Path-based data-informativity conditions for single module identification in dynamic networks

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Abstract— For consistent or minimum variance estimation of a single module in a dynamic network, a predictor model has to be chosen with selected inputs and outputs, composed of a selection of measured node signals and possibly external excitation signals. The predictor model has to be chosen in such a way that consistent estimation of the target module is possible, under the condition that we have data-informativity for the considered predictor model set. Consistent and minimum variance estimation of target modules is typically obtained if we follow a direct method of identification and predictor model selection, characterized by the property that measured node signals are the prime predictor input signals. In this paper the concept of data-informativity for network models will be formalized, and for the direct method the required data-informativity conditions will be specified in terms of path-based conditions on the graph of the network model, guaranteeing data-informativity in a generic sense, i.e. independent on numerical values of the network transfer functions concerned.

I. INTRODUCTION

In our current days scientific and technological environment many dynamical systems that are being considered are interconnected dynamic subsystems. Therefore, the research community involved in data-driven modeling of systems has shown an increasing interest in the problem of identifying a local (single) module in an interconnected dynamic network of which the interconnection structure (topology) is given. In [1] this problem has been formalized in a prediction error identification setting where local subsystems are described by linear dynamic systems in the form of transfer functions, based on the work of [2]. Classical methods for closed-loop identification have been generalized to be applicable in this network situation, typically leading to multiple-input single output type of estimation problems where the target module is embedded in a larger predictor model, and where consistency of the target module estimate is obtained. Reducing the number of to be measured node signals has been addressed in [3], [4]. For direct methods of identification, further specification of these results has been established by handling confounding variables and correlated disturbances and by deriving minimum variance results [5], [6]. This has led to a so-called local direct method of single module identification [7], that provides different scenarios for the

selection of node signals to be measured, and different ways to determine the predictor model. For indirect methods of identification consistency results for local module estimates are provided in [1], [8].

Separate from the introduction of identification methods to identify single modules, the question of single module identifiability has been considered in [9], [10], [11], [12], [13], showing that identifiablity, being independent of the identification method chosen, puts conditions on the presence and location of external signals in the network, and the measurability of node signals in the network. However the single module identifiability problem has so far been solved only for restricted situations: assuming that either all nodes in the network are excited [9], [10] or all are measured [11], [13], or limiting the acquired information from the network to the transfers from measured external excitation signals to internal node signals, and thus neglecting information from (to be estimated) spectral densities of non-measured disturbance signals [12]. The direct and indirect identification methods typically start from a limited set of measured node signals and a selected set of measured external excitation signals, to determine whether a consistent and/or minimum variance estimate of the target module can be obtained. For all of these methods data-informativity conditions need to be satisfied for arriving at consistent module estimates. While for indirect methods these conditions can typically be phrased in terms of persistence of excitation conditions on external excitation signals, see e.g. [8], for direct methods they are typically formulated in terms of a spectral condition on node signals in the network, and thereby harder to interpret for the user who has to set up an experiment. This has also been addressed in [14] where it has been highlighted that the typical spectral conditions will often be conservative in case of modules with finite model order.

Direct methods for single module identification have the advantage that they provide minimum variance estimation properties. And since they can effectively exploit the excitation properties of nonmeasured disturbance signals, they have conditions on the presence of measured external excitation signals that are less strict than those for indirect methods.

In this paper we are going to address the situation of the direct method [7], and we are going to reformulate the data-informativity conditions for this method in terms of excitation conditions of the external excitation signals, together with path-based conditions on the topology of the network model set. In this way the data-informativity conditions become verifiable by the user, rather than remaining implicit. After recalling the dynamic network setup in Section II, we

will highlight the different options for selecting predictor models in Section III. In Section IV data-informativity conditions are specified, for which path-based conditions are being derived in Section V. The results are illustrated with examples. The proofs of all technical results are collected in the appendix.

II. DYNAMIC NETWORKS

Following the setup of [1], a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1, \ldots, L$, and K *external variables* r_k, $k = 1, \ldots K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^{L} G_{jl}(q)w_l(t) + u_j(t) + v_j(t) \quad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1}w_j(t) = w_j(t-1)$;

- G_{jl} are proper rational transfer functions, referred to as *modules*. In order to avoid technicalities in this paper and without loss of generality we will assume that the modules are strictly proper.
- u_j is an input signal, $u_j(t) = \sum_{k=1}^{K} R_{jk}(q)r_k(t)$ with r_k *external variables* that can directly be manipulated by the user.
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase.

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1\,L} \\ G_{L1} & \cdots & G_{L\,L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

$$w = Gw + Rr + He, \quad (2)$$

where by construction the matrix G is hollow, i.e. it has diagonal entries 0. The single module identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of measured time-series of a subset of variables in w, and possibly r.

III. NETWORK ESTIMATION SETUP

We can distinguish two main different prediction error approaches for addressing the single module identification problem, where the target module is indicated by G_{ji}.

1) A *direct method*, that is based on selecting a particular set of predictor input signals w_k, $k \in \mathcal{D}$, and a set of predicted output signals w_ℓ, $\ell \in \mathcal{Y}$, with $i \in \mathcal{D}$, $j \in \mathcal{Y}$, and estimating a dynamic model based on a prediction error:

$$\varepsilon(t, \theta) = \bar{H}(q, \theta)^{-1}[w_{\mathcal{Y}}(t) - \bar{G}(q, \theta)w_{\mathcal{D}}(t)], \quad (3)$$

where $\bar{G}(q, \theta)$ and $\bar{H}(q, \theta)$ are parametrized transfer function matrices. The target module is then embedded in the model $\bar{G}(q, \theta)$, and the objective is to estimate the target module consistently and possibly with minimum variance.

2) An *indirect method*, that is based on selecting a particular set of external excitation signals r_k, $k \in \mathcal{P}$, and a set of predicted node signals w_ℓ, $\ell \in \mathcal{Y}$, that are used in a predictor model, leading to

$$\varepsilon_{\mathcal{Y}}(t, \theta) = w_{\mathcal{Y}}(t) - \bar{T}_{\mathcal{Y}}(q, \theta)r_{\mathcal{P}}(t) \quad (4)$$

Since $\bar{T}_{\mathcal{Y}}$ reflects a mapping from external signals (r) to internal signals $(w_{\mathcal{Y}})$, a processing step is necessary to recover the target module G_{ji} from an estimated $\bar{T}_{\mathcal{Y}}$. Consistency of the target module estimate is the typical objective. Different variations of indirect methods exist, including two-stage and instrumental variable (IV) methods.

3) As introduced in [15], the above two methods can be combined to a generalized approach where both external signals $r_{\mathcal{P}}$ and node signals $w_{\mathcal{D}}$ are used as predictor inputs.

Although the third method provides the most flexibility, here we will primarily focus on the direct method, in order to limit the complexity of the exposition.

For this method to arrive at a consistent and minimum variance estimate of the target module, there are two prime conditions that need to be satisfied.

1) A predictor model needs to be chosen, on the basis of which it is possible to reconstruct the target module G_{ji} from the estimated objects $\bar{G}$ and $\bar{H}$. The predictor model (3) is determined by the selection of signals that appear in $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$.

2) For the chosen predictor model, the data appearing in this model should be sufficiently informative so as to guarantee that consistent estimates of the objects $\bar{G}$, $\bar{H}$ are obtained.

In the sequel of this paper, we recall the results of [7] to cover the first aspect, while in terms of new results we will focus on the data-informativity aspects as mentioned in the second aspect.

IV. DATA-INFORMATIVITY

We consider an estimation setup on the basis of the network equations

$$w_{\mathcal{Y}}(t) = \bar{G}(q)w_{\mathcal{D}}(t) + \bar{R}(q)r_{\mathcal{P}}(t) + \bar{H}(q)\xi_{\mathcal{Y}}(t) \quad (5)$$

with $w_{\mathcal{Y}}$, $w_{\mathcal{D}}$, $r_{\mathcal{P}}$ selected node- and excitation signals and $\xi_{\mathcal{Y}}$ a stationary white noise process.

The one-step ahead predictor for (5) is uniquely defined through[1]

$$\hat{w}_{\mathcal{Y}}(t) := \bar{\mathbb{E}}\{w_{\mathcal{Y}}(t)|w_{\mathcal{Y}}^{t-1}, w_{\mathcal{D}}^{t}, r_{\mathcal{P}}^{t}\} = W(q)z(t) \quad (6)$$

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[1]The notation w^t refers to the past information $\{w(k)\}_{k=-\infty, \cdots, t-1, t}$.

with the predictor filter given by

$$W(q) := \begin{bmatrix} (1 - \bar{H}(q)^{-1}) & \bar{H}(q)^{-1}\bar{G}(q) & \bar{H}(q)^{-1}\bar{R}(q) \end{bmatrix} \tag{7}$$

and

$$z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ w_{\mathcal{D}}(t) \\ r_{\mathcal{P}}(t) \end{bmatrix}. \tag{8}$$

In line with the corresponding definitions in the prediction error literature ([16], Definition 8.1), we can now define the notion of data-informativity for the related network predictor model.

Definition 1: Consider a set of network signals contained in z and a network predictor model

$$\hat{w}_{\mathcal{Y}}(t, \theta) = W(q, \theta)z(t)$$

for a parametrized set of models

$$\mathcal{M} := (\bar{G}(q, \theta), \bar{R}(q, \theta), \bar{H}(q, \theta))_{\theta \in \Theta}.$$

Then a quasi-stationary data set $Z^\infty := \{z(t)\}_{t=0,\cdots}$ with $z(t)$ defined in (8) is *informative enough with respect to the model set* $\mathcal{M}$ if, for any two predictor models $W_1(q)$ and $W_2(q)$ in the model set,

$$\bar{\mathbb{E}}[(W_1(q) - W_2(q))z(t)]^2 = 0$$

implies that $W_1(e^{i\omega}) \equiv W_2(e^{i\omega})$ for almost all ω. □

In line with ([16], Definition 8.2), we formulate:

Definition 2: A quasi-stationary data set Z^∞ is *informative* if it is informative enough with respect to the model set $\mathcal{L}^*$, consisting of all linear time-invariant models.

And in line with ([16], Definition 13.2):

Definition 3: A quasi-stationary signal z is said to be *persistently exciting* if $\Phi_z(\omega) > 0$ for almost all ω.

The essential difference with the classical definitions in [16] is in the composition of the signal vector $z(t)$, being composed according to (8).

A. Classical open-loop case

The classical open-loop case can be represented by the situation that in the predictor model, the predictor input is $w_{\mathcal{D}} = r$. In this case

$$z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ w_{\mathcal{D}}(t) \end{bmatrix}.$$

The well known sufficient condition for data-informativity is now [16]:

$$\Phi_z(\omega) > 0 \quad \text{for almost all } \omega. \tag{9}$$

For estimating *finite-dimensional* models, this sufficient condition can be further relaxed[2] to be satisfied for a sufficient number of frequencies ω. The signal vector z contains both predictor input and predictor outputs. Since there are output disturbances on $w_{\mathcal{Y}}$ that are uncorrelated to $w_{\mathcal{D}}$, the informativity condition simplifies to the condition that $w_{\mathcal{D}}$ should be persistently exciting.

[2] For the network case and considering MISO models this is also addressed in [14].

B. Classical closed-loop case: direct method

The direct method for closed-loop systems is characterized by the situation that in the predictor model

- $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ are distinct signals;
- $r_{\mathcal{P}}$ is not included in the predictor;
- $w_{\mathcal{D}}$ may depend of the present and past samples of $w_{\mathcal{Y}}$ (feedback).

It follows that $z(t) := \begin{bmatrix} w_{\mathcal{Y}}(t) \\ w_{\mathcal{D}}(t) \end{bmatrix}$ and the "open-loop" results of [16] still apply, i.e. the informativity condition of the data is represented by the condition (9).

C. The network case: local direct method

When applying the direct identification method in the network case, a predictor model is constructed with node signals $w_{\mathcal{D}}$ as predictor inputs and $w_{\mathcal{Y}}$ as predicted outputs. According to the results in [6], [7], the selection of signals follows in main line the next reasoning:

- The input w_i and output w_j of the target module, are selected in $w_{\mathcal{D}}$ and $w_{\mathcal{Y}}$ respectively;
- All network paths from w_i to w_j that do not pass through G_{ji} (parallel paths) pass through a signal w_k that is included in $w_{\mathcal{D}}$;
- All paths from w_j to w_j (loop around w_j) pass through a signal w_k that is included in $w_{\mathcal{D}}$;
- An input signal in $w_{\mathcal{D}}$ that is affected by an unmeasured disturbance that is correlated to an unmeasured disturbance on an output in $w_{\mathcal{Y}}$, is blocked by an additional node signal that is included in $w_{\mathcal{D}}$ or the input signal in $w_{\mathcal{D}}$ is added to $w_{\mathcal{Y}}$.

The last step is applied for dealing with correlated disturbances, or confounding variables, i.e. non-measured signals that affect both the input and output of an estimation problem. When adding the respective signals to $w_{\mathcal{Y}}$ a multi-output predictor model can result, as schematically indicated in Figure 1. In this setting we distinguish:
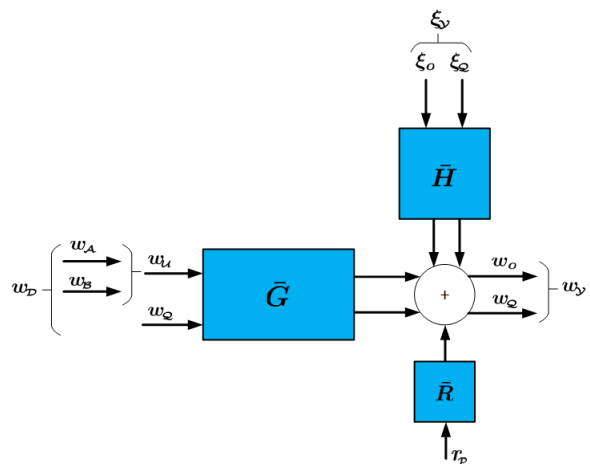

Fig. 1. Predictor model for local direct identification; the set of node signals $w_{\mathcal{Q}}$ appears both at the input and at the output of the predictor model.

- $w_{\mathcal{Y}} = \begin{bmatrix} w_o \\ w_{\mathcal{Q}} \end{bmatrix}$; $\quad w_{\mathcal{D}} = \begin{bmatrix} u_{\mathcal{U}} \\ w_{\mathcal{Q}} \end{bmatrix}$;

- $w_o = w_j$ or w_o is void if w_j is present in $w_{\mathcal{Q}}$;
- $r_{\mathcal{P}}$ contains those measured external excitation signals in r that add directly to measured outputs w_k, $k \in \mathcal{Y}$, i.e. for which $\bar{R}(q)$ is a binary (selection) matrix with known elements, indicating which output signals are excited by signals $r_{\mathcal{P}}$.

Inputs and outputs are allowed to share some common signals, while all node signals are allowed to depend on each other's (present and) past. According to the consistency results in [7] the data-informativity conditions now become:

$$\Phi_\kappa(\omega) > 0 \quad \text{for almost all } \omega, \tag{10}$$

with

$$\kappa(t) := \begin{bmatrix} w_{\mathcal{D}}(t) \\ \xi_{\mathcal{Y}}(t) \end{bmatrix}$$

and $\xi_{\mathcal{Y}}(t)$ the white noise innovation process that relates to output $w_{\mathcal{Y}}(t)$ in (5). In the vector signal κ we collect all the measured node signals that appear as predictor input, and the (external) noise terms $\xi_{\mathcal{Y}}$. The spectrum condition on $\kappa(t)$ can then be interpreted as a condition that requires a full rank spectrum of $w_{\mathcal{D}}$, while using all external signals in the network *except* $\xi_{\mathcal{Y}}$. In other words, $\xi_{\mathcal{Y}}$ can not be used for the "excitation" of the signals $w_{\mathcal{D}}$, but this excitation has to come from other external signals in the network. This mechanism is going to be further elaborated upon in the next Section.

V. Path-based conditions for data-informativity in the direct method

A. General results

The condition (10) for data-informativity in the direct method is compactly formulated, but it is actually implicit and hard to check for the situation of a dynamic network with given topology and unknown dynamics. It would be very attractive to formulate this condition in terms of properties and locations of the external signals in the network (i.e. r and e) together with topological conditions on the interconnections structure in the network models that we consider. In order to achieve this objective, we consider the following Lemma:

Lemma 1: Let $x(t) \in \mathbb{R}^m$ be a quasi-stationary signal that is persistently exciting, and let $F(z) \in \mathbb{R}(z)^{p \times m}$ be the proper rational transfer function of a stable filter. Then the signal $y(t) = F(q)x(t)$ is persistently exciting if and only if filter $F(z)$ has rank p over the field of rational functions. $\square$
If we apply this Lemma with x-signals being the external signals r, e, and y signals being selected node signals w in the network, then the row rank of the considered transfer function $(r, e) \to y$ would need to be evaluated in order to make a statement about data-informativity. In line with the idea of introducing a generic form of identifiability [10], i.e. independent of particular numerical values of coefficients, we can use the same generic type of result for data-informativity, based on the results of [17].

Proposition 1: Consider the situation of Lemma 1. The property that $y(t)$ is persistently exciting holds generically[3]

if in the dynamic network there are p vertex-disjoint paths between the nodes x and y. This is denoted by $b_{x \to y} = p$.

So, a persistently exciting "input" signal x and a sufficient number of vertex-disjoint paths betwen x and y, will generically provide a persistently exciting "output" signal y. This result can be used to translate persistence of excitation conditions on node signals, to persistence of excitation conditions on external network signals.

In order to further specify the data-informativity conditions that apply to the direct method, we need some formal results from [7] that concern the conditions under which the chosen predictor model will lead to results that leave the target module G_{ji} invariant in our estimation setup. This step actually refers to the first set of conditions in Section III, i.e. the choice of predictor model, and specifies conditions under which $\bar{G}_{ji}(q) = G_{ji}^0(q)$. Satisfying these conditions helps to further simplify the topological conditions for data-informativity.

Theorem 1 (Module invariance result [7]): Let G_{ji} be the target network module. In the system's equation (5) conformable to the network model depicted in Figure 1, it holds that $\bar{G}_{ji} = G_{ji}^0$ under the following conditions:

a. Every parallel path from w_i to w_j[4] and every loop around w_j passes through a measured node in $w_{\mathcal{Y} \sqcup \mathcal{U}}$, and
b. $\mathcal{U}$ is decomposed into two disjunct sets, $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$, such that there are no confounding variables[5] for the estimation problems $w_{\mathcal{A}} \to w_{\mathcal{Y}}$ and $w_{\mathcal{A}} \to w_{\mathcal{B}}$, and
c. $i \in \{\mathcal{A} \cup \mathcal{Q}\}$, and
d. Every path from $\{w_i, w_j\}$ to $w_{\mathcal{B}}$ passes through a measured node in $w_{\mathcal{Y} \sqcup \mathcal{U}}$. $\square$

The interpretation of the decomposition of $\mathcal{U}$ into $\mathcal{A} \cup \mathcal{B}$ is that the signals in $w_{\mathcal{B}}$ can be used to block the effect of confounding variables in the estimation problem $w_{\mathcal{A}} \to w_{\mathcal{Y}}$, while confounding variables in the estimation problem $w_{\mathcal{B}} \to w_{\mathcal{Y}}$ are allowed. As a result the transfer functions $\bar{G}_{jk}$ with $k \in \{\mathcal{Q} \cup \mathcal{A}\}$ are invariant, i.e. $\bar{G}_{jk} = G_{jk}^0$. In the remaining part of the paper we will assume that the conditions of Theorem 1 are satisfied.

B. Path-based conditions

The result on vertex-disjoint paths, as formulated in Proposition 1 can now be applied to the particular situation of condition (10). In this step the consequence of having the white noise signal $\xi_{\mathcal{Y}}$ in the condition (10) needs to be translated to conditions on signals in the original network (2). Since the predictor model in the direct method can only include excitation signals $r_{\mathcal{P}}$ that are added directly (with transfer 1) to node signals w, we need to specify which r signals in the original network satisfy this property. Since the transfers that appear in $\bar{R}(q)$ will be dependent on the removal of unmeasured nodes, this is a non-trivial step.

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[3]Generically has to be considered here in terms of a Lebesgue measure 0 of the vector of coefficient values of the rational transfer functions in all modules of the network.

[4]A parallel path is a path from w_i to w_j that does not pass through G_{ji}.
[5]A confounding variable for the estimation problem $w_{\mathcal{A}}$ to $w_{\mathcal{Y}}$ is an unmeasured external or disturbance variable in the network that has unmeasured paths to both $w_{\mathcal{A}}$ and $w_{\mathcal{Y}}$.

Proposition 2: Let $r_{\mathcal{P}}$ be defined as those r-signals in the original network that are directly added to a node signal w_k with $k \in \mathcal{Y}$, such that:

- if $w_k \in w_{\mathcal{Q}}$:
  1) there exist no unmeasured paths from w_k to any $w_\ell, \ell \in \mathcal{B}$; and
  2) all loops through w_k pass through a node in $w_{\mathcal{Q} \cup \mathcal{U}}$.
- if $w_k \in w_o$:
  3) there exists no direct or unmeasured path from w_o to any $w_k, k \in \mathcal{Q}$.

Then $\bar{R}(q)$ is a selection matrix. □

Now we can formulate a path-based condition for verifying the data-informativity.

Theorem 2: Consider a dynamic network with external signals r and e, and let $r_{\mathcal{P}}$ be the r-signals that appear as predictor input in the setting of the local direct method, satisfying the conditions of Proposition 2. Consider the signal vector

$$\eta(t) := \begin{bmatrix} r_{\mathcal{P}} \\ x_{\mathcal{U}} \end{bmatrix}, \quad \text{with}$$

$x_{\mathcal{U}}$: any r- or e-signal that has a direct or unmeasured path to a node signal w_k, $k \in \mathcal{U}$.

Then the transfer function from (r, e) to κ generically has full row rank if there are $n_{\mathcal{D}}$ vertex disjoint paths between external signals η and $w_{\mathcal{D}}$. □

As a direct result of Proposition 1 we can now formulate the following Corollary:

Corollary 1: The data-informativity condition (10) for the local direct method is satisfied if the path-based conditions of Theorem 2 are satisfied and the present excitation signals r are persistently exciting. □

When considering the vector $\eta(t)$ it appears that there are maximally $n_{\mathcal{U}} := dim(u_{\mathcal{U}})$ independent noise signals available that can excite the predictor input $w_{\mathcal{D}}(t)$ while $dim(w_{\mathcal{D}}) = n_{\mathcal{Q}} + n_{\mathcal{U}}$. This is a result of the fact that all of the noises in $x_{\mathcal{U}}$ pass through a node in $u_{\mathcal{U}}$. It leads to the following Corollary.

Corollary 2: There need to be at least $n_{\mathcal{Q}}$ external excitation signals r in the network to satisfy the conditions of Corollary 1. □

The Corollary shows that adding outputs to the predictor model, i.e. increasing $n_{\mathcal{Q}}$ comes at an experimental cost. For every signal that is added to $w_{\mathcal{Q}}$ an extra excitation signal r needs to be present in the network in order to satisfy the data-informativity condition.

We will illustrate the results of this Section in two examples.

Example 1: Consider a classical closed loop system represented by a two-node network as depicted in Figure 2 with v_1 and v_2 being process noises that are correlated. First we consider the situation of having no external excitation signals, $r_1 = r_2 = 0$. The objective is to identify the target module G_{21}. We select w_1 as input and w_2 as output of our predictor model, but due to the correlation between v_1 and v_2, we need to include w_1 also as an output. As a result $w_{\mathcal{Y}} = \{w_1, w_2\}$, $w_o = \{w_2\}$ and $w_{\mathcal{D}} = w_{\mathcal{Q}} = \{w_1\}$. Then $\mathcal{U}$
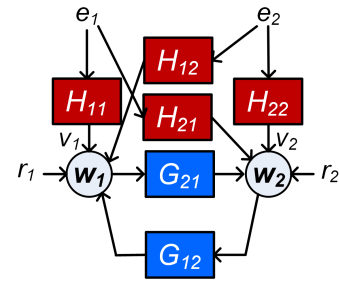

Fig. 2.   Classical closed loop example with two node signals and disturbances v_1 and v_2 being correlated.

is void. In order to satisfy the data informativity condition according to Theorem 2, we need to consider vector $\eta(t)$. Since $\mathcal{U}$ is void and $r_{\mathcal{P}}$ is not present, η is an empty vector, indicating that there are no external signals available for exciting $w_{\mathcal{D}}$. Therefore the data-informativity condition can not be satisfied. The two noise signals e_1 and e_2 constitute the innovation process $\xi_{\mathcal{Y}}$ and according to the definition of $\kappa(t)$ in (10) cannot be used to excite $w_{\mathcal{D}}$. These noise signals are effectively used to estimate the 2×2 noise model.

Adding an external excitation signal r_1 will not lead to a signal in $r_{\mathcal{P}}$ since the loop through w_1 passes only through $w_o = w_2$, and therefore condition 2 of Proposition 2 is not satisfied. In the predictor model $w_1 \rightarrow (w_1, w_2)$, r_1 cannot effectively be used for excitation due to the fact that G_{12} is not modelled, leading to the situation that in the model the contribution of r_1 to w_1 is actually represented by $(1 - G_{12}G_{21})^{-1} r_1$, thus not satisfying the unit transfer that is required for a signal in $r_{\mathcal{P}}$.

Similarly adding an external excitation signal r_2 will not lead to a signal in $r_{\mathcal{P}}$ since there is a direct path from $w_o = w_2$ to $w_{\mathcal{Q}} = w_1$, and thus condition 3 of Proposition 2 is not satisfied. In the predictor model $w_1 \rightarrow (w_1, w_2)$, r_2 cannot effectively be used for excitation due to the fact that G_{12} is not modelled, leading to the situation that in the model the contribution of r_2 to w_1 is actually represented by $(1 - G_{12}G_{21})^{-1} G_{12} r_2$, thus not satisfying the unit transfer that is required for a signal in $r_{\mathcal{P}}$. However, if we include w_2 also as input and model G_{12} with predictor model $(w_1, w_2) \rightarrow (w_1, w_2)$, then $w_{\mathcal{D}}$ changes to $w_{\mathcal{D}} = w_{\mathcal{Q}} = \{w_1, w_2\}$. Then we need two (vertex disjoint) paths from $\eta = r_{\mathcal{P}}$ to $w_{\mathcal{D}}$. With predictor model $(w_1, w_2) \rightarrow (w_1, w_2)$, both r_1 and r_2 lead to a signal in $r_{\mathcal{P}}$ and therefore we need two external excitation signals r_1 and r_2 for achieving data-informativity. This result is in agreement with the observations in [18].

Example 2: Consider the three node network depicted in Figure 3 with v_1 and v_3 being disturbance signals that are correlated. First we consider the situation of having no external excitation signals, $r_1 = r_2 = r_3 = 0$. The objective is to identify the target module G_{12}. According to the local direct method [7], we have multiple ways to choose the predictor model. Following the full input case [6], [7], we choose $w_{\mathcal{Y}} = w_o = w_1$, $w_{\mathcal{A}} = w_2$, and then we choose $w_{\mathcal{B}} = w_3$ in order to block the effect of the confounding
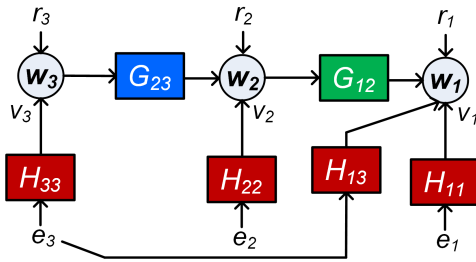
Fig. 3. A three node network example.

variable e_3 for the estimation problem $w_2 \rightarrow w_1$. In this setup $w_{\mathcal{Q}}$ is void and $w_{\mathcal{D}} = u_{\mathcal{U}} = \{w_2, w_3\}$. The data-informativity condition of Theorem 2 now requires two vertex disjoint paths between $\{e_2, e_3\}$ and $\{w_2, w_3\}$. As this can simply be verified from the graph, the data-informativity condition is satisfied without any need for external excitation signals.

When choosing an alternative predictor model, e.g. according to the minimum input case algorithm in [7], we choose $w_{\mathcal{Y}} = \{w_1, w_2\}$ and $w_{\mathcal{Q}} = w_2$, i.e. we model w_2 as output also, in order to deal with the confounding variable e_3 for the estimation problem $w_2 \rightarrow w_1$. In this setup $u_{\mathcal{U}} = w_{\mathcal{A} \cup \mathcal{B}}$ is void. In order to satisfy the data informativity condition according to Theorem 2, we need a path from signal $\eta(t)$ to $w_{\mathcal{D}} = w_2$. Since $\mathcal{U}$ is void and $r_{\mathcal{P}}$ is not present, η is an empty vector, indicating that there are no external signals available for exciting $w_{\mathcal{D}}$. Therefore the data-informativity condition can not be satisfied. Adding external signals r_1 or r_2 will lead to a signal in $\eta = r_p$. But, when $\eta = r_1$ we do not satisfy the data informativity condition since we do not have a path from signal $\eta = r_1$ to $w_{\mathcal{D}} = w_2$. When adding r_2 as external signal, we satisfy the data informativity condition since the path-based condition is satisfied. An external signal r_3 cannot contribute to η because of the non-unity transfer to w_2, and hence does not provide data informativity for the chosen predictor model.

VI. CONCLUSIONS

For consistent identification of a single module that is embedded in a dynamic network it is necessary that the signals that constitute the chosen predictor model satisfy data-informativity conditions. We have formalized the concept of data-informativity for a generalized predictor model that is suited for dynamic network modeling, and that allows for signals to appear both as input and as output in a MIMO predictor model. It generalizes all known situations of indirect and direct methods in closed-loop systems and dynamic networks. The conditions for data-informativity have been specified for a particular identification method, the local direct method, showing that the conditions can be satisfied generically by requiring persistence of excitation of external signals, together with path-based conditions on the topology of the network model set.

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A. *Proof of Lemma 1*

The spectral density of the output signal is given by $\Phi_y(\omega) = F(e^{i\omega})\Phi_x(\omega)F(e^{i\omega})^\star$, with $(\cdot)^\star$ the complex conjugate. For each value of ω in $-\pi \leq \omega \leq \pi$, this is a matrix multiplication for which it holds that $\Phi_y(\omega) > 0$ only if $rank_{\mathbb{R}}(F(e^{i\omega})) = p$ and $\Phi_x(\omega) > 0$. If $rank_{\mathbb{R}(z)}(F(z)) = p$ then $rank_{\mathbb{R}}(F(e^{i\omega})) = p$ for almost all ω. Since $\Phi_x(\omega) > 0$ for almost all ω this implies that $\Phi_y(\omega) > 0$ for almost all ω. If $rank_{\mathbb{R}(z)}(F(z)) < p$ then $rank_{\mathbb{R}}(F(e^{i\omega})) < p$ for all ω and there will be no value of ω for which $\Phi_y(\omega) > 0$. □

B. *Proof of Proposition 2*

On the basis of the decomposition of node signals as defined in section V, we are going to represent the system's equations (2) in the following structured form:

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} G_{\mathcal{QQ}} & G_{\mathcal{Q}o} & G_{\mathcal{QB}} & G_{\mathcal{QA}} & G_{\mathcal{QZ}} \\ G_{o\mathcal{Q}} & G_{oo} & G_{o\mathcal{B}} & G_{o\mathcal{A}} & G_{o\mathcal{Z}} \\ G_{\mathcal{BQ}} & G_{\mathcal{B}o} & G_{\mathcal{BB}} & G_{\mathcal{BA}} & G_{\mathcal{BZ}} \\ G_{\mathcal{AQ}} & G_{\mathcal{A}o} & G_{\mathcal{AB}} & G_{\mathcal{AA}} & G_{\mathcal{AZ}} \\ G_{\mathcal{ZQ}} & G_{\mathcal{Z}o} & G_{\mathcal{ZB}} & G_{\mathcal{ZA}} & G_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \\ w_{\mathcal{Z}} \end{bmatrix}
$$
$$
+ \begin{bmatrix} H_{\mathcal{QQ}} & H_{\mathcal{Q}o} & H_{\mathcal{QB}} & H_{\mathcal{QA}} & H_{\mathcal{QZ}} \\ H_{o\mathcal{Q}} & H_{oo} & H_{o\mathcal{B}} & H_{o\mathcal{A}} & H_{o\mathcal{Z}} \\ H_{\mathcal{BQ}} & H_{\mathcal{B}o} & H_{\mathcal{BB}} & H_{\mathcal{BA}} & H_{\mathcal{BZ}} \\ H_{\mathcal{AQ}} & H_{\mathcal{A}o} & H_{\mathcal{AB}} & H_{\mathcal{AA}} & H_{\mathcal{AZ}} \\ H_{\mathcal{ZQ}} & H_{\mathcal{Z}o} & H_{\mathcal{ZB}} & H_{\mathcal{ZA}} & H_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{B}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix}
$$
$$
+ \begin{bmatrix} R_{\mathcal{QQ}} & 0 & 0 & 0 & 0 \\ 0 & R_{oo} & 0 & 0 & 0 \\ 0 & 0 & R_{\mathcal{BB}} & 0 & 0 \\ 0 & 0 & 0 & R_{\mathcal{AA}} & 0 \\ 0 & 0 & 0 & 0 & R_{\mathcal{ZZ}} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{B}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}} \end{bmatrix} \quad (11)
$$

where, without loss of generality, we make the notation agreement that $R_{\mathcal{QQ}}$ is diagonal, the matrix H is not necessarily monic, and the scaling of the white noise process e is such that $cov(e) = I$.

The objective is to rewrite the system's equations into a form that is conformable to Figure 1 where the output variables $w_{\mathcal{Y}}$ can be written as:

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \end{bmatrix} = \begin{bmatrix} \bar{G}_{\mathcal{QQ}} & 0 & \bar{G}_{\mathcal{QB}} & \bar{G}_{\mathcal{QA}} \\ \bar{G}_{o\mathcal{Q}} & 0 & \bar{G}_{o\mathcal{B}} & \bar{G}_{o\mathcal{A}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{A}} \\ w_{\mathcal{B}} \end{bmatrix} +
$$
$$
+ \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} & 0 & 0 \\ \bar{H}_{21} & \bar{H}_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{\mathcal{Q}} \\ \xi_o \\ \xi_{\mathcal{B}} \\ \xi_{\mathcal{A}} \end{bmatrix} + \bar{R}r_p \quad (12)
$$

withe the vector ξ representing the innovation process of the disturbance signals on the measured variables w, with the submatrix $\bar{H}$ forming a monic stable and stably invertible noise model. Subsequently we need to show that $\bar{R}$ is a selection matrix. The proof will be executed in a number of subsequent steps.

Step 1: Removing the unmeasured nodes $w_{\mathcal{Z}}$

Starting with the network representation (11), we eliminate (immerse) the non-measured node variables $w_{\mathcal{Z}}$ from the equations, by writing the last (block) row of (11) into an explicit expression for $w_{\mathcal{Z}}$:

$$
w_{\mathcal{Z}} = (I - G_{\mathcal{ZZ}})^{-1} \left[\sum_{k \in \mathcal{Q} \cup \{o\} \cup \mathcal{U}} G_{\mathcal{Z}k} w_k + \sum_{\ell \in \mathcal{Q} \cup \{o\} \cup \mathcal{U} \cup \mathcal{Z}} H_{\mathcal{Z}\ell} w_\ell + u_{\mathcal{Z}} \right],
$$

with $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$. By substituting this into the expressions for the remaining w-variables we obtain

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \breve{G}_{\mathcal{QQ}} & \breve{G}_{\mathcal{Q}o} & \breve{G}_{\mathcal{QB}} & \breve{G}_{\mathcal{QA}} \\ \breve{G}_{o\mathcal{Q}} & \breve{G}_{oo} & \breve{G}_{o\mathcal{B}} & \breve{G}_{o\mathcal{A}} \\ \breve{G}_{\mathcal{BQ}} & \breve{G}_{\mathcal{B}o} & \breve{G}_{\mathcal{BB}} & \breve{G}_{\mathcal{BA}} \\ \breve{G}_{\mathcal{AQ}} & \breve{G}_{\mathcal{A}o} & \breve{G}_{\mathcal{AB}} & \breve{G}_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \breve{v} + \breve{u},
$$

with

$$
\breve{v} = \underbrace{\begin{bmatrix} \breve{H}_{\mathcal{QQ}} & \breve{H}_{\mathcal{Q}o} & \breve{H}_{\mathcal{QB}} & \breve{H}_{\mathcal{QA}} & \breve{H}_{\mathcal{QZ}} \\ \breve{H}_{o\mathcal{Q}} & \breve{H}_{oo} & \breve{H}_{o\mathcal{B}} & \breve{H}_{o\mathcal{A}} & \breve{H}_{o\mathcal{Z}} \\ \breve{H}_{\mathcal{BQ}} & \breve{H}_{\mathcal{B}o} & \breve{H}_{\mathcal{BB}} & \breve{H}_{\mathcal{BA}} & \breve{H}_{\mathcal{BZ}} \\ \breve{H}_{\mathcal{AQ}} & \breve{H}_{\mathcal{A}o} & \breve{H}_{\mathcal{AB}} & \breve{H}_{\mathcal{AA}} & \breve{H}_{\mathcal{AZ}} \end{bmatrix}}_{\breve{H}} \begin{bmatrix} e_{\mathcal{Q}} \\ e_o \\ e_{\mathcal{B}} \\ e_{\mathcal{A}} \\ e_{\mathcal{Z}} \end{bmatrix},
$$

$$
\breve{u} = \underbrace{\begin{bmatrix} R_{\mathcal{QQ}} & 0 & 0 & 0 & \breve{R}_{\mathcal{QZ}} \\ 0 & R_{oo} & 0 & 0 & \breve{R}_{o\mathcal{Z}} \\ 0 & 0 & R_{\mathcal{BB}} & 0 & \breve{R}_{\mathcal{BZ}} \\ 0 & 0 & 0 & R_{\mathcal{AA}} & \breve{R}_{\mathcal{AZ}} \end{bmatrix}}_{\breve{R}} \underbrace{\begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{A}} \\ r_{\mathcal{B}} \\ r_{\mathcal{Z}} \end{bmatrix}}_{r}. \quad (13)
$$

and where

$$
\breve{G}_{kh} = G_{kh} + G_{k\mathcal{Z}}(I - G_{\mathcal{ZZ}})^{-1} G_{\mathcal{Z}h} \quad (14)
$$

with $k, h \in \{\mathcal{Q} \cup \{o\} \cup \mathcal{U}\}$, and

$$
\breve{H}_{k\ell} = H_{k\ell} + G_{k\mathcal{Z}}(I - G_{\mathcal{ZZ}})^{-1} H_{\mathcal{Z}\ell}, \quad (15)
$$
$$
\breve{R}_{k\ell} = R_{k\ell} + G_{k\mathcal{Z}}(I - G_{\mathcal{ZZ}})^{-1} R_{\mathcal{Z}\ell}, \quad (16)
$$

with $\ell \in \{\mathcal{Q} \cup \{o\} \cup \mathcal{U} \cup \mathcal{Z}\}$.

In order to be able to arrive at a term $\bar{R}r_p$ in (12) where $\bar{R}$ is a selection matrix, we are going to separate the matrix $\breve{R}$ in two parts:

$$
\breve{R} = \breve{R}_1 + \breve{R}_2
$$

where the term $\breve{R}_1 r$ is going to be modelled as a disturbance term, and $\breve{R}_2 r$ will be used to convert to the explicit term $\bar{R}r_p$ present in (12). This separation is done as follows:

$$
\breve{R}_1 = \begin{bmatrix} \breve{R}_{\mathcal{QQ}} & 0 & 0 & 0 & \breve{R}_{\mathcal{QZ}} \\ 0 & \breve{R}_{oo} & 0 & 0 & \breve{R}_{o\mathcal{Z}} \\ 0 & 0 & R_{\mathcal{BB}} & 0 & \breve{R}_{\mathcal{BZ}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad (17)
$$

$$
\breve{R}_2 = \begin{bmatrix} \tilde{R}_{\mathcal{QQ}} & 0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{oo} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{\mathcal{AA}} & \breve{R}_{\mathcal{AZ}} \end{bmatrix} \quad (18)
$$

while the terms $\breve{R}_{\mathcal{QQ}}, \breve{R}_{oo}, \tilde{R}_{\mathcal{QQ}}, \tilde{R}_{oo}$ are determined as follows:

- For each $k \in \mathcal{Q}$, if w_k satisfies the conditions 1) and 2) of the Proposition, then $\tilde{R}_{kk} = 1$ and $\check{R}_{kk} = 0$; else $\check{R}_{kk} = 1$ and $\tilde{R}_{kk} = 0$.
- If w_0 satisfies condition 3) of the Proposition, then $\tilde{R}_{oo} = 1$ and $\check{R}_{oo} = 0$; else $\check{R}_{oo} = 1$ and $\tilde{R}_{oo} = 0$.

We are now going to add the term $\check{R}_1 r$ to the disturbance model.

On the basis of (13), the spectral density of $\tilde{v} = \check{H}e + \check{R}_1 r$ is given by $\Phi_{\tilde{v}} = \check{H}\check{H}^* + \check{R}_r \Phi_r \check{R}_1^*$. Applying a spectral factorization [19] to $\Phi_{\tilde{v}}$ will deliver $\Phi_{\tilde{v}} = \tilde{H}\tilde{\Lambda}\tilde{H}^*$ with $\tilde{H}$ a monic, stable and minimum phase rational matrix, and $\tilde{\Lambda}$ a positive definite (constant) matrix. Then there exists a white noise process $\tilde{\xi}$ defined by

$$\tilde{\xi} := \tilde{H}^{-1}(\check{H}e + \check{R}_1 r) \tag{19}$$

such that $\tilde{H}\tilde{\xi} = \tilde{v}$, with $\mathrm{cov}(\tilde{\xi}) = \tilde{\Lambda}$. As a result, (13) can be rewritten as

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \check{G}_{\mathcal{QQ}} & \check{G}_{\mathcal{Q}o} & \check{G}_{\mathcal{QB}} & \check{G}_{\mathcal{QA}} \\ \check{G}_{o\mathcal{Q}} & \check{G}_{oo} & \check{G}_{o\mathcal{B}} & \check{G}_{o\mathcal{A}} \\ \check{G}_{\mathcal{BQ}} & \check{G}_{\mathcal{B}o} & \check{G}_{\mathcal{BB}} & \check{G}_{\mathcal{BA}} \\ \check{G}_{\mathcal{AQ}} & \check{G}_{\mathcal{A}o} & \check{G}_{\mathcal{AB}} & \check{G}_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \tilde{H} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{B}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix} + \check{R}_2 r. \tag{20}$$

Under conditions as formulated in Theorem 1, the noise model can now be shown to satisfy particular structural properties.

Lemma 2 (Lemma 3 in [7]): Consider a dynamic network as defined in (13) where the non-measured node signals $w_{\check{z}}$ are immersed, and let $\mathcal{U}$ be decomposed in sets $\mathcal{A}$ and $\mathcal{B}$ satisfying Condition (b) in Theorem 1. Then the spectral density $\Phi_{\tilde{v}}$ has the unique spectral factorization $\Phi_{\tilde{v}} = \tilde{H}\tilde{\Lambda}\tilde{H}^*$ with $\tilde{\Lambda}$ constant and $\tilde{H}$ monic, stable, minimum phase, and of the form

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & 0 \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & 0 \\ 0 & 0 & 0 & \Lambda_{44} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} & 0 \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} & 0 \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & 0 \\ 0 & 0 & 0 & \tilde{H}_{44} \end{bmatrix} \tag{21}$$

where the block dimensions are conformable to the dimensions of $w_{\mathcal{Q}}$, w_o, $w_{\mathcal{B}}$ and $w_{\mathcal{A}}$ respectively. $\square$

Step 2: Complying with the system's structure (12)

For complying with the system's structure (12) the terms $\check{G}_{\mathcal{Q}o}$, $\check{G}_{oo}$, $\tilde{H}_{13}$ and $\tilde{H}_{23}$ need to removed from the expressions (20)-(21). First we treat the terms in $\tilde{H}$ by premultiplying (20) with

$$\begin{bmatrix} I & 0 & -\check{H}_{13} & 0 \\ 0 & I & -\check{H}_{23} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{22}$$

where

$$\begin{bmatrix} \check{H}_{13} \\ \check{H}_{23} \end{bmatrix} := \begin{bmatrix} \tilde{H}_{13}\tilde{H}_{33}^{-1} \\ \tilde{H}_{23}\tilde{H}_{33}^{-1} \end{bmatrix}. \tag{23}$$

When we only keep the identity terms on the left hand side of the equation, we obtain the equivalent network equation:

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \check{G}'_{\mathcal{QQ}} & \check{G}'_{\mathcal{Q}o} & \check{G}'_{\mathcal{QB}} & \check{G}'_{\mathcal{QA}} \\ \check{G}'_{o\mathcal{Q}} & \check{G}'_{oo} & \check{G}'_{o\mathcal{B}} & \check{G}'_{o\mathcal{A}} \\ \check{G}'_{\mathcal{BQ}} & \check{G}'_{\mathcal{B}o} & \check{G}'_{\mathcal{BB}} & \check{G}'_{\mathcal{BA}} \\ \check{G}'_{\mathcal{AQ}} & \check{G}'_{\mathcal{A}o} & \check{G}'_{\mathcal{AB}} & \check{G}'_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \tilde{H}'_{11} & \tilde{H}'_{12} & 0 & 0 \\ \tilde{H}'_{21} & \tilde{H}'_{22} & 0 & 0 \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & 0 \\ 0 & 0 & 0 & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{B}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{R}_{\mathcal{QQ}} & 0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{oo} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{\mathcal{AA}} & \check{R}_{\mathcal{AZ}} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{B}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}} \end{bmatrix}, \tag{24}$$

with

$$\check{G}'_{\mathcal{BA}} = \check{G}_{\mathcal{BA}} - \check{H}_{13}\check{G}_{\mathcal{BB}} + \check{H}_{13} \tag{25}$$

$$\check{G}'_{\mathcal{Q}\star} = \check{G}_{\mathcal{Q}\star} - \check{H}_{13}\check{G}_{\mathcal{B}\star} \tag{26}$$

$$\check{G}'_{o\star} = \check{G}_{o\star} - \check{H}_{23}\check{G}_{\mathcal{B}\star} \tag{27}$$

$$\check{G}'_{\mathcal{A}\mathcal{A}} = \check{G}_{\mathcal{A}\mathcal{A}} - \check{H}_{23}\check{G}_{\mathcal{BB}} + \check{H}_{23} \tag{28}$$

$$\tilde{H}'_{1\bullet} = \tilde{H}_{1\bullet} - \check{H}_{13}\tilde{H}_{3\bullet} \tag{29}$$

$$\tilde{H}'_{2\bullet} = \tilde{H}_{2\bullet} - \check{H}_{23}\tilde{H}_{3\bullet} \tag{30}$$

where $\star \in \{\mathcal{Q} \cup \{o\} \cup \mathcal{A}\}$ and $\bullet \in \{1, 2\}$.

The next step is now to show that the block elements $\check{G}'_{\mathcal{Q}o}$ and $\check{G}'_{oo}$ in G can be made 0. This can be done by variable substitution as follows:

The second row in (24) is replaced by an explicit expression for w_o according to

$$w_o = (1 - \check{G}'_{oo})^{-1}[\check{G}'_{o\mathcal{Q}}w_{\mathcal{Q}} + \check{G}'_{o\mathcal{B}}w_{\mathcal{B}} + \check{G}'_{o\mathcal{A}}w_{\mathcal{A}} + \tilde{H}'_{21}\tilde{\xi}_{\mathcal{Q}} + \tilde{H}'_{22}\tilde{\xi}_o + \tilde{R}_{oo}r_o]. \tag{31}$$

Additionally, this expression for w_o is substituted into the first block row of (24), to remove the w_o-dependent term on the right hand side, leading to

$$\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \check{G}''_{\mathcal{QQ}} & 0 & \check{G}''_{\mathcal{QB}} & \check{G}''_{\mathcal{QA}} \\ \bar{G}_{o\mathcal{Q}} & 0 & \bar{G}_{o\mathcal{B}} & \bar{G}_{o\mathcal{A}} \\ \check{G}_{\mathcal{BQ}} & \check{G}_{\mathcal{B}o} & \check{G}_{\mathcal{BB}} & \check{G}_{\mathcal{BA}} \\ \check{G}_{\mathcal{AQ}} & \check{G}_{\mathcal{A}o} & \check{G}_{\mathcal{AB}} & \check{G}_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} + \begin{bmatrix} \tilde{H}''_{11} & \tilde{H}''_{12} & 0 & 0 \\ \tilde{H}''_{21} & \tilde{H}''_{22} & 0 & 0 \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & 0 \\ 0 & 0 & 0 & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{B}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{R}_{\mathcal{QQ}} & \check{R}''_{\mathcal{Q}o} & 0 & 0 & 0 \\ 0 & \bar{R}_{oo} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{\mathcal{AA}} & \check{R}_{\mathcal{AZ}} \end{bmatrix} \begin{bmatrix} r_{\mathcal{Q}} \\ r_o \\ r_{\mathcal{B}} \\ r_{\mathcal{A}} \\ r_{\mathcal{Z}} \end{bmatrix}, \tag{32}$$

with

$$\bar{G}_{o\star} = (I - \check{G}'_{oo})^{-1}\check{G}'_{o\star} \tag{33}$$

$$\tilde{H}''_{2\star} = (I - \check{G}'_{oo})^{-1}\tilde{H}'_{2\star} \tag{34}$$

$$\check{G}''_{\mathcal{Q}\star} = \check{G}'_{\mathcal{Q}\star} + \check{G}'_{\mathcal{Q}o}\bar{G}_{o\star} \tag{35}$$

$$\tilde{H}''_{1\star} = \tilde{H}'_{1\star} + \check{G}'_{\mathcal{Q}o}\tilde{H}''_{2\star} \tag{36}$$

$$\bar{R}_{oo} = (I - \check{G}'_{oo})^{-1}\tilde{R}_{oo} \tag{37}$$

$$\check{R}''_{\mathcal{Q}o} = \check{G}'_{\mathcal{Q}o}\bar{R}_{oo}. \tag{38}$$

Due to these operations, the matrix $\check{G}''_{\mathcal{QQ}}$ might not be hollow, we move any diagonal terms of this matrix to the left

hand side of the equation, and premultiply the first (block) equation by the diagonal matrix $(I-\mathrm{diag}(\breve{G}''_{QQ}))^{-1}$, to obtain the expression

$$
\begin{bmatrix} w_Q \\ w_o \\ w_B \\ w_A \end{bmatrix} = \begin{bmatrix} \bar{G}_{QQ} & 0 & \bar{G}_{QB} & \bar{G}_{QA} \\ \bar{G}_{oQ} & 0 & \bar{G}_{oB} & \bar{G}_{oA} \\ \breve{G}_{BQ} & \breve{G}_{Bo} & \breve{G}_{BB} & \breve{G}_{BA} \\ \breve{G}_{AQ} & \breve{G}_{Ao} & \breve{G}_{AB} & \breve{G}_{AA} \end{bmatrix} \begin{bmatrix} w_Q \\ w_o \\ w_B \\ w_A \end{bmatrix} + \begin{bmatrix} \tilde{H}'''_{11} & \tilde{H}'''_{12} & 0 & 0 \\ \tilde{H}''_{21} & \tilde{H}''_{22} & 0 & 0 \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} & 0 \\ 0 & 0 & 0 & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_Q \\ \tilde{\xi}_o \\ \tilde{\xi}_B \\ \tilde{\xi}_A \end{bmatrix}
$$

$$
+ \underbrace{\begin{bmatrix} \bar{R}_{QQ} & \bar{R}_{Qo} & 0 & 0 & 0 \\ 0 & \bar{R}_{oo} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{AA} & \breve{R}_{Az} \end{bmatrix} \begin{bmatrix} r_Q \\ r_o \\ r_B \\ r_A \\ r_z \end{bmatrix}}_{\bar{u}}
\tag{39}
$$

with

$$
\bar{G}_{QQ} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}(\breve{G}''_{QQ} - \mathrm{diag}(\breve{G}''_{QQ})), \tag{40}
$$
$$
\bar{G}_{Qi} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\breve{G}''_{Qi} \tag{41}
$$
$$
\tilde{H}'''_{1\star} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\tilde{H}''_{1\star} \tag{42}
$$
$$
\bar{R}_{QQ} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\tilde{R}_{QQ} \tag{43}
$$
$$
\bar{R}_{Qo} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\breve{R}''_{Qo}. \tag{44}
$$

Step 3: Adjusting the noise model

Since the noise model $\tilde{H}_r := \begin{bmatrix} \tilde{H}'''_{11} & \tilde{H}'''_{12} \\ \tilde{H}''_{21} & \tilde{H}''_{22} \end{bmatrix}$ is not necessarily monic, stable and minimum phase, we apply spectral factorization to rewrite

$$
\tilde{v}_y = \tilde{H}_r \tilde{\xi}_y = \bar{H}\xi_y. \tag{45}
$$

with $\bar{H}$ a monic stable and stably invertible rational matrix, and ξ_y a white noise process. Therefore an equivalent representation of the first two block rows of (39) that maintains the second order properties of the node signals, is given by

$$
\begin{bmatrix} w_Q \\ w_o \end{bmatrix} = \begin{bmatrix} \bar{G}_{QQ} & 0 & \bar{G}_{QB} & \bar{G}_{QA} \\ \bar{G}_{oQ} & 0 & \bar{G}_{oB} & \bar{G}_{oA} \end{bmatrix} \begin{bmatrix} w_Q \\ w_o \\ w_B \\ w_A \end{bmatrix} + \\
+ \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{H}_{22} \end{bmatrix} \begin{bmatrix} \xi_Q \\ \xi_o \end{bmatrix} + \begin{bmatrix} \bar{R}_{QQ} & \bar{R}_{Qo} \\ 0 & \bar{R}_{oo} \end{bmatrix} \begin{bmatrix} r_Q \\ r_o \end{bmatrix}. \tag{46}
$$

Step 4: Verifying whether $\bar{R}$ is a selection matrix

From (46) it follows that

$$
\bar{R}r_p = \begin{bmatrix} \bar{R}_{QQ} & \bar{R}_{Qo} \\ 0 & \bar{R}_{oo} \end{bmatrix} \begin{bmatrix} r_Q \\ r_o \end{bmatrix}. \tag{47}
$$

First we evaluate the term $\bar{R}_{Qo}$. With (44) and (38) it follows that

$$
\bar{R}_{Qo} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\breve{G}'_{Qo}(I - \breve{G}'_{oo})^{-1}\tilde{R}_{oo},
$$

while with (27):

$$
\breve{G}'_{Qo} = \breve{G}_{Qo} - \tilde{H}_{13}\breve{G}_{Bo}.
$$

From condition (d) in Theorem 1 it follows that $\breve{G}_{Bo} = 0$, and from condition (3) in Proposition 2 it follows that $\breve{G}_{Qo} = 0$. As a result, $\breve{G}'_{Qo} = 0$ leading to $\bar{R}_{Qo} = 0$.

For the term $\bar{R}_{oo}$, we have according to (37) that $\bar{R}_{oo} = (I - \breve{G}'_{oo})^{-1}\tilde{R}_{oo}$, with, by definition, $\tilde{R}_{oo} \in \{0, 1\}$. From condition (a) and (d) of Theorem 1 it follows that $\breve{G}'_{oo} = 0$, leading to $\bar{R}_{oo} \in \{0, 1\}$.

Finally, for the term $\bar{R}_{QQ}$ we consider according to (43), $\bar{R}_{QQ} = (I - \mathrm{diag}(\breve{G}''_{QQ}))^{-1}\tilde{R}_{QQ}$ and from (35):

$$
\breve{G}''_{QQ} = \breve{G}'_{QQ} + \breve{G}'_{Qo}\bar{G}_{oQ},
$$

which when substituting (27) and (28) becomes for $k \in Q$:

$$
\breve{G}''_{kk} = \breve{G}_{kk} - \tilde{H}_{13}\breve{G}_{Bk} + \\
+ (\breve{G}_{ko} - \tilde{H}_{13}\breve{G}_{Bo})(I - \breve{G}'_{oo})^{-1}(\breve{G}_{ok} - \tilde{H}_{23}\breve{G}_{Bk}).
$$

From conditions (2) and (1) in Proposition 2 it follows that $\breve{G}_{kk} = 0$ and $\breve{G}_{Bk} = 0$, and from conditions (a) and (d) in Theorem 1, we have $\breve{G}_{Bo} = 0$ and $\breve{G}'_{oo} = 0$. Substituting this in the previous equation leads to $\breve{G}'''_{kk} = \breve{G}_{ko}\breve{G}_{ok}$. Since condition (1) of Proposition 2 excludes the possibility that there is a loop through $w_k, k \in Q$ that passes through w_o, it follows that $\breve{G}''_{kk} = 0$. With (43) and given the fact that $\tilde{R}_{QQ}$ is a selection matrix, it follows that $\bar{G}_{QQ}$ is a selection matrix too. This concludes the final part of the Proof. $\square$

C. Proof of Theorem 2

The results of Lemma 1 and Proposition 1 indicate that the transfer function from external signals to κ generically has full row rank, if there are $n_Q + n_u + n_y$ vertex disjoint paths between the external signals and κ.

In the system's setting with all unmeasured nodes removed, and with some excitation signals r being encapsulated in the disturbance process, we first have to determine which external signals should be considered. According to the term $\tilde{u}$ in system representation (39), the external signals r that remain as external excitation signals are r_P (being the terms in $\bar{R}_{QQ}$ and $\bar{R}_{oo}$ that are equal to 1), r_A, and $r_z^A := \breve{R}_{Az}r_z$, being the r signals in r_z that have an unmeasured path to a node signal in w_A. These external excitation signals need to be complemented by disturbances sources.

Characterizing ξ: disturbance sources in the transformed network

When premultiplying (39) with

$$
P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\tilde{H}'_{31} & -\tilde{H}'_{32} & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
$$

where $[\tilde{H}'_{31} \quad \tilde{H}'_{32}] = [\tilde{H}_{31} \quad \tilde{H}_{32}] \begin{bmatrix} \tilde{H}'''_{11} & \tilde{H}'''_{12} \\ \tilde{H}'''_{21} & \tilde{H}'''_{22} \end{bmatrix}^{-1}$, while only keeping the identity terms on the left hand side, we

obtain an equivalent network equation

$$
\begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \bar{G}_{\mathcal{QQ}} & 0 & \bar{G}_{\mathcal{QB}} & \bar{G}_{\mathcal{QA}} \\ \bar{G}_{o\mathcal{Q}} & 0 & \bar{G}_{o\mathcal{B}} & \bar{G}_{o\mathcal{A}} \\ \breve{G}'_{\mathcal{BQ}} & \breve{G}'_{\mathcal{Bo}} & \breve{G}'_{\mathcal{BB}} & \breve{G}'_{\mathcal{BA}} \\ \breve{G}_{\mathcal{AQ}} & \breve{G}_{\mathcal{Ao}} & \breve{G}_{\mathcal{AB}} & \breve{G}_{\mathcal{AA}} \end{bmatrix} \begin{bmatrix} w_{\mathcal{Q}} \\ w_o \\ w_{\mathcal{B}} \\ w_{\mathcal{A}} \end{bmatrix} +
$$

$$
+ \begin{bmatrix} \tilde{H}'''_{11} & \tilde{H}'''_{12} & 0 & 0 \\ \tilde{H}''_{21} & \tilde{H}''_{22} & 0 & 0 \\ 0 & 0 & \tilde{H}_{33} & 0 \\ 0 & 0 & 0 & \tilde{H}_{44} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{\mathcal{Q}} \\ \tilde{\xi}_o \\ \tilde{\xi}_{\mathcal{B}} \\ \tilde{\xi}_{\mathcal{A}} \end{bmatrix} + P'\bar{u}. \quad (48)
$$

where the third equation has been scaled to maintain a hollow matrix $\breve{G}'_{\mathcal{BB}}$. The disturbance term in this equation can, after spectral factorization and creating a monic, stable and minimum phase noise model, be rewritten into

$$
\begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} & 0 & 0 \\ \bar{H}_{21} & \bar{H}_{22} & 0 & 0 \\ 0 & 0 & \bar{H}_{33} & 0 \\ 0 & 0 & 0 & \bar{H}_{44} \end{bmatrix} \begin{bmatrix} \xi_{\mathcal{Q}} \\ \xi_o \\ \xi_{\mathcal{B}} \\ \xi_{\mathcal{A}} \end{bmatrix}, \quad (49)
$$

showing that $\xi_{\mathcal{U}}$ is a filtered version of $\tilde{\xi}_{\mathcal{U}}$.

Writing the disturbance $\xi_{\mathcal{U}}$ in terms of external signals
According to (19) we have

$$
\tilde{\xi} = \tilde{H}^{-1}(\breve{H}e + \breve{R}_1 r) = \tilde{H}^{-1}(\breve{v} + \breve{R}_1 r). \quad (50)
$$

where $\tilde{H}$ is a monic, stable and minimum phase rational matrix and $\breve{v}$ is the process noise on the nodes in the immersed network, i.e. the network that results after removing the unmeasured node signals. Following Lemma 3 in [7] (see also Lemma 2), if condition b in Theorem 1 is satisfied, then $\tilde{H}$ is block diagonal and of the form

$$
\tilde{H} = \begin{bmatrix} \tilde{H}_b & 0 \\ 0 & \tilde{H}_a \end{bmatrix}; \quad \breve{v} = \begin{bmatrix} \breve{v}_{\mathcal{Y} \cup \mathcal{B}} \\ \breve{v}_{\mathcal{A}} \end{bmatrix}.
$$

where $\tilde{H}_b$ combines the three block rows and columns related to the nodes in $\mathcal{Q} \cup \{o\} \cup \mathcal{B} = \mathcal{Y} \cup \mathcal{B}$.

Since $\tilde{H}_b$ is monic, the matrix inverse definitely has nonzero diagonal terms, implying that with (50), $\breve{v}_{\mathcal{B}}$ is affecting $\tilde{\xi}_{\mathcal{B}}$, and with a similar reasoning $\breve{v}_{\mathcal{A}}$ is affecting $\tilde{\xi}_{\mathcal{A}}$. Consequently the disturbance terms that appear in $\tilde{\xi}_{\mathcal{U}}$ are given by $\breve{u}_{\mathcal{U}}$, which are the noise signals on $u_{\mathcal{U}}$ in the immersed network (13) with unmeasured nodes removed, and hence is a filtered version of all signals in e that have a direct or unmeasured path to a node in $u_{\mathcal{U}}$.

The external excitation signals that appear in $\tilde{\xi}_{\mathcal{U}}$ can be analysed from (50) also. Because of the structure of $\breve{R}_1$ as provided in (17), and the fact that because of monicity of $\tilde{H}_b$ the matrix inverse definitely has nonzero diagonal terms, the external excitation signals appearing in $\tilde{\xi}_{\mathcal{B}}$ (i.e. the third row of $\breve{R}_1 r$) are: $r_{\mathcal{B}}$ and $r_{\mathcal{Z}}^{\mathcal{B}}$, being the r signals in $r_{\mathcal{Z}}$ that have an unmeasured path to a node in $w_{\mathcal{B}}$.

Combining the above result, and using the fact that $\xi_{\mathcal{U}}$ is a filtered version of $\tilde{\xi}_{\mathcal{U}}$ it follows that the following external signals appear in $\xi_{\mathcal{U}}$:

- $e^{\mathcal{U}}$: all signals in e that have a direct or unmeasured path to a node in $w_{\mathcal{U}}$

- $r_{\mathcal{B}}$ and $r_{\mathcal{Z}}^{\mathcal{B}}$.

Finalizing the proof
The mapping that we need to evaluate for verifying the number of vertex disjoint paths is given by

$$
(r_{\mathcal{P}}, r_{\mathcal{A}}, r_{\mathcal{Z}}^{\mathcal{A}}, \xi) \to (w_{\mathcal{D}}, \xi_{\mathcal{Y}}).
$$

Since $\xi_{\mathcal{Y}}$ appears on both sides of the mapping, the path condition can equivalently be formulated for the mapping

$$
(r_{\mathcal{P}}, r_{\mathcal{A}}, r_{\mathcal{Z}}^{\mathcal{A}}, \xi_{\mathcal{U}}) \to w_{\mathcal{D}}.
$$

Given the external signals that affect $\xi_{\mathcal{U}}$ as analyzed above, it is sufficient to evaluate the mapping

$$
(r_{\mathcal{P}}, r_{\mathcal{A}}, r_{\mathcal{Z}}^{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{Z}}^{\mathcal{B}}, e^{\mathcal{U}}) \to w_{\mathcal{D}}.
$$

In this mapping, $r_{\mathcal{A}}, r_{\mathcal{B}}$ unite all r signals that have a direct path to a node in $u_{\mathcal{U}}$, and $r_{\mathcal{Z}}^{\mathcal{A}}, r_{\mathcal{Z}}^{\mathcal{B}}$ unite all r signals that have an unmeasured path to an node in $u_{\mathcal{U}}$. With the definition of $x_{\mathcal{U}}$ in the Theorem, it follows that $x_{\mathcal{U}} = (r_{\mathcal{A}}, r_{\mathcal{Z}}^{\mathcal{A}}, r_{\mathcal{B}}, r_{\mathcal{Z}}^{\mathcal{B}}, e^{\mathcal{U}})$, showing that we need to evaluate the mapping: $(r_{\mathcal{P}}, x_{\mathcal{U}}) \to w_{\mathcal{D}}$. $\square$