On the choice of uncertainty structure in identification for robust control

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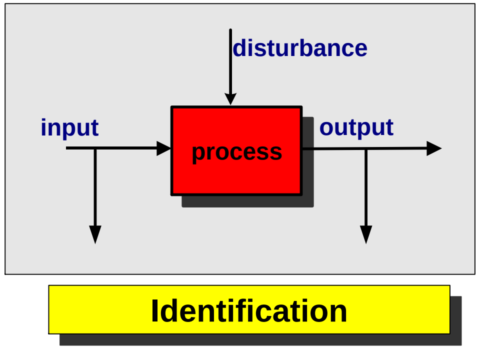
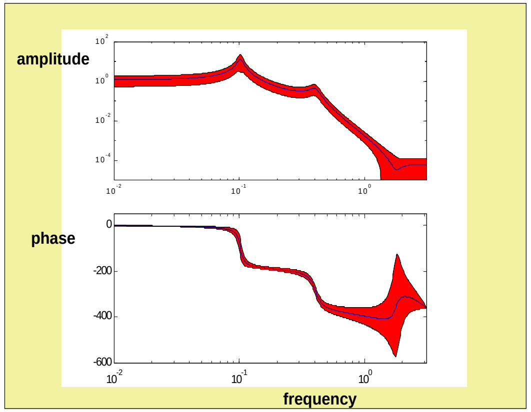
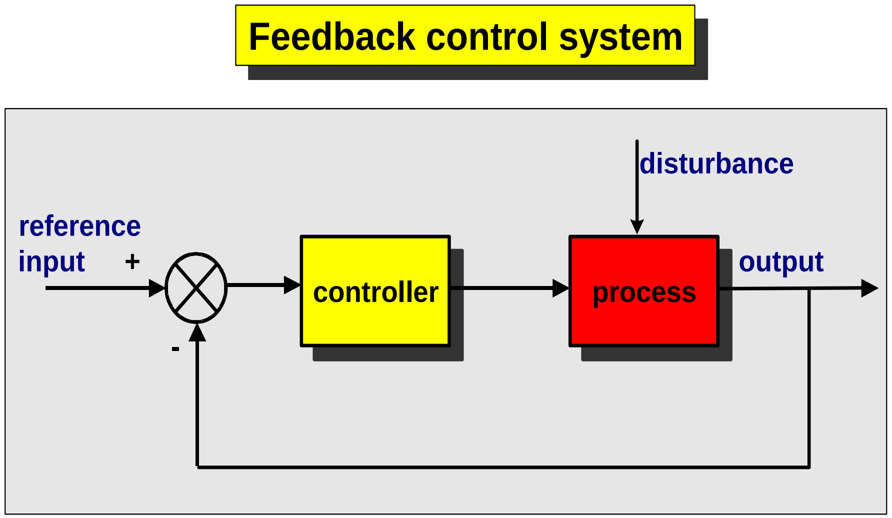
System Identification for Robust Control



Data → Model



Model → Controller



Relevance of model uncertainty

- **Determines the achievable robust stability/performance $\rightarrow$ reduce uncertainty in control-relevant area**
- **Guidelines for appropriate identification after dedicated experiment design**
  - **experimental data and priors determine set of unfalsified models**
  - **identification technique determines nominal estimate**
  - **model uncertainty bound additionally determined by choice of representation**

Model uncertainty representation

In Control

- **unstructured additive, multiplicative ($\mathcal{H}_\infty$-norm bounded)**
- **real parametric**
- **Youla parameter**
- **gap, ν-gap metric**

In Identification

- **parametric uncertainty (statistical or worst-case)**
  (e.g. Ljung (1987), Milanese et al. (1996), Bombois et al. (2001))
- **additive frequency response bounds ($\mathcal{H}_\infty$-norm)**
  (e.g. Goodwin et al. (1992), Hakvoort et al. (1997), Chen and Gu (2000)) **on open-loop or closed-loop model**

Problem Formulation

Question to be considered:

Do robust stability/performance requirements in a particular control problem motivate the use of a specific uncertainty structure in identification?

Is there a best uncertainty structure for identification?

In this presentation:

- **Some (relevant) thoughts and aspects for SISO LTI systems**
- **Equivalences / differences between uncertainty sets**
- **Analytical expressions for performance (analysis / synthesis)**

Uncertainty Structures

- **Additive uncertainty set**

$$\mathcal{G}_a(G_x, W_a) := \{G_\Delta(s) \mid G_\Delta(s) = G_x(s) + \Delta_a(s) \ , \quad |\Delta_a(i\omega)| \le |W_a(i\omega)| \quad \forall \omega \in \mathbb{R}\}$$

- **Dual-Youla uncertainty set**

$$\mathcal{G}_Y(G_x, C, Q, Q_c, W_Y) := \left\{ G_\Delta(s) \mid G_\Delta(s) = \frac{\bar{N}_x(s) + \bar{D}_c(s)\Delta_G(s)}{\bar{D}_x(s) - \bar{N}_c(s)\Delta_G(s)} \ , \quad \left|Q_c^{-1}(i\omega)\Delta_G(i\omega)Q(i\omega)\right| \le |W_Y(i\omega)| \quad \forall \omega \in \mathbb{R} \right\} .$$

Both define (for each frequency) circular uncertainty regions in the complex plane.

Uncertainty Structures

The same circular property holds for

- ν-gap sets

$$\mathcal{G}_\nu(G_x, W_\nu) := \{G_\Delta(s) \mid \kappa\left(G_\Delta(i\omega), G_x\left(i\omega\right)\right) \leq |W_\nu(i\omega)| \quad \forall \omega \in \mathbb{R}\}$$

with κ the chordal distance,

$$\kappa(G_\Delta\left(i\omega\right), G_x\left(i\omega\right)) := \frac{\left|G_x\left(i\omega\right) - G_\Delta\left(i\omega\right)\right|}{\sqrt{\left(1 + \left|G_\Delta\left(i\omega\right)\right|^2\right)\left(1 + \left|G_x\left(i\omega\right)\right|^2\right)}}$$

Equivalence of Uncertainty Sets

For the Dual-Youla uncertainty set:

$$\mathcal{G}_Y(G_x, C, Q, Q_c, W_Y) = \mathcal{G}_a(G_{centre}, W_a)$$

with

$$G_{centre} = C^{-1}\left(\frac{|N_c W_Y|^2}{|D_x|^2 - |N_c W_Y|^2}\right) + G_x\left(\frac{|D_x|^2}{|D_x|^2 - |N_c W_Y|^2}\right)$$

$$W_a = \frac{|\Lambda|}{|D_x|^2 - |N_c W_Y|^2}\,|W_Y|\,.$$

$\Lambda = N_x N_c + D_c D_x$; $(N_x, D_x) = (\bar{N}_x, \bar{D}_x)Q$; $(N_c, D_c) = (\bar{N}_c, \bar{D}_c)Q_c$

Equivalence of the Uncertainty Sets

For the ν-gap uncertainty set:

$$\mathcal{G}_\nu(G_x, W_\nu) = \mathcal{G}_a(G_{centre}, W_a)$$

with

$$G_{centre} = \frac{G_x}{1 - \left(1 + |G_x|^2\right)|W_\nu|^2}$$

$$W_a = \frac{\sqrt{\left(1 - |W_\nu|^2\right)\left(|G_x|^2 + 1\right)}\,|W_\nu|}{1 - \left(1 + |G_x|^2\right)|W_\nu|^2}.$$

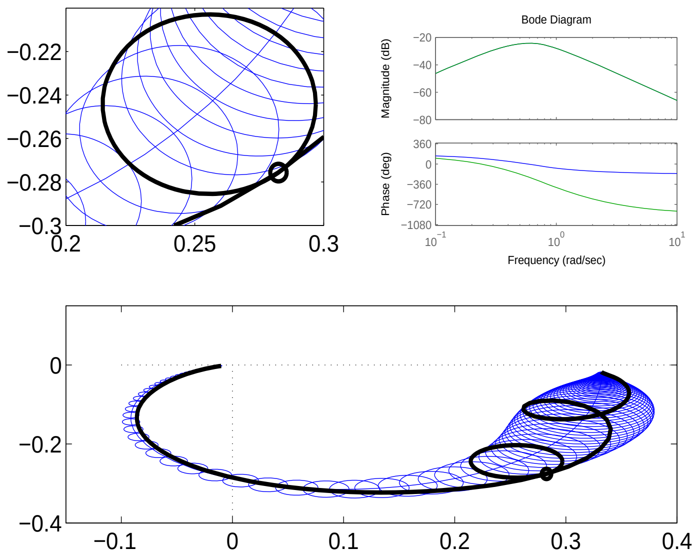
Equivalence of the Uncertainty Sets

For robustness analysis, usually additional conditions on unstable poles and zeros are imposed

- **additive:** $\Delta_a \in \mathbb{R}H_\infty$
- **Youla:** $\Delta_G \in \mathbb{R}H_\infty$
- **ν-gap:** $wno(\bar{N}_x{}^* \bar{N}_\Delta + \bar{D}_x{}^* \bar{D}_\Delta) = 0$

Equivalence of the Uncertainty Sets

For different pole/zero conditions on the transfer functions, the uncertainty set becomes a subset of the union of circles in the frequency domain.



However, every point in the union of circles in the frequency domain is always attained by at least one member of the subset.

Equivalence of the Uncertainty Sets

Every point on the boundary of the circles is reached by at least one member of the set.

Consequence:

No difference between uncertainty structures with respect to

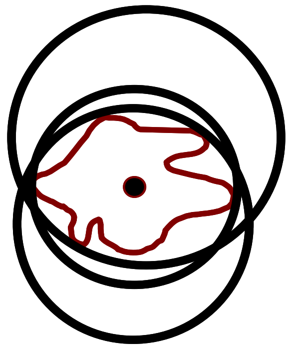
- robust stability condition $C(i\omega) \neq -G_{\Delta}^{-1}(i\omega) \quad \forall\omega$
- worst-case performance ($\|T\|_\infty < \gamma$)

Remark:

From an identification point of view realistic conditions on unstable poles and zeros are those for:
additive for open-loop and **Youla for closed-loop**.

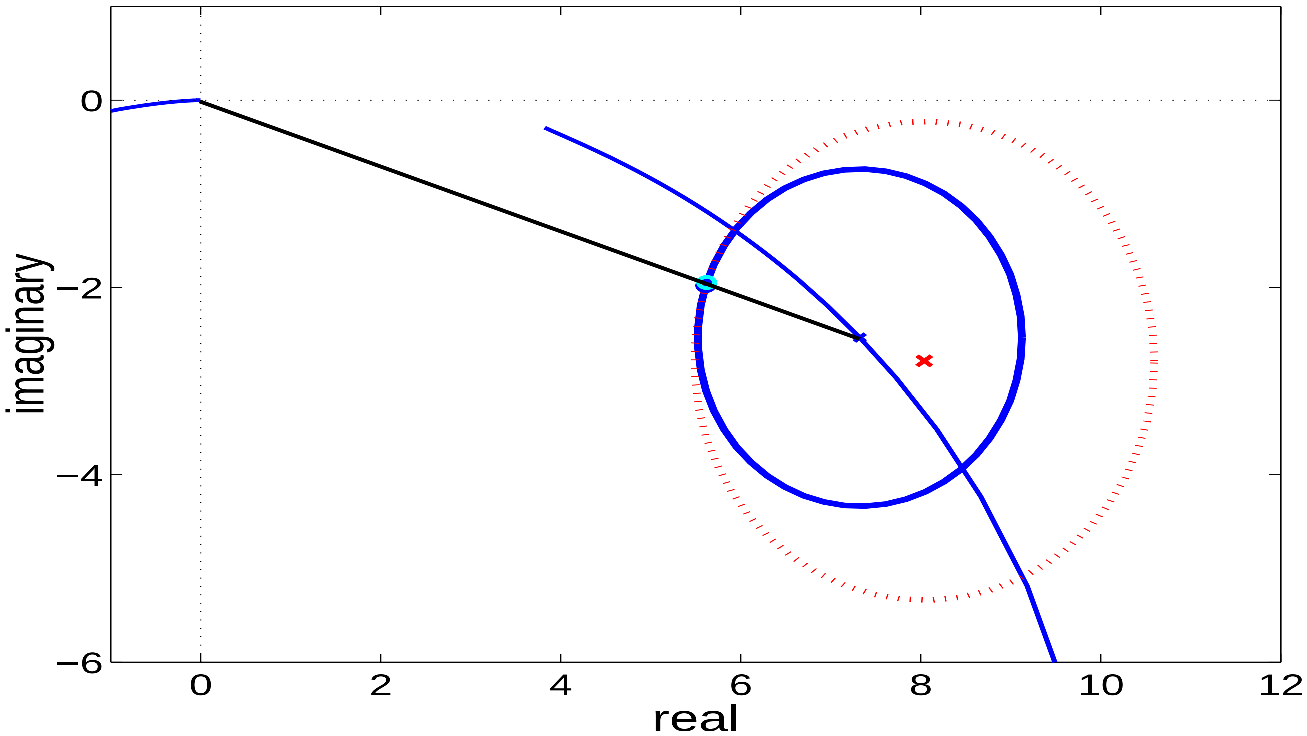
Observations from an identification perspective

- **For identification of model uncertainty** *sets* **from data, the choice of structure "does not matter"**
- **Differences occur in complexities of G_{centre} and the weighting functions**
- **For a fixed/estimated nominal model $\hat{G}_x$ bounding the uncertainty in different structures leads to different results, affecting achievable robust performance**



Observations from an identification perspective

Embedding $\mathcal{G}_a(\hat{G}_x)$ with a ν-gap set $\mathcal{G}_\nu(\hat{G}_x)$



$\star$: $\hat{G}_x$; solid: $\mathcal{G}_a(\hat{G}_x)$; dotted: $\mathcal{G}_\nu(\hat{G}_x)$; $\star$: additive center of $\mathcal{G}_\nu$

Performance analysis and synthesis

Performance functions:

- weighted H_∞-norm-bounded (bounds on amplitude or maximum singular value)
- linear fractional transformations

$$\bar{\sigma}\left(VT(G_\Delta, C)W\right) < 1$$

$$\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} G_\Delta \\ 1 \end{pmatrix} (1 + CG_\Delta)^{-1} \begin{pmatrix} C & 1 \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}.$$

LFT: circles are mapped into circles

A set based on a linear fractional transformation

$$F_u(P,\Delta) = P_{22} + P_{21}\Delta(1+P_{11}\Delta)^{-1}P_{12} \;, \quad \text{with } \left|W^{-1}\Delta\right| \leq 1$$

can equivalently be described in an additive structure:

$$F_u\,(P,\Delta) = F_{centre} + \Delta_a \quad , \quad \left|W_a^{-1}\Delta_a\right| \leq 1,$$

with

$$F_{centre} = P_{22} + \frac{-P_{21}P_{12}P_{11}^{*}\,|W|^2}{1-|P_{11}W|^2}$$

and

$$W_a = \frac{|P_{21}P_{12}|}{\left(1-|P_{11}W|^2\right)}\,|W|$$

Example

Set of complementary sensitivity functions T_Δ for a controller C and an additive uncertainty set $\mathcal{G}_a(G_x, W_a)$:

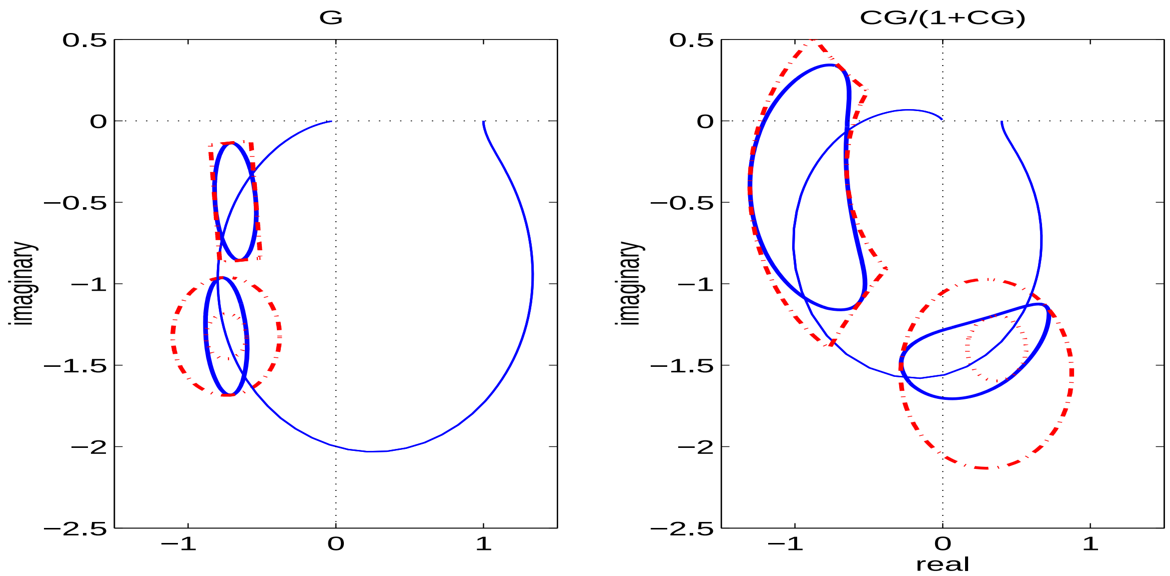
$$
\begin{aligned}
T_\Delta &= \frac{(G_x + \Delta_a)C}{1 + (G_x + \Delta_a)\,C}, \quad \left|W_a^{-1}\Delta_a\right| \leq 1 \\
&= 1 - \frac{(1 + CG_x)^{-1}}{1 - \left|(1 + CG_x)^{-1}\,CW_a\right|^2} + \Delta_T
\end{aligned}
$$

$$
|\Delta_T| \leq |W_T| = \frac{\left|(1 + CG_x)^{-2}\,C\right|}{1 - \left|(1 + CG_x)^{-1}\,CW_a\right|^2}\,|W_a|\,.
$$

Non-circular bounds

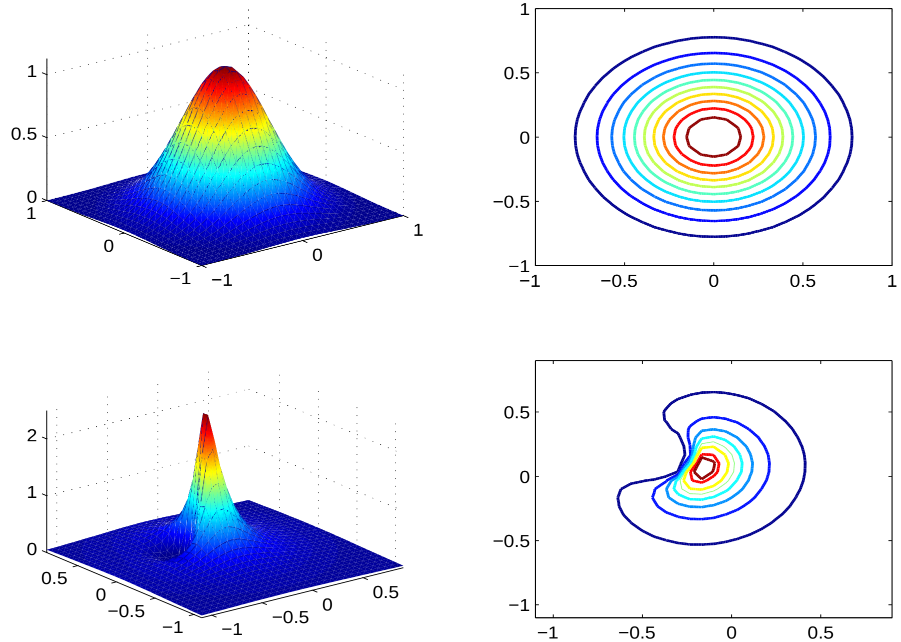
General shapes are not maintained under LFT.

For example, non-parametric uncertainty regions, e.g. confidence regions (ellipsoidal,boxed) in Nyquist curve, following a pdf:



Exception for Youla: $\frac{G_\Delta C}{1+CG_\Delta} = \frac{G_x C}{1+CG_x} + \frac{N_c D_c}{D_c D_x + N_c N_x}\Delta_G$

Mapping of probability density functions



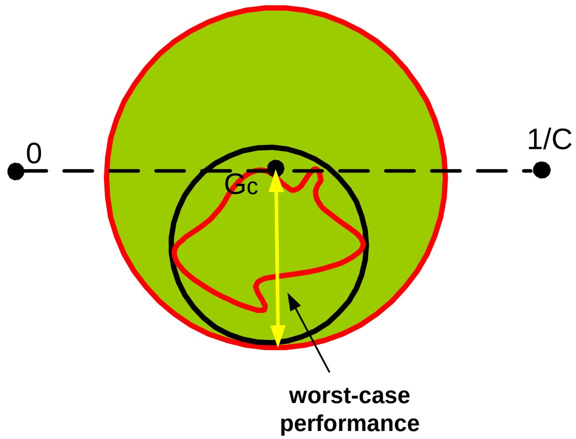
Consequences (Heath 2000)

- probability density function changes
- unbiased estimate does not imply unbiased transform

Robust Performance Analysis - analytical expressions

All plants G_Δ which achieve $\bar{\sigma}\left(VT(G_\Delta, C)W\right) < 1$ are characterized by

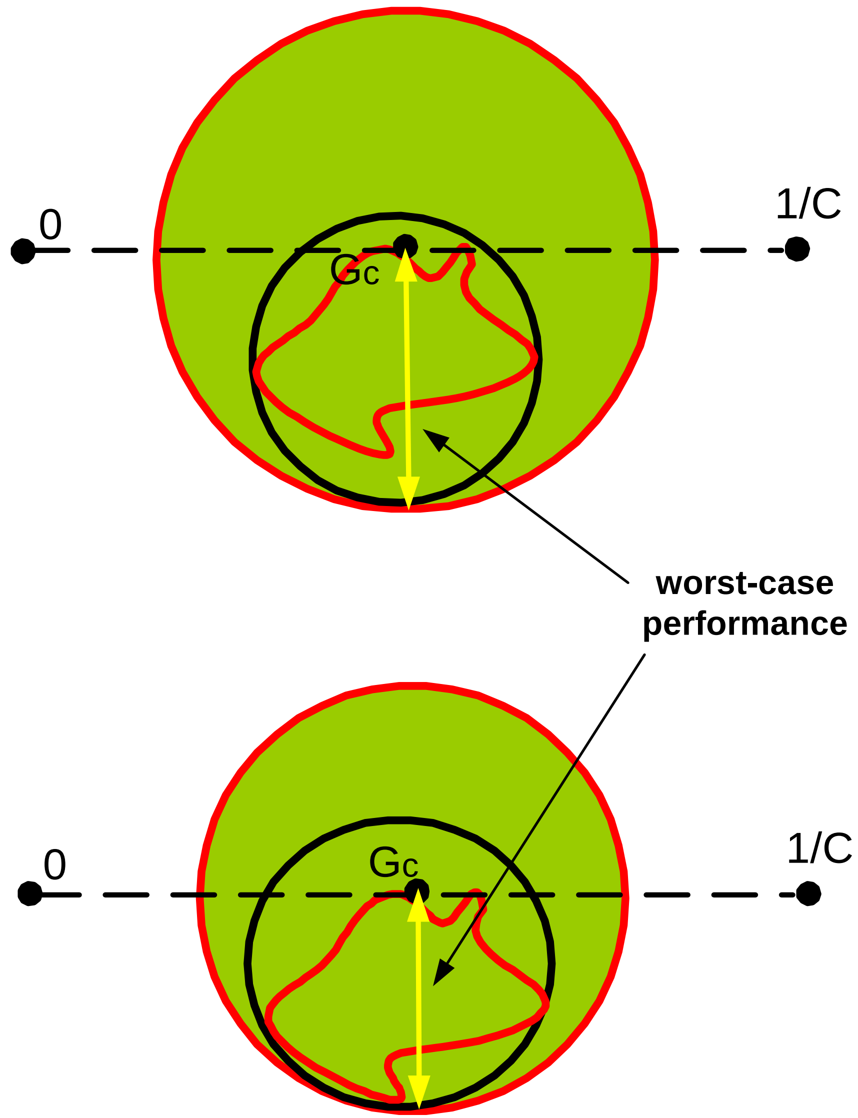
$$G_\Delta = G_{centre} + \Delta_a \quad , \quad \left|W_a^{-1}\Delta_a\right| \leq 1$$



$$G_{centre} = C^{-1}\frac{|W_Y|^2 + |V_2|^2|C|^2|V_1|^2}{|V_1|^4 - |W_Y|^2}$$

$$W_a = \left|C^{-1}\right| \frac{\left||V_2|^2|C|^2 + |V_1|^2\right|}{|V_1|^4 - |W_Y|^2} W_Y.$$

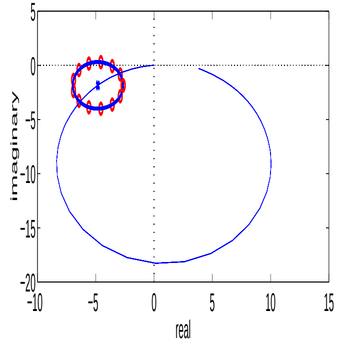
$$W_Y = |C|\sqrt{\left(\frac{\left(|V_1|^2 + |V_2|^2|C|^2\right)}{\left(|W_2|^2 + |W_1|^2|C|^2\right)} - |V_1|^2|V_2|^2\right)}$$


0
1/C
Gc
worst-case performance
0
Gc
1/C

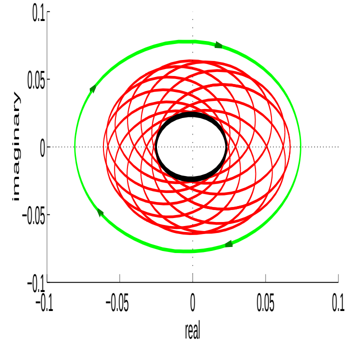
Robust Performance Synthesis

Similarly, all controllers C which achieve $\bar{\sigma}\left(VT(G_\Delta, C)W\right) < 1$ are characterized by a circular region.

Synthesis: union of circles.



Special case:
Loop-shaped performance
$\bar{\sigma}\left(T(WG_\Delta, W^{-1}C)\right)$



Vinnicombe (1993)

$$\max_{G_\Delta \in G_\nu(G_x, W_v)} \bar{\sigma}\left(T(G_\Delta, C)\right) =$$
$$\sin\left(arc\sin\left(\bar{\sigma}\left(T(G_x, C)\right)\right)^{-1} - arc\sin\left(W_v\right)\right)^{-1}$$

Conclusions

- Circular uncertainty regions in the frequency domain equivalently described in additive, dual Youla and ν-gap uncertainty structure.
- If SYSID is split in (a) estimating nominal model and (b) bounding the uncertainty: easily non-optimal.
- Transforms from open-loop to closed-loop model uncertainty sets (and vice versa): OK, but only for circular areas.
- Loop-shaped performance measure allows for easy worst-case optimization.
- ... Work in progress.