Graph-Theoretical Methods for Identifiability Analysis and Synthesis in Dynamic Networks

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The 2021 European Control Conference (Virtual Conference)

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Outline

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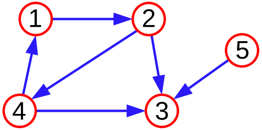
2. Vertex-Disjoint Paths
3. Disconnecting Set
   - Generic Identifiability Analysis for Single Module
   - Excitation Allocation for Single Module Identifiability
4. Pseudotree Covering
   - Generic Identifiability Analysis for Full Network
   - Excitation Allocation For Full Network Identifiability
5. Conclusions

Graph Theory

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with node set $\mathcal{V} = \{1, 2, \cdots, L\}$, and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$



$$\mathcal{V} = \{1, 2, 3, 4, 5\}$$

$$\mathcal{E} = \{(1, 2), (2, 3), (2, 4), (4, 1), (4, 3), (5, 3)\}$$

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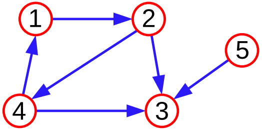

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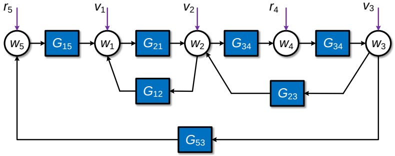
Terminologies:

- $\mathcal{N}_k^-$: in-neighbors of k, e.g., $\mathcal{N}_3^- = \{2, 4, 5\}$
  $\mathcal{N}_k^+$: out-neighbors of k, e.g., $\mathcal{N}_2^+ = \{3, 4\}$
- source: a node without in-neighbors, e.g., 5;
  sink: a node without out-neighbors, e.g., 3;
- **path**: a sequence of nodes where each node appears at most once
  e.g., $1 \to 2 \to 4 \to 3$ (*a single node can also be viewed as a path*)

Graph Representation of Dynamic Networks

Consider a dynamic network $w(t) = G(q,\theta)w(t) + Rr(t) + H(q,\theta)e(t)$[1]



Dynamic network with process noises v_1 and v_2 correlated

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 \\ 0 & 0 & H_{33}(\theta) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[1] In this talk, all the modules in $G(q)$ are parameterized

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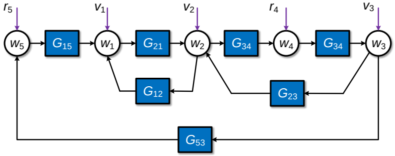

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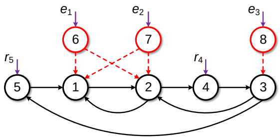


Extended graph $\mathcal{G}$: dashed edges indicate the noise corelation strucutre

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, H(\theta) = \begin{bmatrix} H_{11}(\theta) & H_{12}(\theta) & 0 \\ H_{21}(\theta) & H_{22}(\theta) & 0 \\ 0 & 0 & H_{33}(\theta) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{ji} \neq 0 \Rightarrow (i, j) \in \mathcal{G}$$

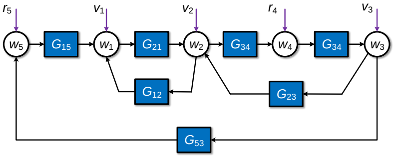
$$H_{i,k} \neq 0 \Rightarrow (i, k + L) \in \mathcal{G}$$

$\mathcal{R} = \{4, 5, 6, 7, 8\}$: excited nodes

[1] In this talk, all the modules in $G(q)$ are parameterized

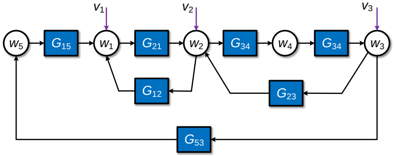
Problems

Analysis problem: generic identifiability condition



Under what conditions (i.e., the number and locations of external excitation signals), $\mathcal{M}$ is generically identifiable?

Synthesis problem: allocation of excitation signals



Given a network without external excitation signals, where to allocate excitation signals to achieve generic identifiability?

Vertex-Disjoint Paths

- Two paths in a graph $\mathcal{G}$ are vertex-disjoint if they do not share any common node.

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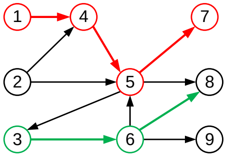
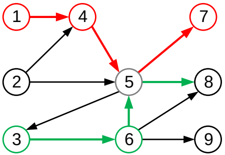
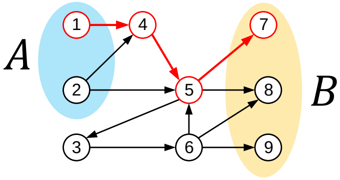


Node-disjoint



Not Node-disjoint

- $b_{\mathcal{A}\to\mathcal{B}}$: the maximum number of vertex-disjoint paths from node set $\mathcal{A}$ to node set $\mathcal{B}$



$$\mathcal{A} = \{1, 2\},\ \mathcal{B} = \{7, 8, 9\}$$
$$\Rightarrow b_{\mathcal{A}\to\mathcal{B}} = 1$$

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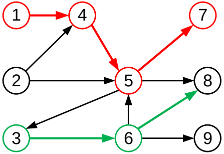

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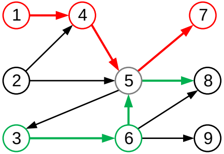

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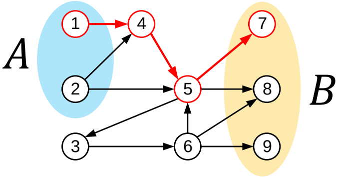


$$\mathcal{A} = \{1,2\},\ \mathcal{B} = \{7,8,9\}$$
$$\Rightarrow b_{\mathcal{A}\to\mathcal{B}} = 1$$

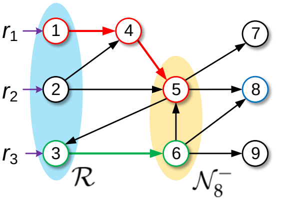
The largest set of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$ is not unique!

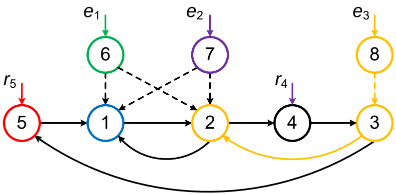
Path-Based Condition

Let $\mathcal{R} \subseteq \mathcal{V}$ be the set of excited nodes, and all the nodes in $\mathcal{V}$ are measured.

Theorem (Path-based condition)

The transfer functions from $\mathcal{N}_j^-$ to j, i.e. $G_{j,\mathcal{N}_j^-}$ are generically identifiable if and only if there are $|\mathcal{N}_j^-|$ node disjoint paths from $\mathcal{R}$ to $\mathcal{N}_j^-$.
The model set $\mathcal{M}$ is generically identifiable, if the above condition holds for all $i \in \mathcal{V}$.



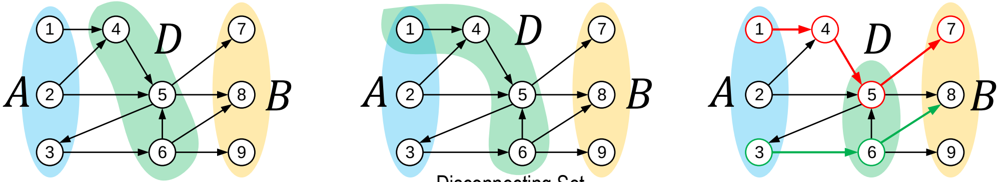


$\mathcal{R} = \{4, 5, 6, 7, 8\}$,
$\mathcal{N}_1^- = \{2, , 5, 67\}$,
$\Rightarrow b_{\mathcal{R}\to\mathcal{N}_1^-} = 4 = |\mathcal{N}_1^-|$

[2][1] Van der Woude, 1991; [2] Hendrickx, Gevers & Bazanella, CDC 2017; [3] Weerts et al., CDC 2018

Disconnecting Set

- In $\mathcal{G}$, a set of nodes $\mathcal{D}$ is a **disconnecting set** from $\mathcal{A}$ to $\mathcal{B}$, if removing $\mathcal{D}$ leads to no paths from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{D}$ may share common vertices in the sets $\mathcal{A}$ and $\mathcal{B}$.)
- $\mathcal{D}$ is a **minimum disconnecting set** from $\mathcal{A}$ to $\mathcal{B}$ if $|\mathcal{D}|$ is minimum



Disconnecting Set

Disconnecting Set
(A and D have overlap)

Minimum Disconnecting Set

[3] Schrijver 2003

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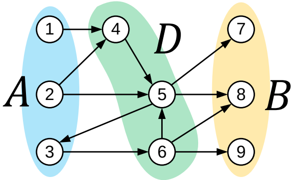

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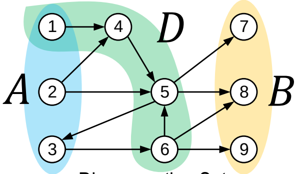

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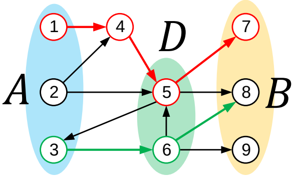


Minimum Disconnecting Set

- Menger's theorem: $|\mathcal{D}| = b_{\mathcal{X}\to\mathcal{Y}}$
  - $\mathcal{D}$: a minimum disconnecting set from $\mathcal{A}$ to $\mathcal{B}$
  - $b_{\mathcal{X}\to\mathcal{Y}}$: the maximum number of vertex-disjoint paths from $\mathcal{A}$ to $\mathcal{B}$.

[3] Schrijver 2003

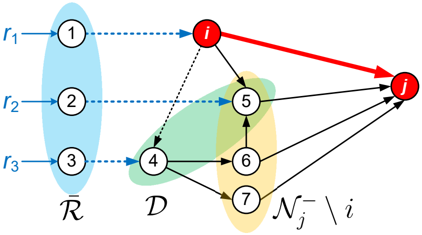
A Disconnecting Set Condition (Single Module)

Let $\mathcal{R} \subseteq \mathcal{V}$ be the set of excited nodes, and all the nodes in $\mathcal{V}$ are measured.

Theorem (Disconnecting set condition)

A single module G_{ji} is generically identifiable if and only if for some $\bar{\mathcal{R}} \subseteq \mathcal{R}$, there is a disconnecting set $\mathcal{D}$ from $i \cup \bar{\mathcal{R}}$ to the other in-neighbors of j (i.e. $\mathcal{N}_j^- \setminus i$) such that

$$b_{\bar{\mathcal{R}} \to \mathcal{D} \cup i} = |D| + 1$$



$\bar{\mathcal{R}} = \{1, 2, 3\}$, $\mathcal{N}_j^- \setminus i = \{5, 6, 7\}$,

$\mathcal{D} = \{4, 5\}$

$\Rightarrow b_{\bar{\mathcal{R}} \to \mathcal{D} \cup i} = b_{\{1,2,3\} \to \{4,5,i\}} = |D| + 1 = 3$

G_{ji} is generic identifiable!

[4] [1] Shi et al., IFAC-PapersOnLine 2020; [2] Shi et al., arXiv:2008.01495

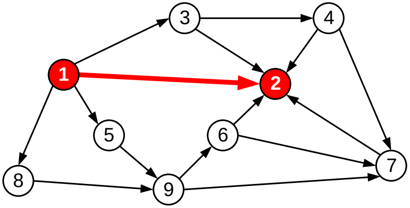
Allocation of Excitation Signals

Problem

How to allocate a minimum number of excitation signals for G_{ji} to be generic identifiable?

The disconnecting set result leads to a simple synthesis procedure:

- **Step 1**: Find a minimum disconnecting set $\mathcal{D}$ from $\mathcal{N}_i^+$ to $\mathcal{N}_j^- \setminus i$.
- **Step 2**: Assign independent external signals at $\mathcal{D}$ and i.



Target module: G_{21}

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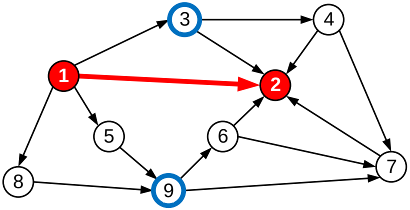

Target module: G_{21}

- Observe that $\mathcal{N}_1^+ = \{2, 3, 5, 8\}$, $\mathcal{N}_2^- \setminus \{1\} = \{3, 4, 6, 7\}$
- Find a minimum disconnecting set from $\mathcal{N}_1^+$ to $\mathcal{N}_2^- \setminus \{1\}$: $\mathcal{D} = \{3, 9\}$

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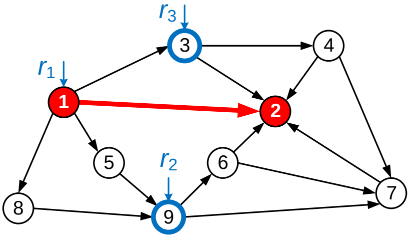

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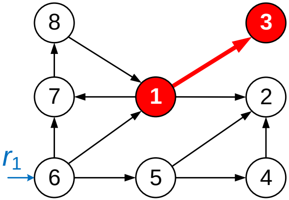
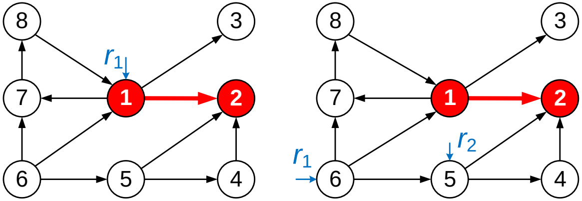
- Find a minimum disconnecting set from $\mathcal{N}_1^+$ to $\mathcal{N}_2^- \setminus \{1\}$: $\mathcal{D} = \{3, 9\}$
- Excite $\mathcal{D}$ and node 1

Allocation of Excitation Signals - Special Cases

Find a disconnecting set $\mathcal{D}$ from $i \cup \bar{\mathcal{R}}$ to $\mathcal{N}_j^- \setminus i$ such that $b_{\bar{\mathcal{R}} \to \mathcal{D} \cup i} = |D| + 1$ holds for some excited nodes $\bar{\mathcal{R}}$.



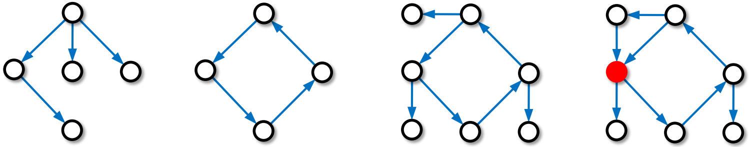


Case 1: $\mathcal{N}_j^- \setminus i = \emptyset$, i.e. i is the only in-neighbor of j
$\mathcal{D} = \emptyset$, then $b_{\bar{\mathcal{R}} \to i} = 1$.
Excite any node that has a path to i

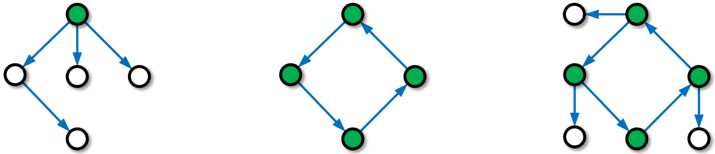
Case 2: there is no path from i to $\mathcal{N}_j^- \setminus i$
(1) Excite node i, $\mathcal{D} = \emptyset$, and $b_{\bar{\mathcal{R}} \to i} = 1$.
(2) Excite any node k having a path to i and
If k has a path to $\mathcal{N}_j^- \setminus i$, then excite a disconnecting set $\mathcal{D}$ from $\mathcal{N}_k^+$ to $\mathcal{N}_j^- \setminus i$

Pseudotrees

- A connected directed graph $\mathcal{T}$ is called a (directed) **pseudotree** if $|\mathcal{N}_i^-| \leq 1$, for every node i in $\mathcal{T}$.
  (including directed tree, directed cycle, or directed cycle with out-going branches)



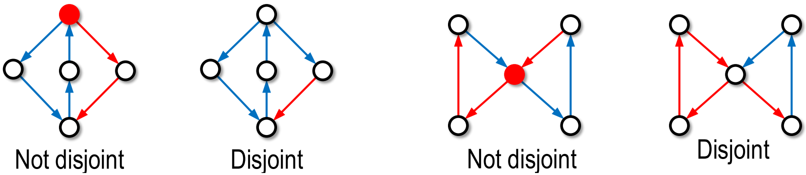
- **Roots**: nodes having directed paths to all nodes in the pseudotree



[5] [1]Cheng et al., CDC2019, [2] Cheng et al., TAC2022, to appear

Disjoint Pseudotrees

- A set of pseudotrees are called **disjoint** if
  1. The pseudotrees *do not share common edges*
  2. All the edges leaving from the same node are in the same pseudotree



- Any directed network can be decomposed into a set of disjoint pseudotrees

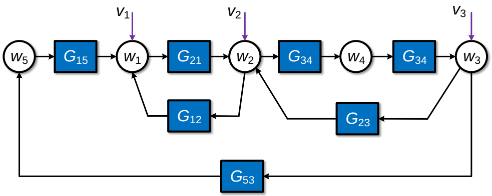
[5] [1]Cheng et al., CDC2019, [2] Cheng et al., TAC2022, to appear

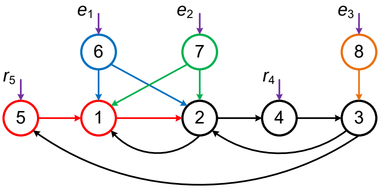
A Pseudotree Covering Condition (Full Network)

Theorem (Pseudo-tree covering condition)

A network model set $\mathcal{M}$ is generically identifiable if

- *there are a set of disjoint pseudotrees covering all the edges of $\mathcal{G}$, and*
- *and at least one of the roots in each pseudotree is an excited node.*





Generically identifiable!

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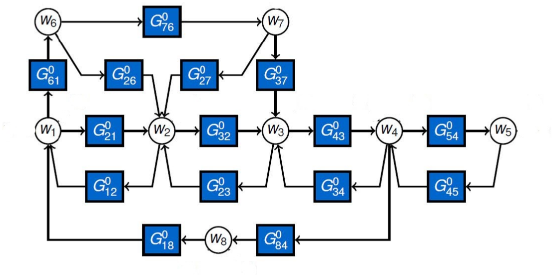
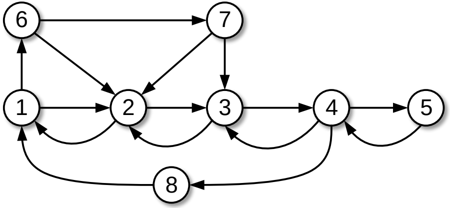
Excitation Allocation in a Full Network

Procedure to allocate a minimum number of excitation signals:

- **Step 1**: Find a minimum number of disjoint pseudotrees that cover all the edges.
- **Step 2**: Assign an independent excitation signal at a root of each pseudotree.
- **Step 3**: Remove redundant excitation signals if generic identifiability remains.





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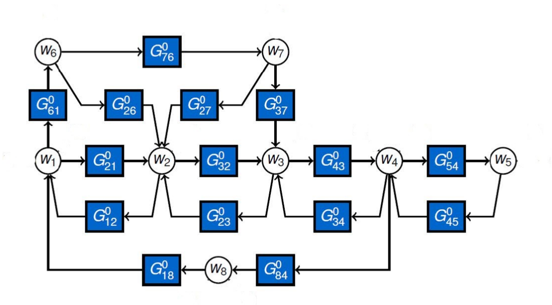
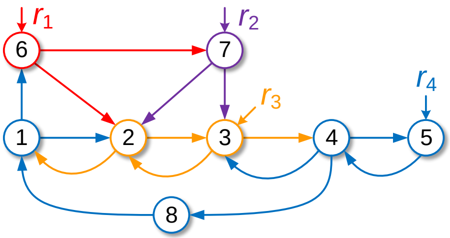
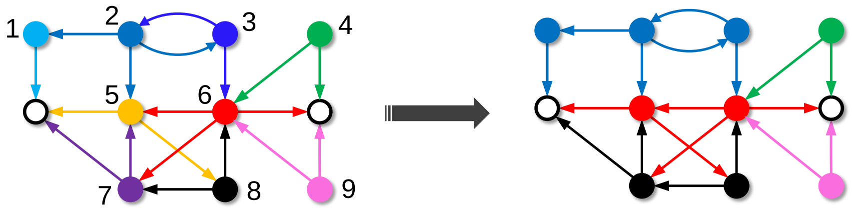




Pseudotree Merging Algorithm

Algorithm Steps:

1. Partition $\mathcal{G}$ into a set of minimal pseudotree.
   A minimal pseudotree contains one root and all the edges from it to its out-neighbors.
2. Iteratively merge two pseudotrees if they are mergeable.
   $\mathcal{T}_i$ is mergeable to $\mathcal{T}_j$ if their union remains a pseudotree, and the roots of $\mathcal{T}_j$ have paths to every node in $\mathcal{T}_i$. (e.g., $\mathcal{T}_1$ is mergeable to $\mathcal{T}_2$, but not vise versa)



[6] Any directed network can be covered by a set of disjoint minimal pseudotrees.

Mergeability

- Define the set $\mathbb{M} := \{1, \varnothing, 0\}$ with **operation rule** on $\mathbb{M}$:

$$1 \odot 1 = 1,\ 1 \odot 0 = 0,\ 1 \odot \varnothing = 1,\ 0 \odot 0 = 0,\ \varnothing \odot 0 = 0,\ \varnothing \odot \varnothing = \varnothing.$$

- **Mergeability matrix** $\mathcal{M}$:

$$\mathcal{M}_{ij} = \begin{cases} 1 & \text{if } \mathcal{T}_i \text{ is mergeable to } \mathcal{T}_j; \\ \varnothing & \text{if } \mathcal{T}_i \text{ and } \mathcal{T}_j \text{ do not share nodes;} \\ 0 & \text{otherwise.} \end{cases}$$

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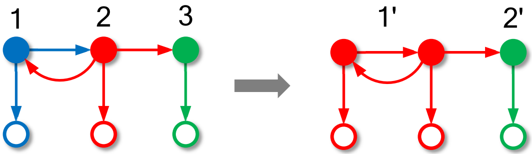


$$\mathcal{M} = \begin{bmatrix} 0 & 1 & \varnothing \\ 1 & 0 & 0 \\ \varnothing & 1 & 0 \end{bmatrix} \Rightarrow \hat{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$0 \odot 1 \odot 1 \odot 0 = 0$

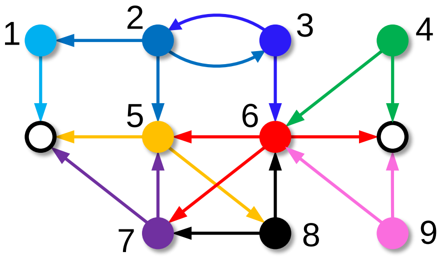
$\varnothing \odot 0 = 0$

$\varnothing \odot 1 = 1$

Algebraic operations on the matrix correspond to the merging in the graph!

Pseudotree Merging Algorithm

- **Procedure**: find an entry $\mathcal{M}_{ij} = 1$, and merge the i-th and j-th columns and rows
- **Policy**: First select $\mathcal{M}_{ij} = 1$ when row i has less "1" but more "$\varnothing$"
- **Stop**: if there is no "1" entry in $\mathcal{M}$



$$
\begin{array}{c}
\begin{array}{ccccccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_7 & \mathcal{T}_8 & \mathcal{T}_9 \end{array} \\
\begin{pmatrix}
0 & 1 & \varnothing & \varnothing & 0 & \varnothing & 0 & \varnothing & \varnothing \\
0 & 0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\
\varnothing & 1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
\varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
0 & 1 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\
\varnothing & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varnothing & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\
\varnothing & \varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0
\end{pmatrix}
\begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \\ \mathcal{T}_7 \\ \mathcal{T}_8 \\ \mathcal{T}_9 \end{array}
\end{array}
$$

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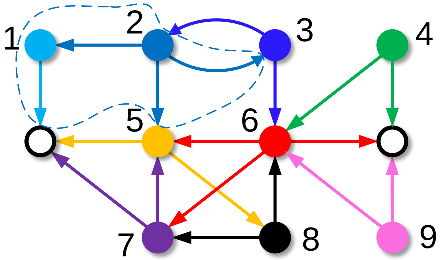

$$
\begin{array}{c c}
\begin{matrix} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_7 & \mathcal{T}_8 & \mathcal{T}_9 \end{matrix} & \\
\begin{pmatrix}
0 & 1 & \varnothing & \varnothing & 0 & \varnothing & 0 & \varnothing & \varnothing \\
0 & 0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\
\varnothing & 1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
\varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
0 & 1 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\
\varnothing & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varnothing & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\
\varnothing & \varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\varnothing & \varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0
\end{pmatrix} &
\begin{matrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \\ \mathcal{T}_7 \\ \mathcal{T}_8 \\ \mathcal{T}_9 \end{matrix}
\end{array}
$$

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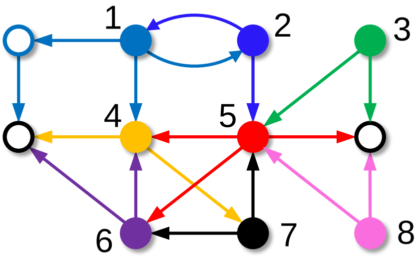

$$
\begin{array}{c c}
\begin{array}{cccccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_7 & \mathcal{T}_8 \end{array} & \\
\begin{pmatrix}
0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\
1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
\varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
0 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varnothing & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\
\varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0
\end{pmatrix} &
\begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \\ \mathcal{T}_7 \\ \mathcal{T}_8 \end{array}
\end{array}
$$

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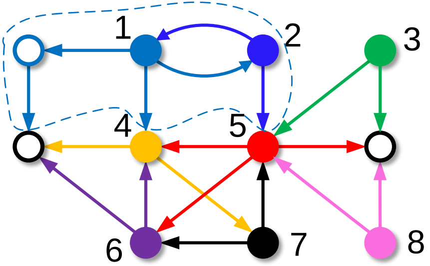

$$
\begin{array}{c}
\begin{array}{cccccccc}
\mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_7 & \mathcal{T}_8
\end{array} \\
\begin{pmatrix}
0 & 1 & \varnothing & 0 & 0 & 0 & \varnothing & \varnothing \\
1 & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
\varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
0 & \varnothing & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varnothing & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\
\varnothing & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\varnothing & 0 & 0 & \varnothing & 0 & \varnothing & 0 & 0
\end{pmatrix}
\end{array}
\begin{array}{c}
\\ \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \\ \mathcal{T}_7 \\ \mathcal{T}_8
\end{array}
$$

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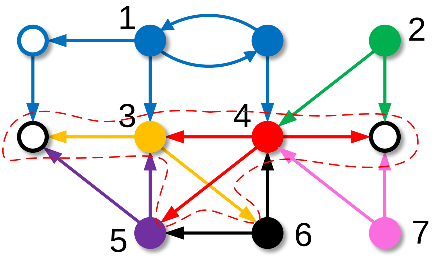

$$
\begin{array}{c}
\begin{array}{ccccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 & \mathcal{T}_7 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varnothing & 0 & \varnothing & 0 & 0 \\
0 & \varnothing & 0 & 1 & 0 & 0 & \varnothing \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varnothing & 0 & 0 & 0 & 1 & \varnothing \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \varnothing & 0 & \varnothing & 0 & 0
\end{pmatrix}
\end{array}
\begin{array}{c} \\ \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \\ \mathcal{T}_7 \end{array}
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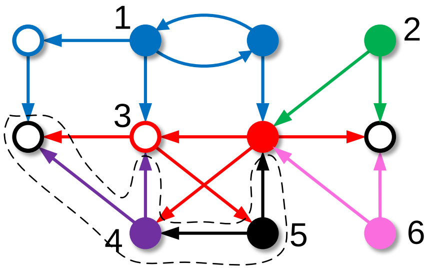

$$
\begin{array}{c c}
\begin{array}{cccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 & \mathcal{T}_6 \end{array} & \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varnothing & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \varnothing & 0 & 0 & 1 & \varnothing \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varnothing & 0 & 0
\end{pmatrix} &
\begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \\ \mathcal{T}_6 \end{array}
\end{array}
$$

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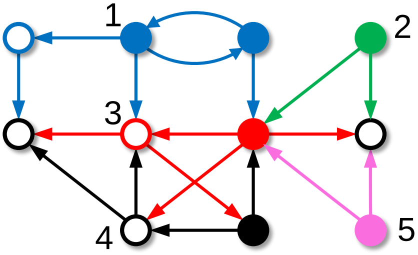
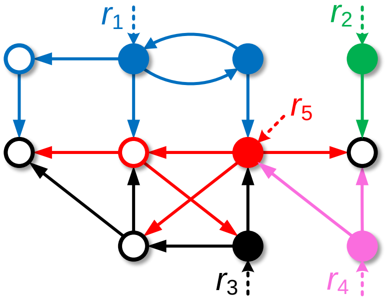


$$
\begin{array}{c|c}
 & \mathcal{T}_1 \quad \mathcal{T}_2 \quad \mathcal{T}_3 \quad \mathcal{T}_4 \quad \mathcal{T}_5 \\
\begin{matrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \end{matrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

Excitation Allocation Based on Pseudotrees Covering

1. Excite a root in every pseudotree
2. Remove redundant excitation signals
   - r_k is redundant, if the network remains generic identifiability after removing r_k.
   - Only check $\mathcal{T}_k$ using the generic identifiability conditions based on vertex-disjoint paths



After removing r_5, the nodes in the red pseudotree satisfy the path-based condition

$\Rightarrow$ r_5 is redundant and thus can be removed

Excitation Allocation Based on Pseudotrees Covering

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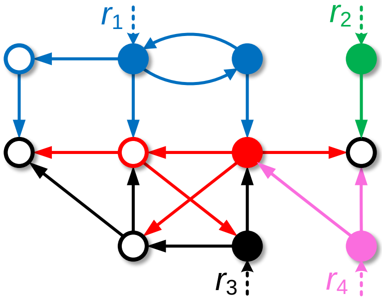

Generic identifiability with 4 excitations!

1. Problem Setting
2. Vertex-Disjoint Paths
3. Disconnecting Set
   - Generic Identifiability Analysis for Single Module
   - Excitation Allocation for Single Module Identifiability
4. Pseudotree Covering
   - Generic Identifiability Analysis for Full Network
   - Excitation Allocation For Full Network Identifiability
5. Conclusions

Conclusions

- **Graph-theoretical analysis** for identifiability of dynamic network
  (1) *vertex-disjoint paths*; (2) *disconnecting sets*; (3) *disjoint pseudotree covering*
- **Signal allocation** for generic identifiability: *minimum disconnecting set* (single module), *pseudotree merging algorithm* (full network)

Relevant works and generalizations

- Identifiability for more general model sets: correlated noises, non-parameterized (fixed) or switching modules [1], [2], [3]
- Identifiability results for dynamic networks in the setting of partial excitation and partial measurement [2], [4], [5]
- Conditions for global identifiability [6] and local generic identifiability [7]

[1] [1] Cheng et al., TAC2022; [2] Shi et al., arXiv:2008.01495; [3] Dreef et al. SYSID 2021 [4] Bazanella, CDC 2019; [5] Cheng et al., arXiv:2105.03187; [6] van Waarde et al., TAC 2019 [7] Legat et al., CDC2020

Relevant References

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Thanks for your attention!