \{w_i\}$ through vertex disjoint paths, while $\mathcal{W}_d$ remains a disconnecting set from the added signals to $\mathcal{W}_j \setminus \{w_i\}$. Note that finding maximum vertex disjoint paths and a minimum disconnecting set in steps 1, 2 and 7 can be conducted by standard graphic algorithms (Schrijver, 2003), whose solutions are generally not unique. Thus, the obtained result from Algorithm 1 may not be unique as well. The correctness of Algorithm 1 is proved in Corollary 10.

Corollary 10. In the returned $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$ from Algorithm 1, the target module G_{ji} is generically identifiable.

We demonstrate the scheme of Algorithm 1 by the following example.

Example 3. In the network model in Fig. 3(a), the problem is to allocate excitation signals such that G_{73}, indicated by the thick arrow, becomes generically identifiable. $\mathcal{U}^0_7 = \{e\}$ is the only external signal that is initially present. Firstly, a disconnecting set from $\mathcal{U}^0_7 \cup \{w_3\} = \{w_3, e\}$ to the other in-neighbors of w_7, i.e. $\mathcal{W}_7 \setminus \{w_3\} = \{w_4, w_5, w_6\}$, is constructed as $\mathcal{W}_d = \{w_4, w_7\}$, indicated by the red vertices in Fig. 3(b). Based on Theorem 5, generic identifiability of G_{73} requires that there are three vertex disjoint paths from external signals to $\mathcal{W}_d \cup \{w_3\} = \{w_3, w_4, w_7\}$, while $\mathcal{W}_d$ remains a disconnecting set from the external signals to $\mathcal{W}_7 \setminus \{w_3\}$. Following step 2 in Algorithm 1, we find a path $e \to w_4$ from $\mathcal{U}^0_7$ to $\mathcal{W}_d \cup \{w_3\}$ while the path is internally vertex disjoint with $\mathcal{W}_d \cup \{w_3\}$ and is colored blue in Fig. 3(b). Thus we only need to allocate extra excitation signals to create two vertex disjoint paths to $\{w_3, w_7\}$, which are vertex disjoint with $e \to w_4$. As in step 5, the potential locations to allocate excitation signals is $\bar{\mathcal{W}} = \{w_1, w_2, w_3, w_4, w_7, w_8\}$, which satisfies that $\mathcal{W}_d$ remains a disconnecting set from $\bar{\mathcal{W}}$ to $\mathcal{W}_7 \setminus \{w_3\}$. After removing $e \to w_4$, we choose $\mathcal{W}_{exp} = \{w_1, w_8\} \subseteq \bar{\mathcal{W}}$ to be excited. It can be verified that there are two vertex disjoint paths from $\mathcal{W}_{exp}$ to $\{w_3, w_7\}$, and the paths are also vertex disjoint with $e \to w_4$, as indicated by the blue paths in Fig. 3(c). In this situation, G_{73} is generically identifiable. ■

Note that at step 2 of Algorithm 1, if the internal vertex disjointness is not required using Lemma 4, then in the above example, the path $e \to w_7$ in Fig. 3(b) instead of $e \to w_4$ can be chosen at step 2. However, since $e \to w_7$ also contains vertex w_4, both w_7 and w_4 will be removed at step 6, which incorrectly removes two elements in $\mathcal{W}_d \cup \{w_3\}$ while there is actually only one vertex disjoint path from $\mathcal{U}_7^0 = \{e\}$ to $\mathcal{W}_d \cup \{w_3\}$.

Even if Algorithm 1 requires less signals than the approaches in Proposition 7 and Corollary 9, there is not yet guarantee for a minimal number of additional signals when $\mathcal{U}_j^0 \neq \emptyset$. However, since the number of allocated signals in Algorithm 1 is $|\mathcal{W}_d| + 1 - b_{\mathcal{U}_j^0 \to \mathcal{W}_d \cup \{w_i\}}$, the following bound on the number of additional signals required for generic identifiability of a single module can be derived.

Corollary 11. Given a network model set $\mathcal{M}$ with its graph $\mathcal{G}$ and a set of initial external signals $\mathcal{U}_j^0$ for G_{ji}, let c denote the number of excitation signals available for allocation. If

$$c \geqslant |\mathcal{W}_d| + 1 - b_{\mathcal{U}_j^0 \to \mathcal{W}_d \cup \{w_i\}},$$

where $\mathcal{W}_d$ is a minimum disconnecting set from $\mathcal{N}_{w_i}^+ \cup \mathcal{U}_j^0$ to $\mathcal{W}_j \setminus \{w_i\}$, then there exist vertices in $\mathcal{G}$ where the excitation signals can be allocated such that G_{ji} is generically identifiable.

Algorithm 1 Signal allocation for a single module

INPUT: A network model set $\mathcal{M}$ with graph $\mathcal{G}$, a target parameterized module G_{ji}, and a set of initial external signals $\mathcal{U}_j^0$;

OUTPUT: A new model set $\mathcal{M}_{out}$ with its graph $\mathcal{G}_{out}$

1: Compute a minimum disconnecting set $\mathcal{W}_d$ from $\mathcal{N}_{w_i}^+ \cup \mathcal{U}_j^0$ to $\mathcal{W}_j \setminus \{w_i\}$;
2: Based on Lemma 4, compute a set $\mathcal{P}$ of maximum vertex disjoint paths from $\mathcal{U}_j^0$ to $\mathcal{W}_d \cup \{w_i\}$ while the paths are internally vertex disjoint with $\mathcal{W}_d \cup \{w_i\}$;
3: Find set $\bar{\mathcal{U}}_{j1} \subseteq \mathcal{U}_j^0$, which collects all the vertices that can reach $\mathcal{W}_d \cup \{w_i\}$ through distinct paths in $\mathcal{P}$. Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$ be the set of these distinct paths and $\bar{\mathcal{W}}_d \subseteq \mathcal{W}_d \cup \{w_i\}$ be the set of vertices can be reached;
4: **if** $|\bar{\mathcal{U}}_{j1}| < |\mathcal{W}_d| + 1$ **then**
5: Find the set $\bar{\mathcal{W}} \subseteq \mathcal{W}$ such that $\mathcal{W}_d$ is a disconnecting set from $\bar{\mathcal{W}}$ to $\mathcal{W}_j \setminus \{w_i\}$;
6: Build a subgraph $\bar{\mathcal{G}} \subseteq \mathcal{G}$ by removing all vertices in $\bar{\mathcal{P}}$ and their corresponding edges;
7: Find the set $\mathcal{W}_{exp} \subseteq \bar{\mathcal{W}}$ such that there are $|\mathcal{W}_d| + 1 - |\bar{\mathcal{U}}_{j1}|$ vertex disjoint paths from $\mathcal{W}_{exp}$ to $\mathcal{W}_d \cup \{w_i\} \setminus \bar{\mathcal{W}}_d$ in $\bar{\mathcal{G}}$;
8: In $\mathcal{G}$, assign distinct excitation signals to every vertex in $\mathcal{W}_{exp}$, which leads to a new model set $\mathcal{M}_{out}$ with a new graph $\mathcal{G}_{out}$;
9: Return $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$;
10: **else**
11: $\mathcal{M}_{out} \leftarrow \mathcal{M}$ and $\mathcal{G}_{out} \leftarrow \mathcal{G}$;
12: Return $\mathcal{M}_{out}$ with the graph $\mathcal{G}_{out}$.
13: **end if**

Note that the presented methods can also handle the presence of nonparameterized modules. This will affect the construction of $\mathcal{W}_j$, defined as the set of in-neighbors of w_j that are input to a parametrized module that maps to w_j. The consequence of having non-parametrized modules is shown in the following example.

Example 4. Consider a network model set with the same topology as the one in Fig. 1, G_{43} and G_{42} are known and thus are nonparameterized in the model set, as shown in Fig. 4. Based on the formulation of $\mathcal{W}_4$ in Theorem 2, w_3 and w_2 do not belong to set $\mathcal{W}_4$ now, as we have $\mathcal{W}_4 = \{w_1\}$ in Fig. 4. Thus, in this example, the minimum disconnecting set will be an empty set, and generic identifiability of G_{41} can be achieved by a single excitation on w_1, compared to two excitation signals in Fig. 1(b). ■

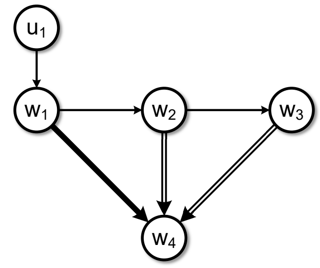


Fig. 4. Generic identifiability of G_{41} (thick line) with nonparameterized modules (double-line arrows) can be achieved by a single excitation signal u_1.

5. CONCLUSION

The synthesis problem to achieve generic identifiability of one module in a dynamic network by allocating excitation signals is addressed in this work. A new condition is derived based on the concepts of disconnecting set, and then synthesis approaches are developed to allocated excitation signals at the vertices in the disconnecting set. It has been shown that generic identifiability of a single module can be guaranteed by the synthesis approaches, and the situation with both nonparameterized modules and parameterized modules can also be handled.

REFERENCES

Bazanella, A.S., Gevers, M., Hendrickx, J.M., and Parraga, A. (2017). Identifiability of dynamical networks: which nodes need be measured? In *Proc. 56th IEEE Conf. Decis Control (CDC)*, 5870–5875. IEEE.

Cheng, X., Shi, S., and Van den Hof, P.M.J. (2019). Allocation of Excitation Signals for Generic Identifiability of Linear Dynamic Networks. *arXiv e-prints*, arXiv:1910.04525.

Dankers, A.G., Van den Hof, P.M.J., Bombois, X., and Heuberger, P.S.C. (2015). Identification of dynamic models in complex networks with prediction error methods: Predictor input selection. *IEEE Trans. Autom. Control*, 61(4), 937–952.

Gonçalves, J. and Warnick, S. (2008). Necessary and sufficient conditions for dynamical structure reconstruction of LTI networks. *IEEE Trans. Automatic Control*, 53(7), 1670–1674.

Hendrickx, J.M., Gevers, M., and Bazanella, A.S. (2019). Identifiability of dynamical networks with partial node measurements. *IEEE Trans. Autom. Control*, 64(6), 2240–2253.

Materassi, D. and Innocenti, G. (2010). Topological identification in networks of dynamical systems. *IEEE Trans. Autom. Control*, 55(8), 1860–1871.

Ramaswamy, K.R., Bottegal, G., and Van den Hof, P.M.J. (2018). Local module identification in dynamic networks using regularized kernel-based methods. In *Proc. 57th IEEE Conf. Decis Control (CDC)*, 4713–4718. IEEE.

Ramaswamy, K.R. and Van den Hof, P.M.J. (2019). A local direct method for module identification in dynamic networks with correlated noise. *arXiv preprint arXiv:1908.00976*.

Schrijver, A. (2003). Disjoint paths. In *Combinatorial optimization: polyhedra and efficiency*, 131–132. Springer Science & Business Media.

Shi, S., Bottegal, G., and Van den Hof, P.M.J. (2019). Bayesian topology identification of linear dynamic networks. In *2019 18th European Control Conference (ECC)*, 2814–2819.

Van den Hof, P.M.J., Dankers, A.G., Heuberger, P.S.C., and Bombois, X. (2013). Identification of dynamic models in complex networks with prediction error methods basic methods for consistent module estimates. *Automatica*, 49(10), 2994–3006.

van Waarde, H.J., Tesi, P., and Camlibel, M.K. (2018). Necessary and sufficient topological conditions for identifiability of dynamical networks. *arXiv preprint arXiv:1807.09141*.

Weerts, H.H.M., Van den Hof, P.M.J., and Dankers, A.G. (2018a). Identifiability of linear dynamic networks. *Automatica*, 89, 247–258.

Weerts, H.H.M., Van den Hof, P.M.J., and Dankers, A.G. (2018b). Prediction error identification of linear dynamic networks with rank-reduced noise. *Automatica*, 98, 256–268.

Weerts, H.H.M., Van den Hof, P.M.J., and Dankers, A.G. (2018c). Single module identifiability in linear dynamic networks. In *Proc. 57th IEEE Conf. Decis Control (CDC)*, 4725–4730. IEEE.

Appendix A. PROOF OF LEMMA 4

Proof. We prove the lemma by showing that there always exists a $\mathcal{P}_{new}$ by modifying the paths in $\mathcal{P}$. Let $w_i \to w_j$ be an arbitrary path in $\mathcal{P}$ which contains an internal vertex in $\mathcal{V}$, then we can always replace $w_i \to w_j$ by its subpath which contains a starting vertex in $\mathcal{V}_1$ and an ending vertex in $\mathcal{V}_2$, while the other vertices in the subpath are not in $\mathcal{V}$. This includes the special case that the obtained subpath has no internal vertex. Applying the above modification to all the paths in $\mathcal{P}$ which contain internal vertices in $\mathcal{V}$, we obtain $\mathcal{P}_{new}$, with $|\mathcal{P}_{new}| = |\mathcal{P}|$, in which all the paths are still vertex disjoint.

Appendix B. PROOF OF THEOREM 5

We first introduce a new notation. Given two subsets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ in a graph, $\mathcal{B}_{\mathcal{V}_1\to\mathcal{V}_2}$ denotes a set of maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$.

Proof. To prove the "if" part, we show that if there exists $\bar{\mathcal{U}}_j$ and $\mathcal{W}_d$ such that equation (2) holds, then G_{ji} is generically identifiable based on condition (3) in Theorem 2, i.e. there exists a directed path from $\mathcal{U}_j$ to w_i and a set of maximum vertex disjoint paths $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$, which is also vertex disjoint with the path from $\mathcal{U}_j$ to w_i.

Equation (2) implies that there is a path $u_i \to w_i$ and a set of vertex disjoint paths from $\bar{\mathcal{U}}_j \setminus \{u_i\}$ to $\mathcal{W}_d$, denoted by $\mathcal{P}_0$, while $\mathcal{P}_0$ is also vertex disjoint with $u_i \to w_i$. Given the path $u_i \to w_i$, to prove condition (3) in Theorem 2, we only need to show that there exists a set $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$ which is vertex disjoint with $u_i \to w_i$. In the following, we find $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$ based on $\mathcal{P}_0$. We will first introduce a new set of paths $\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}}$ where $\widetilde{\mathcal{U}}_j \triangleq \mathcal{U}_j \setminus \bar{\mathcal{U}}_j$, and then the paths in $\mathcal{P}_0$ and the paths in $\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}}$ will be concatenated such that the target $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$ is obtained. Note that based on Lemma 4, $\mathcal{P}_0$ is chosen such that its paths $\bar{\mathcal{U}}_j \setminus \{u_i\} \to \mathcal{W}_d$ are internally vertex disjoint with $\mathcal{W}_d$.

Based on Lemma 4, $\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}}$ is chosen such that all paths in $\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}}$ are internally vertex disjoint with $\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j$ and $\mathcal{W}_j \setminus \{w_i\}$. Let $\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}}$ be divided into two disjoint subsets, i.e.

$$\mathcal{B}_{\mathcal{W}_d\cup\widetilde{\mathcal{U}}_j\to\mathcal{W}_j\setminus\{w_i\}} = \mathcal{P}_{1a} \cup \mathcal{P}_{1b}, \tag{B.1}$$

where

- $\mathcal{P}_{1a}$ contains all paths from $\mathcal{W}_d$ to $\mathcal{W}_j \setminus \{w_i\}$, and let $\mathcal{W}_{j1}$ be the set of all the ending vertices of the paths;
- $\mathcal{P}_{1b}$ contains all paths from $\widetilde{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and let $\mathcal{W}_{j2}$ be the set of all the ending vertices of the paths in $\mathcal{P}_{1b}$.

Then to construct $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$, a subset $\bar{\mathcal{P}}_0 \subseteq \mathcal{P}_0$ and all paths in $\mathcal{P}_{1a}$ can be concatenated, which leads to a set of walks $\mathcal{P}_2$ from $\bar{\mathcal{U}}_j \setminus \{u_i\}$ via $\mathcal{W}_d$ to $\mathcal{W}_{j1}$, with $|\bar{\mathcal{P}}_0| = |\mathcal{P}_2| = |\mathcal{P}_{1a}|$. Note that the concatenation is always feasible because based on equation (2), every vertex in $\mathcal{W}_d$ belongs to a distinct path in $\mathcal{P}_0$. Next, we claim that $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is the desired $\mathcal{B}_{\mathcal{U}_j\to\mathcal{W}_j\setminus\{w_i\}}$. The claim is true if 1) $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is a set of maximum vertex disjoint paths from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$; 2) $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is vertex disjoint with $u_i \to w_i$. In the following, we show the proof for the two statements.

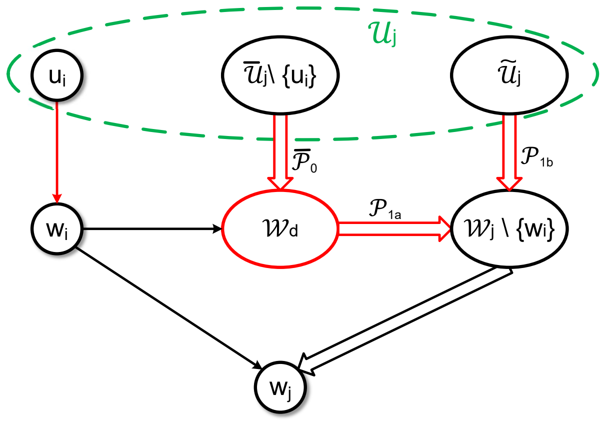


Fig. B.1. Relevant sets of vertices and sets of paths defined in the proof of Theorem 5.

The statement 1) is shown as follows. (i) Firstly, we need to show that the walks in $\mathcal{P}_2$ from concatenation are valid directed paths and thus do not have repeated vertices, i.e. all the paths $\bar{\mathcal{U}}_j \setminus \{u_i\} \to \mathcal{W}_d$ in $\bar{\mathcal{P}}_0$ and all the paths $\mathcal{W}_d \to \mathcal{W}_{j1}$ in $\mathcal{P}_{1a}$ are vertex disjoint except at the vertices in $\mathcal{W}_d$. Assuming that they are not vertex disjoint, a directed path from $\bar{\mathcal{U}}_j \setminus \{u_i\}$ to $\mathcal{W}_{j1}$ will exist without intersecting with $\mathcal{W}_d$, which contradicts the condition that $\mathcal{W}_d$ is a disconnecting set and thus proves (i); (ii) Secondly, we show that $\mathcal{P}_2$ and $\mathcal{P}_{1b}$ are vertex disjoint, which suffices to show that $\mathcal{P}_{1a}$ and $\bar{\mathcal{P}}_0$ are vertex disjoint with $\mathcal{P}_{1b}$. It is clear that $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$ are vertex disjoint due to (B.1). Furthermore, $\bar{\mathcal{P}}_0$ and $\mathcal{P}_{1b}$ are also vertex disjoint, because if they are vertex joint, a directed path from $\bar{\mathcal{U}}_j \setminus \{u_i\}$ to $\mathcal{W}_{j2}$ exists without intersecting $\mathcal{W}_d$, which contradicts the fact that $\mathcal{W}_d$ is a disconnecting set. (iii) Finally, we prove that $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| = b_{\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Since $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ contains a set of vertex disjoint paths from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ based on (ii), we have $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| \leqslant b_{\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Recall that $|\mathcal{P}_2 \cup \mathcal{P}_{1b}| = |\mathcal{P}_{1a} \cup \mathcal{P}_{1b}| = b_{\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j \to \mathcal{W}_j \setminus \{w_i\}}$ in equation (B.1), and thus we can prove (iii) by showing the following inequality:

$$b_{\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j \to \mathcal{W}_j \setminus \{w_i\}} \geqslant b_{\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}}. \tag{B.2}$$

Let $\mathcal{C}$ be a minimum disconnecting set from $\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and based on Menger's theorem, it holds that $|\mathcal{C}| = b_{\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Since all directed paths from $\bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ intersect with $\mathcal{W}_d$, and $\mathcal{C}$ intersects with all paths from $\mathcal{W}_d$ to $\mathcal{W}_j \setminus \{w_i\}$, $\mathcal{C}$ is also a disconnecting set from $\bar{\mathcal{U}}_j \cup \widetilde{\mathcal{U}}_j = \mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Then $|\mathcal{C}| \geqslant b_{\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}}$ as $\mathcal{C}$ is not necessarily a minimum disconnecting set from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and thus equation (B.2) is also proved due to $|\mathcal{C}| = b_{\mathcal{W}_d \cup \widetilde{\mathcal{U}}_j \to \mathcal{W}_j \setminus \{w_i\}}$. Consequently, we conclude that $\mathcal{P}_2 \cup \mathcal{P}_{1b} = \mathcal{B}_{\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}}$, i.e. $\mathcal{P}_2 \cup \mathcal{P}_{1b}$ is a set of maximum vertex disjoint paths from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$.

We further show the statement 2), which is sufficient to show that $\bar{\mathcal{P}}_0$, $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$ are vertex disjoint with $u_i \to w_i$. Recall that $u_i \to w_i$ is vertex disjoint with $\bar{\mathcal{P}}_0$ based on equation (2). $u_i \to w_i$ is also vertex disjoint with $\mathcal{P}_{1a}$ and $\mathcal{P}_{1b}$, because if they are joint, there exists a directed path from u_i to $\mathcal{W}_j \setminus \{w_i\}$ or a directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$ without intersecting with $\mathcal{W}_d$, which contradicts the condition that $\mathcal{W}_d$ is a disconnecting set. Thus, the $u_i \to w_i$ is vertex disjoint with $\mathcal{P}_2 \cup \mathcal{P}_{1b}$. In conclusion, it has been shown that if equation (2) holds, a set of maximum vertex disjoint paths from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ exists, i.e. set $\mathcal{P}_2 \cup \mathcal{P}_{1b}$, which is also vertex disjoint with a path from $\mathcal{U}_j$ to w_i, i.e. the $u_i \to w_i$ path. Then based on condition (3) of Theorem 2, G_{ji} is generically identifiable. This concludes the proof of the "if" part.

For the "only if" part, given that G_{ji} is generically identifiable, the proof is to find a $\bar{\mathcal{U}}_j$ and $\mathcal{W}_d$ satisfying equation (2). Based on Theorem 2, if G_{ji} is generically identifiable, there exist a directed path $\mathcal{U}_j \to w_i$ and a set of maximum vertex disjoint paths $\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}$, which is also vertex disjoint with $\mathcal{U}_j \to w_i$. We choose $\bar{\mathcal{U}}_j \triangleq \mathcal{U}_j$ and $\mathcal{W}_d$ as the minimum disconnecting set from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Then the proof is to show that $\mathcal{W}_d$ is also a disconnecting set from $\{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$, and the equality in (2) is also satisfied.

To show that $\mathcal{W}_d$ is also a disconnecting set from w_i to $\mathcal{W}_j \setminus \{w_i\}$, we discuss two cases: (i) If there is no directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$, it is trivial that $\mathcal{W}_d$ is a disconnecting set from $\mathcal{U}_j \cup \{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$; (ii) If there are paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$, the paths should intersect with $\mathcal{W}_d$, because if not, a directed path from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ through w_i exists without intersecting $\mathcal{W}_d$, which contradicts the condition that $\mathcal{W}_d$ is a minimum disconnecting set from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. Based on (i) and (ii), we can conclude that the chosen $\mathcal{W}_d$ is a disconnecting set from $\mathcal{U}_j \cup \{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$.

Finally, we prove equality (2). Since $\mathcal{W}_d$ is a minimum disconnecting set from $\mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, we have $b_{\mathcal{U}_j \to \mathcal{W}_d} = |\mathcal{W}_d|$, and the vertex disjoint paths counted in $b_{\mathcal{U}_j \to \mathcal{W}_d}$ can be regarded as subpaths of the paths $\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}$. Since $\mathcal{U}_j \to w_i$ and $\mathcal{U}_j \to \mathcal{W}_j \setminus \{w_i\}$ are vertex disjoint, the paths counted in $b_{\mathcal{U}_j \to \mathcal{W}_d}$ are also vertex disjoint with $\mathcal{U}_j \to w_i$, which leads to $b_{\mathcal{U}_j \to \mathcal{W}_d \cup \{w_i\}} = |\mathcal{W}_d| + 1$. This concludes the "only if" part.

Appendix C. PROOF OF PROPOSITION 6

The "if" part is trivial based on Theorem 5. For the "only if" part, if G_{ji} is generically identifiable, based on Theorem 5, there exists a $\bar{\mathcal{U}}_j$ and a disconnecting set $\mathcal{W}_{d1}$ from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ such that equation (2) holds with $\mathcal{W}_{d1}$. Then $\mathcal{W}_d \triangleq \mathcal{W}_{d1} \cup (\mathcal{U}_j \setminus \bar{\mathcal{U}}_j)$ becomes a disconnecting set from $\{w_i\} \cup \mathcal{U}_j$ to $\mathcal{W}_j \setminus \{w_i\}$, and it holds that $b_{\mathcal{U}_j \to \mathcal{W}_d \cup \{w_i\}} = |\mathcal{W}_d| + 1$ because a single vertex is regarded to have a path to itself, which concludes the proof.

Appendix D. PROOF OF PROPOSITION 7

Proof. We first prove that there exists a directed path from w_i to $\mathcal{W}_j \setminus \{w_i\}$ if and only if there exist a parallel path from w_i to w_j or a cycle around the output w_j. Note that due to model (1), $\mathcal{G}$ is a *simple graph*, i.e. there is no self-loop such as (w_i, w_i), and no parallel directed edges from one vertex to another vertex. For "if" part, if there exists a parallel path from w_i to w_j, this parallel path has to intersect with $\mathcal{W}_j \setminus \{w_i\}$. Then we can find a directed path from w_i to one vertex in $\mathcal{W}_j \setminus \{w_i\}$ as a subpath of the parallel path. If a cycle around w_j exists, it will also intersect with $\mathcal{W}_j \setminus \{w_i\}$, and thus the cycle contains a subpath from w_j to one vertex in $\mathcal{W}_j \setminus \{w_i\}$. Linking this subpath and the edge (w_i, w_j) leads to a path from w_i to $\mathcal{W}_j \setminus \{w_i\}$.

For "only if" part, for any directed path from w_i to $w_k \in \mathcal{W}_j \setminus \{w_i\}$, if the path does not contain edge (w_i, w_j), then combining the the path and the edge (w_k, w_j) will create a parallel path. If the path contains (w_i, w_j), then combining the path and the edge (w_k, w_j) while excluding (w_i, w_j) will lead to a cycle around w_j. This concludes the relationship between the parallel paths, the cycles around the output and the paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$.

As seen from the above relationship, since we collect an internal vertex into $\mathcal{W}_d$ from each parallel path and cycle around the output, and thus $\mathcal{W}_d$ is a disconnecting set from w_i to $\mathcal{W}_j \setminus \{w_i\}$ with $w_i \notin \mathcal{W}_d$. Assigning distinct excitation signals to each vertex in $\mathcal{W}_d \cup \{w_i\}$ then achieves generic identifiability of G_{ji} based on Corollary 6.

Appendix E. PROOF OF PROPOSITION 8

We will first show that for any vertex set $\mathcal{C}$ subject to $w_i \notin \mathcal{C}$, $\mathcal{C}$ is a disconnecting set from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ if and only if it is also a disconnecting set from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. The "only if" part holds because if $\mathcal{C}$ intersects with all paths from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{U}}_j$, then it also intersects with the paths from $\{w_i\} \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$. For the "if" part, since $w_i \notin \mathcal{C}$ and $\mathcal{C}$ intersects with all the paths from $\{w_i\} \cup \bar{\mathcal{U}}_j$

to $\mathcal{W}_j \setminus \{w_i\}$, those paths from w_i to $\mathcal{W}_j \setminus \{w_i\}$ have to intersect with $\mathcal{C}$ at their internal vertices or the ending vertices. Since the first internal vertices of the paths belong to set $\mathcal{N}_{w_i}^+$, then $\mathcal{C}$ is a also disconnecting set from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$.

Having the above result, the proposition is proved by showing that a minimum disconnecting set $\mathcal{W}_d$ from $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{U}}_j$ to $\mathcal{W}_j \setminus \{w_i\}$ does not contain w_i, because if it does, it remains a disconnecting set after w_i is excluded, which contradicts the condition that $\mathcal{W}_d$ is a minimum disconnecting set.

Appendix F. PROOF OF COROLLARY 10

Proof. From step 1 to 2 in Algorithm 1, by construction, it can be found that $\mathcal{W}_d$ is a disconnecting set from $\bar{\mathcal{U}}_{j1} \cup \mathcal{N}_{w_i}^+$ to $\mathcal{W}_j \setminus \{w_i\}$ where $\bar{\mathcal{U}}_{j1} \subseteq \mathcal{U}_j^0$, and $\bar{\mathcal{U}}_{j1} \leqslant |\mathcal{W}_d| + 1$. When $\bar{\mathcal{U}}_{j1} = |\mathcal{W}_d| + 1$, equation (2) holds with $\bar{\mathcal{U}}_j \triangleq \bar{\mathcal{U}}_{j1}$, and thus the module is generically identifiable in the original model set $\mathcal{M}$.

When $\bar{\mathcal{U}}_{j1} < |\mathcal{W}_d| + 1$, based on Theorem 5, we need to allocate extra $|\mathcal{W}_d| + 1 - |\bar{\mathcal{U}}_{j1}|$ signals, such that: (i) there are $|\mathcal{W}_d| + 1 - |\bar{\mathcal{U}}_{j1}|$ vertex disjoint paths from these signals to $\mathcal{W}_d \cup \{w_i\} \setminus \bar{\mathcal{W}}_d$ which are also vertex disjoint with $\bar{\mathcal{P}}$; (ii) $\mathcal{W}_d$ remains a disconnecting set from the added signals to $\mathcal{W}_j \setminus \{w_i\}$.

Then we can find in the algorithm, steps 5 guarantees (ii), and steps 3, 6, 7 and 8 guarantee (i). This concludes that we obtain a new model set with generically identifiable G_{ji} after applying the algorithm.