On dynamic network modeling of stationary processes

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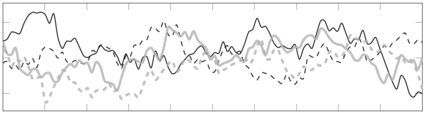
EE / Control Systems

Problem statement

Focus: p-dimensional stationary process $w(t) \longrightarrow$ spectral density $\Phi(z)$



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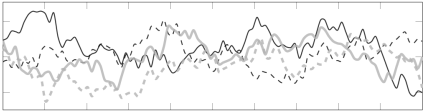
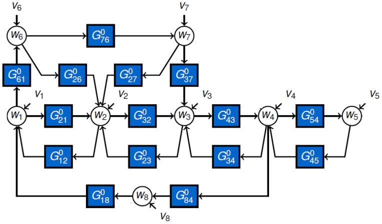
Main question

Can we model the dynamics of $w(t)$ as the output of a dynamic network?





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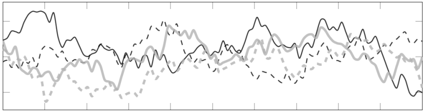
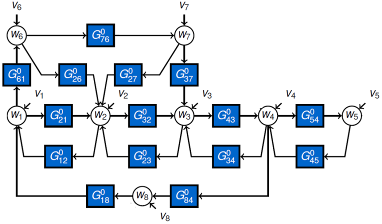
Main question

Can we model the dynamics of $w(t)$ as the output of a dynamic network?

- ... But first, what is a dynamic network?
- How many networks can model $w(t)$?
- Are there network models that we should "prefer"?





Dynamic networks

Rationale

- Each $w_j(t) =$ combination of past values of $w_i(t)$, $i \neq j$
  $\hookrightarrow$ gain understanding of observed phenomenon via topology

Applications

- Econometrics: how stock price w_i influences price w_j
- Brain networks: how brain area w_i influences area w_j

Dynamic networks

Rationale

- Each $w_j(t) =$ combination of past values of $w_i(t)$, $i \neq j$
  $\hookrightarrow$ gain understanding of observed phenomenon via topology
- ... Plus a noise component $\equiv$ how $w_j(t)$ differs from $w_i(t)$

Applications

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What is exactly the role of noise?

Related work

Identifiability

Find conditions such that a network is uniquely determined from spectral data
$\hookrightarrow$ Weerts et al. (2018), Hendrickx et al. (2018)

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Connect representations in state space (manifest + latent variables) to transfer-function ones
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Connect representations in state space (manifest + latent variables) to transfer-function ones
$\hookrightarrow$ Goncalves & Warnick (2008), Hayden et al. (2016)

Topology reconstruction

Find connections among network components from data
$\hookrightarrow$ Materassi & Innocenti (2010), Materassi & Salapaka (2012)

Innovation model

Characterization of $w(t)$

Innovation Model (IM) of $w(t)$:

$$w(t) = \Gamma(z)\varepsilon(t)$$

with:

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with:

- $\varepsilon(t)$ white noise, $\mathbf{E}[\varepsilon(t)\varepsilon(t)^\top] = \Lambda$
- $\Gamma(z) \in \mathbb{C}^{p\times p}$ such that ($\equiv$ canonical spectral factor):
    - $\Phi(z) = \Gamma(z)\Lambda\Gamma^*(z)$
    - $\Gamma(z)$ and $\Gamma^{-1}(z)$ are stable
    - $\Gamma(\infty) = I_p$

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$\varepsilon(t) =$ **innovation of** $w(t)$

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Definition: Dynamic Network Model (DNM)

$$w(t) = \tilde{G}(z)w(t) + \tilde{H}(z)\tilde{e}(t)$$

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An Innovation–driven DDM (IDNM) is a DNM with $\tilde{e}(t) = \varepsilon(t)$

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- $H(z) \in \mathbb{C}^{p \times p}$ is such that
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Definition

An Innovation-driven DDNM (IDDNM) is a DDNM with $e(t) = \varepsilon(t)$.

Examples of innovation-driven networks

IM:

$$w(t) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix} \varepsilon(t)$$

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DNM:

$$w(t) = \begin{bmatrix} 0 & \tilde{G}_{12} & 0 \\ \tilde{G}_{21} & 0 & 0 \\ 0 & \tilde{G}_{32} & 0 \end{bmatrix} w(t) + \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} \\ \tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} \\ \tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33} \end{bmatrix} \varepsilon(t)$$

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Which of these networks are “unique”?

Innovation-driven representations: Properties

Proposition

A DNM such that

(i) $\tilde{H}^{-1}(z)$, $\tilde{H}^{-1}(z)\tilde{G}(z)$ and $(I - \tilde{G}(z))^{-1}\tilde{H}(z)$ are stable

is an IDNM, i.e. $\tilde{e}(t) = \varepsilon(t)$

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The converse is not true: It is only for DDNMs

Proposition

For a DDNM, the following two conditions are equivalent:

(i) $H^{-1}(z)$, $H^{-1}(z)G(z)$ and $(I - G(z))^{-1}H(z)$ are stable

(ii) $e(t) = \varepsilon(t)$, i.e. the model is an IDDNM

Construction of an IDDNM

Proposition

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- The relation $(G(z), H(z)) \longleftrightarrow \Gamma(z)$ is

$$
\begin{aligned}
G_{ij}(z) &= -\left(\left[\Gamma^{-1}(z)\right]_{ii}\right)^{-1}\left[\Gamma^{-1}(z)\right]_{ij} \\
H_{ii}(z) &= \left[\Gamma^{-1}(z)\right]_{ii}
\end{aligned}
$$

$\implies$ an IDDNM is unique

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$\Longrightarrow$ an IDDNM is unique

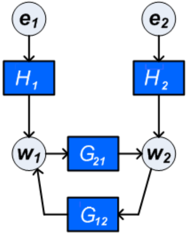
We can build a unique dynamic network driven by innovation from spectral data

Two-node case

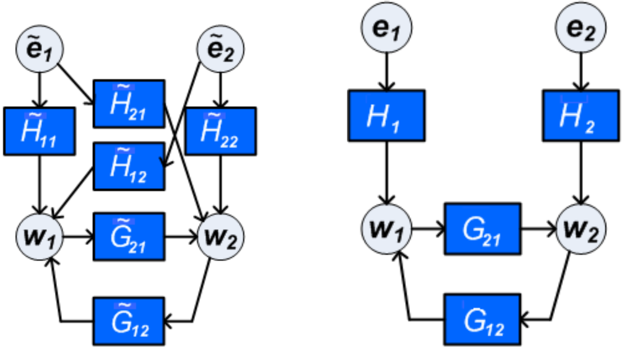
If $p = 2$, then

$$G_{12}(z) = \Gamma_{22}^{-1}(z)\Gamma_{12}(z) \qquad H_{11}(z) = \Gamma_{11}(z) - \Gamma_{12}(z)\Gamma_{22}^{-1}(z)\Gamma_{21}(z)$$
$$G_{21}(z) = \Gamma_{11}^{-1}(z)\Gamma_{21}(z) \qquad H_{22}(z) = \Gamma_{22}(z) - \Gamma_{21}(z)\Gamma_{11}^{-1}(z)\Gamma_{12}(z)$$

$\longrightarrow$ Expression of Anderson & Gevers (1981) for feedback representation of processes



Two-node network



Problem

How to transform network in the left (DNM) to network in the right (DDNM)?

Transformation

Proposition

Fix $e(t) = \tilde{e}(t)$. DNM $\longrightarrow$ DDNM via

$$
\begin{aligned}
G_{21} &= \frac{\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21}}{\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12}} \qquad G_{12} = \frac{\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12}}{\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21}} \\
H_1 &= \frac{(\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})}{(1 - \tilde{G}_{12}\tilde{G}_{21})} - \frac{(\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21})(\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12})}{(1 - \tilde{G}_{12}\tilde{G}_{21})(\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})} \\
H_2 &= \frac{(\tilde{H}_{22} + \tilde{H}_{12}\tilde{G}_{21})}{(1 - \tilde{G}_{12}\tilde{G}_{21})} - \frac{(\tilde{H}_{12} + \tilde{H}_{22}\tilde{G}_{12})(\tilde{H}_{21} + \tilde{H}_{11}\tilde{G}_{21})}{(1 - \tilde{G}_{12}\tilde{G}_{21})(\tilde{H}_{11} + \tilde{H}_{21}\tilde{G}_{12})}.
\end{aligned}
$$

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Implications: identifiability with confounding variables

P.M.J. Van Den Hof, A.G. Dankers, H.H.M. Weerts. From closed-loop identification to dynamic networks: generalization of the direct method *IEEE CDC 2017*

Summary

- New insights on dynamic network modeling of stationary processes
- Diagonal noise $\Longrightarrow$ Unique model
- Particularly interesting when noise $\equiv$ innovation

Conclusions and open questions

Summary

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- Diagonal noise $\Longrightarrow$ Unique model
- Particularly interesting when noise $\equiv$ innovation

Open questions

- Relation with other models (Materassi, Goncalves, ...)?
- Can we use the results for estimation?
- How to include reference signals?

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