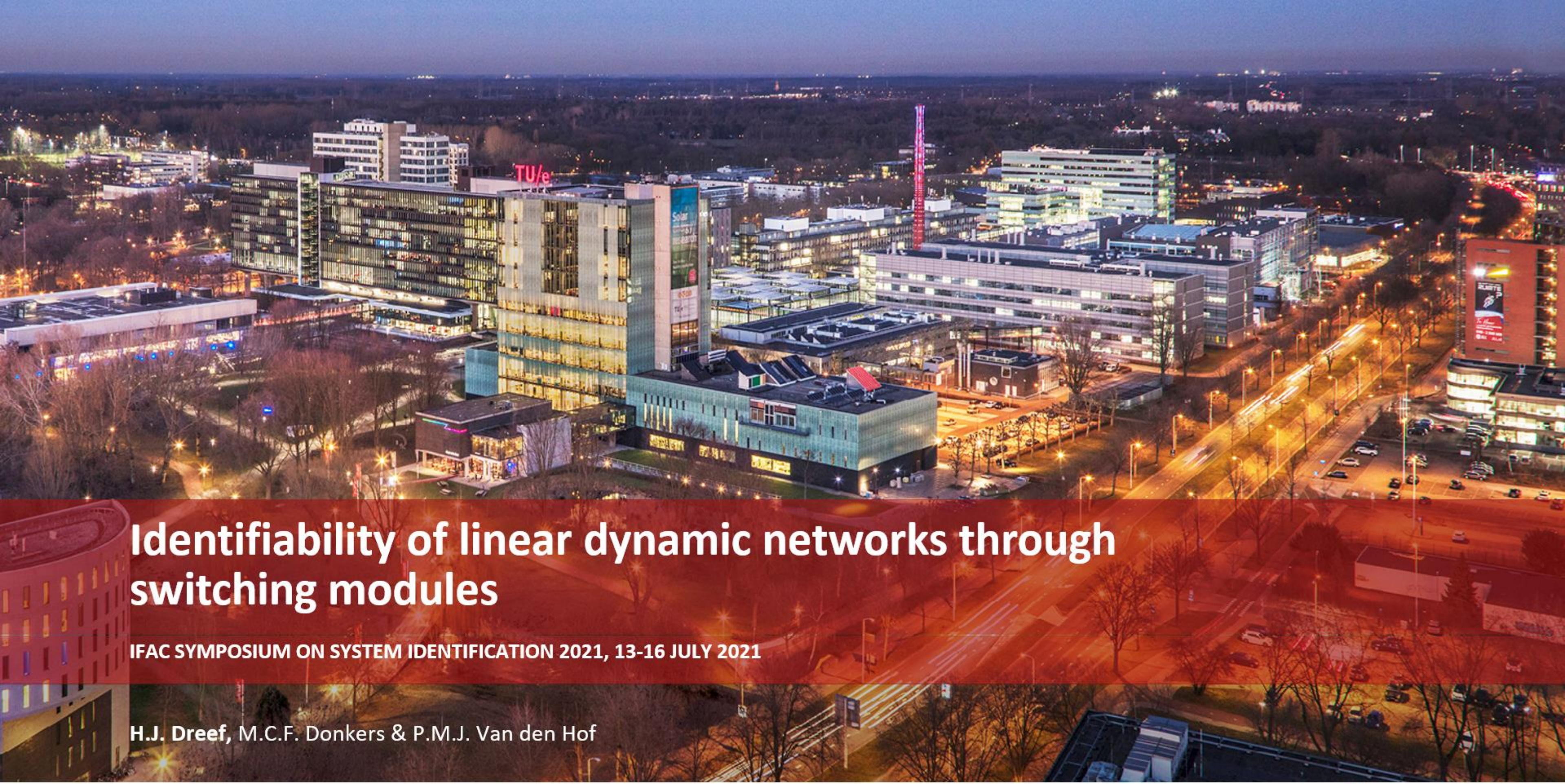

Identifiability of linear dynamic networks through switching modules

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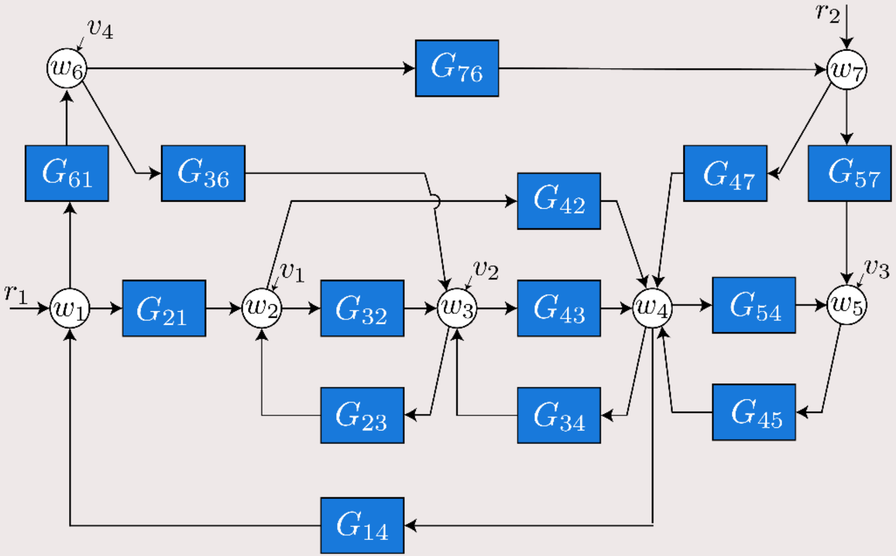
Dynamic network model

- Measured node signals w_i
- Transfer functions G_{ji}
- Process noise signals v_i
- External excitation signals r_i



[1] Materassi, D. *et al., TAC,* 2010
[2] Chiuso, A. *et al., Automatica,* 2012
[3] Van den Hof, P.M.J. *et al., Automatica,* 2013
[4] Goncalves, J. *et al.,* TAC, 2008

Identifiability in dynamic networks

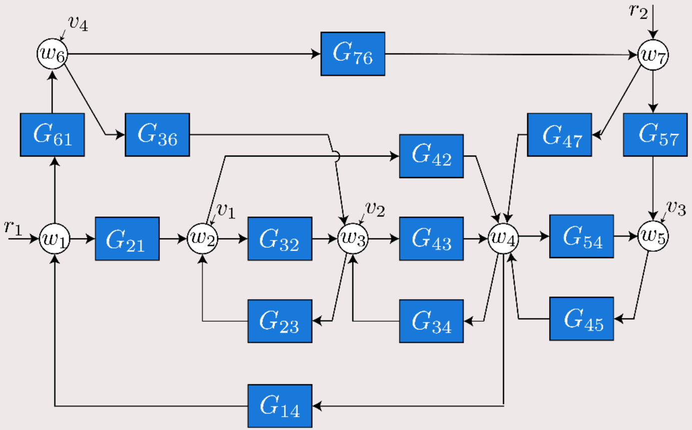
Given:

- Interconnection structure
- Measured nodes
- Locations of signals

$$\Longrightarrow T : \begin{pmatrix} r \\ e \end{pmatrix} \rightarrow w$$

Identifiability: $\Longleftarrow ?$

- Are there multiple models that generate the same T?
- Independent of data & identification method



Focus: the role of excitation on identifiability

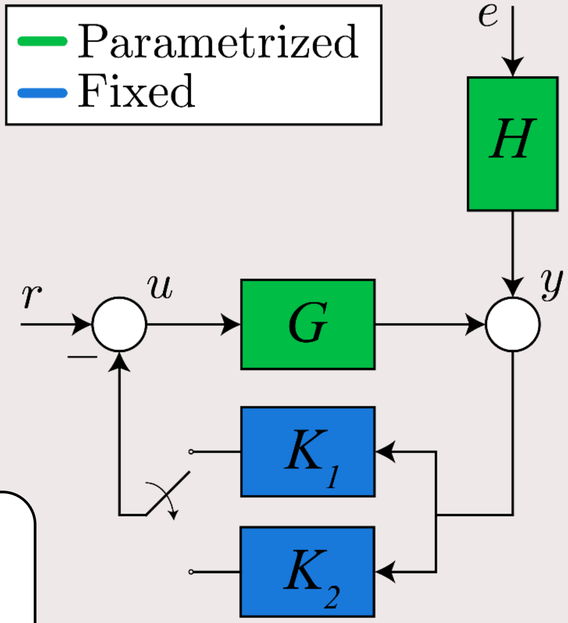
Excitation by switching modules

Closed-loop setup

- Identify G and H
- White noise signal e

Excitation requirement

- Persistently exciting r signal
- Alternative: Switching controllers [1]



Under which conditions do switching modules provide excitation for network identifiability?

[1] Söderström, T. *et al.* TAC, 1976

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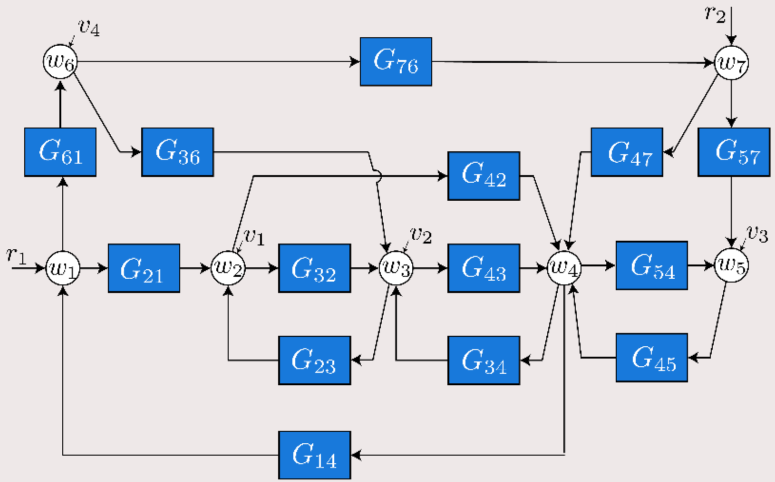
Standard dynamic network model

Network model: $M = (G, H, R, \Lambda)$

$$\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1L} \\ G_{L1} & \cdots & G_{LL-1} & 0 \end{bmatrix}}_{G} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + H \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}}_{e} + R \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix}}_{r}$$



- $G \in \mathbb{R}^{L \times L}(z)$, diagonal entries 0
- $H \in \mathbb{R}^{L \times p}(z)$
- $R \in \mathbb{R}^{L \times K}(z)$
- $\Lambda \in \mathbb{R}^{p \times p}, \Lambda > 0$
- Well-posed: $(I - G)^{-1}$ proper and stable

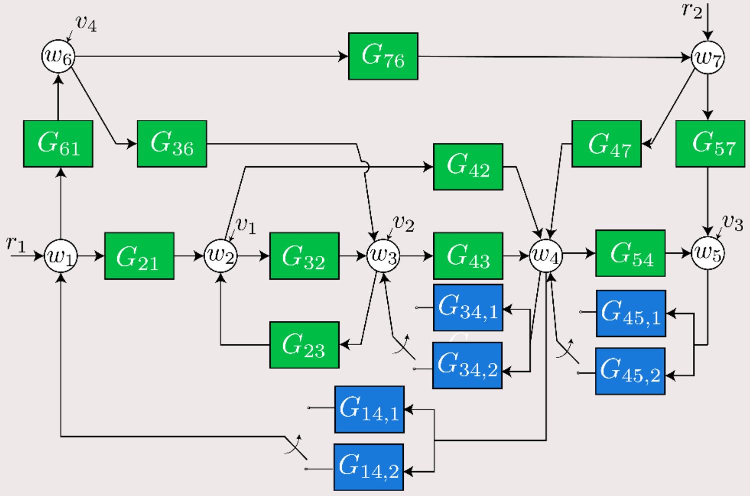
[1] Materassi, D. *et al., TAC,* 2010
[2] Chiuso, A. *et al., Automatica,* 2012
[3] Van den Hof, P.M.J. *et al., Automatica,* 2013
[4] Goncalves, J. *et al.,* TAC, 2008

Switching dynamic network model

Network model per mode $\ell \in \{1, \ldots, m\}$: $M_\ell = (G_\ell, H, R, \Lambda)$

$$G_\ell(\theta) = G^{inv}(\theta) + G_\ell^s$$

- G^{inv} are mode-invariant modules
- G_ℓ^s are switching modules
- For each (j, i) either $G_{ji}^{inv} = 0$ or $G_{ji,\ell}^s = 0$



Model set $\mathcal{M} := \{\mathbb{M}(\theta), \theta \in \Theta\}$

with model

$\mathbb{M}(\theta) := \{M_\ell(\theta)\}_{\ell \in \{1, \ldots, m\}}$

where

$M_\ell(\theta) = (G^{inv}(\theta) + G_\ell^s, H(\theta), R(\theta), \Lambda(\theta))$

Module switching network identifiability

Transfer matrix $\begin{pmatrix} r \\ e \end{pmatrix} \rightarrow w$:

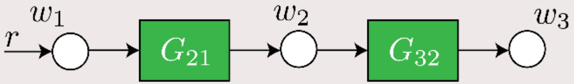
$$T_\ell(\theta) := (I - G_\ell(\theta))^{-1} U(\theta) \ \text{ with } \ U(\theta) := \begin{bmatrix} R(\theta) & H(\theta) \end{bmatrix}$$

Definition: Full *generic* network identifiability

The network model set is *generically* identifiable from (r, w) if for *almost all* models $\mathbb{M}(\theta_1), \mathbb{M}(\theta_0)$ it holds that

$$T_\ell(\theta_1) = T_\ell(\theta) \ \ \forall \ell \in \{1, \ldots, m\} \implies \mathbb{M}(\theta_1) = \mathbb{M}(\theta_0)$$

Generic identifiability [1-3]:



G_{32} not identifiable when $G_{21} = 0$

[1] Bazanella, A.S., *et al., CDC,* 2017
[2] Weerts, H.H.M. *et al., Automatica,* 2018
[3] Hendrickx, J.M., *et al., TAC,* 2019

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Rank conditions: derivation

Rewrite the transfer matrix:

$$T_\ell(\theta) = (I - G_\ell(\theta))^{-1} U(\theta)$$

$$(I - G_\ell(\theta)) T_\ell(\theta) = U(\theta)$$

$$(I - G_\ell(\theta))_{j\star} T_\ell(\theta) = U(\theta)_{j\star}$$

$$(I - G_\ell(\theta))_{j\star} P_j P_j^{-1} T_\ell(\theta) Q_j = U(\theta)_{j\star} Q_j$$

$$\begin{bmatrix} -G(\theta)^{(1)}_{j\star} & (I - G_\ell)^{(2)}_{j\star} \end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j = \begin{bmatrix} U^{(1)}_{j\star} & 0 \end{bmatrix} + U(\theta)^{(2)}_{j\star} V_j$$

$$\begin{bmatrix} G(\theta)^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,\ell}(\theta) \\ V_j \end{bmatrix} = \rho_{j,\ell}$$

Permutation of parametrized entries:

$$(I - G_\ell(\theta))_{j\star} P_j = \begin{bmatrix} -G(\theta)^{(1)}_{j\star} & (I - G_\ell)^{(2)}_{j\star} \end{bmatrix}$$

$$U(\theta)_{j\star} Q_j = \begin{bmatrix} U^{(1)}_{j\star} & 0 \end{bmatrix} + U(\theta)^{(2)}_{j\star} V_j$$

with $V_j = \begin{bmatrix} 0 & I_{\beta_j} \end{bmatrix}$

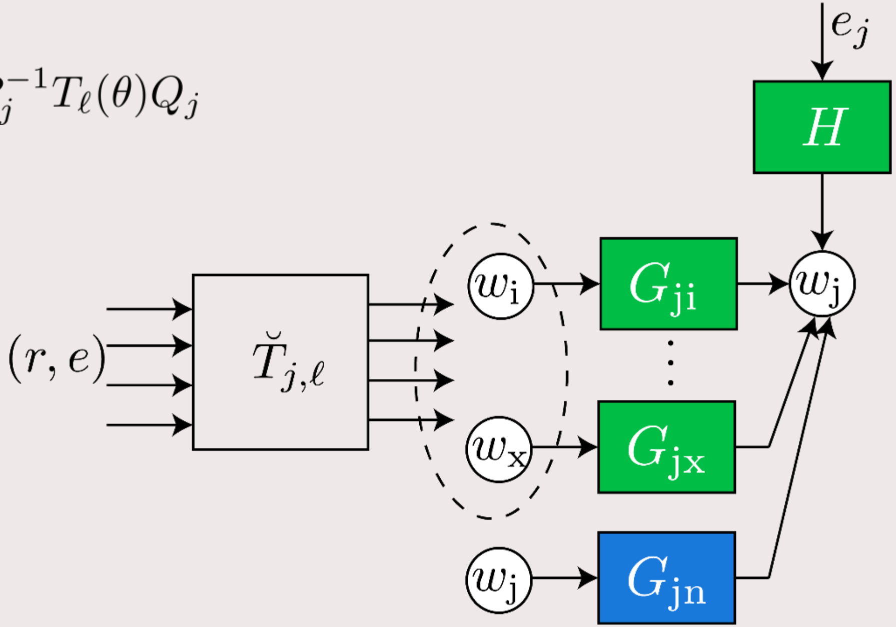
Number of par. entries:

- In $G(\theta)_{j\star}$: α_j
- In $U(\theta)_{j\star}$: β_j

$$\breve{T}_{j,\ell}(\theta) = \begin{bmatrix} I_{\alpha_j} & 0 \end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j$$

Transfer matrix interpretation

$$\breve{T}_{j,\ell}(\theta) = \begin{bmatrix} I_{\alpha_j} & 0 \end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j$$
$$V_j = \begin{bmatrix} 0 & I_{\beta_j} \end{bmatrix}$$



Rank conditions: results

For each mode:

$$\begin{bmatrix} G(\theta)_{j\star}^{(1)} & U(\theta)_{j\star}^{(2)} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,\ell}(\theta) \\ V_j \end{bmatrix} = \rho_{j,\ell}$$

$$\breve{T}_{j,\ell}(\theta) = \begin{bmatrix} I_{\alpha_j} & 0 \end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j$$

$$V_j = \begin{bmatrix} 0 & I_{\beta_j} \end{bmatrix}$$

Juxtaposition of all the m network modes:

$$\begin{bmatrix} G(\theta)_{j\star}^{(1)} & U(\theta)_{j\star}^{(2)} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,1}(\theta) & \cdots & \breve{T}_{j,m}(\theta) \\ V_j & \cdots & V_j \end{bmatrix} = \begin{bmatrix} \rho_{j,1} & \cdots & \rho_{j,m} \end{bmatrix}$$

Theorem: Rank conditions for switching networks

A switching network model set $\mathcal{M}$ is generically network identifiable from (r, w) if for each row j the matrix

$$\begin{bmatrix} \breve{T}_{j,1}(\theta) & \dots & \breve{T}_{j,m}(\theta) \\ V_j & \dots & V_j \end{bmatrix}$$

has full row rank for almost all $\theta \in \Theta$.

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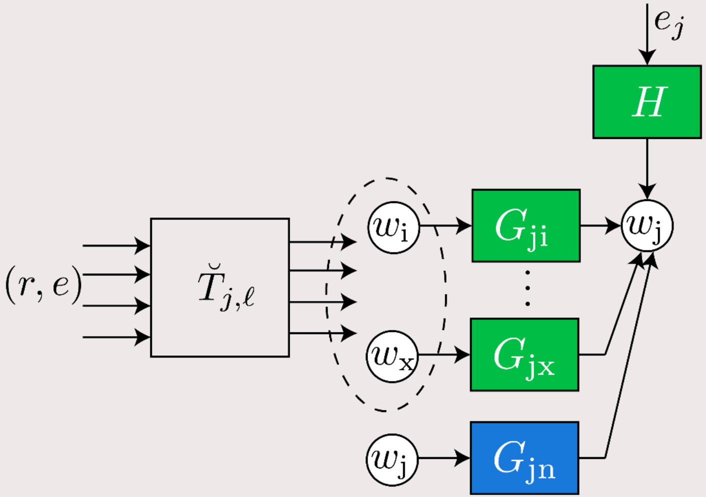
Non-switching networks

Rank condition for $m = 1$:

$$\begin{bmatrix} \breve{T}_j(\theta) \\ V_j \end{bmatrix} = \begin{bmatrix} \breve{T}_j^{(1)}(\theta) & \breve{T}_j^{(2)}(\theta) \\ 0 & I_{\beta_j} \end{bmatrix}$$

$$\text{full row rank} \begin{bmatrix} \breve{T}_j(\theta) \\ V_j \end{bmatrix} \Leftrightarrow \text{full row rank } \breve{T}_j^{(1)}(\theta)$$

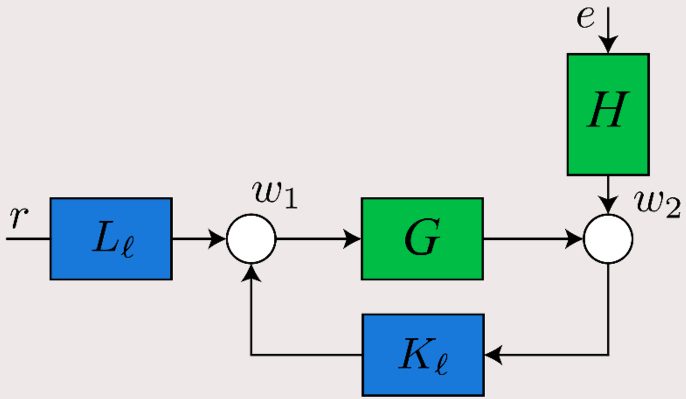


Condition in [1]: $\text{full row rank } \breve{T}_j^{(1)}(\theta)$

[1] Weerts, H.H.M. *et al., Automatica,* 2018

Switching controllers in closed-loop

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & K_\ell \\ G(\theta) & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} L_\ell & 0 \\ 0 & H(\theta) \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix}$$



Form the matrix for the rank condition:

$$\begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix} = \begin{bmatrix} \hat{S}_1 L_1 & K_1 \hat{S}_1 \hat{H} & \cdots & \hat{S}_m L_m & K_m \hat{S}_m \hat{H} \\ 0 & I & \cdots & 0 & I \end{bmatrix}$$

with $\hat{S}_\ell = (I - \hat{G} K_\ell)^{-1}$

$$\operatorname{rank} \begin{bmatrix} \breve{T}_{j,1} \cdots \breve{T}_{j,m} \\ V_j \cdots V_j \end{bmatrix} = \operatorname{rank} \Xi_1 \underbrace{\begin{bmatrix} I & \cdots & I & 0 & \cdots & 0 \\ K_1 & \cdots & K_m & L_1 & \cdots & L_m \end{bmatrix}}_{[1]} \Xi_3,$$

with Ξ_1 and Ξ_3 full rank

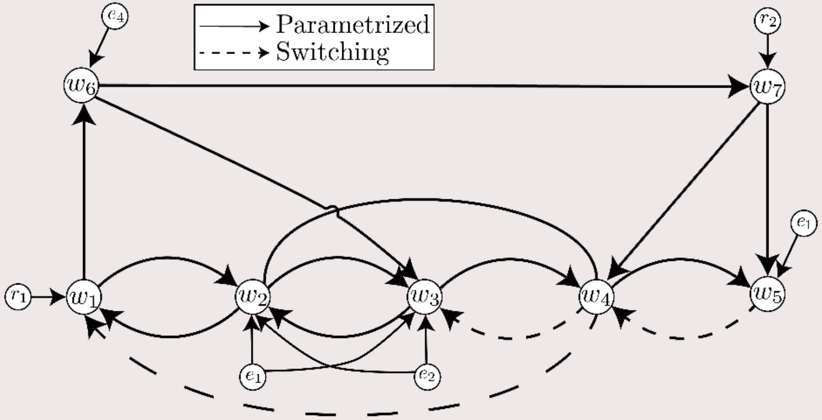
[1] Söderström, T. *et al.* TAC, 1976

Content

Path-based conditions

Advantages:

- Insightful conditions
- Only requires structural information
- Efficiently verifiable



Definition: Extended graph[1] with switching edges

$$\mathcal{V} := \{L \text{ nodes}, K \text{ signals } \& \ p \text{ noise}\}$$

$$\mathcal{E}_{inv} := \{(j,i) \in \mathcal{V} \mid \{G_{ji}^{inv}, R_{ji}, H_{ji}\} \neq 0\}$$

$$\mathcal{E}_s := \{(j,i) \in \mathcal{V} \mid G_{ji,\ell}^{s} \neq 0\}$$

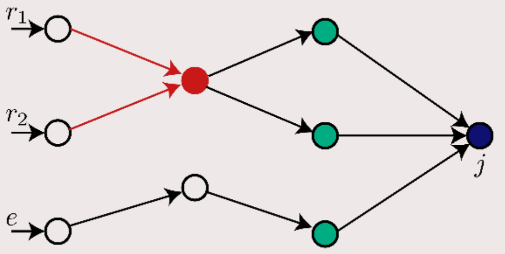
[1] Cheng, X. *et al., TAC,* 2021

Relation of paths with rank

The sets of excited nodes and in-neighbors are crucial

$$\mathcal{U} := \{i \in \mathcal{V} \mid i \in \{L, \dots, L+K+P\}\}$$

$$\mathcal{W}_j := \{i \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}_{inv} \text{ or } U_{j,i-L} \text{ is parametrized}\}$$



Relation between rank and graph[1]: $b_{\mathcal{U}\to\mathcal{W}_j} = \text{rank}\, T_{\mathcal{W}_j\mathcal{U},1}$

Maximum vertex-disjoint paths: $b_{\mathcal{U}\to\mathcal{W}_j}$ $\quad T_{\mathcal{W}_j\mathcal{U},\ell} = \begin{bmatrix} \breve{T}_{j,\ell} \\ V_j \end{bmatrix}$

Theorem: path-based conditions for non-switching network[2]

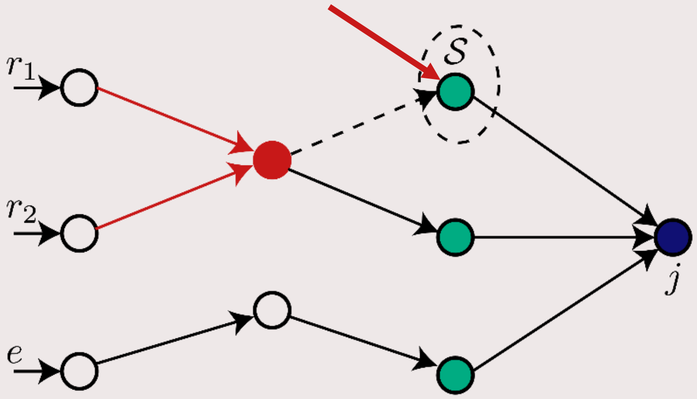
A network model set $\mathcal{M}$ (with $m = 1$) is generically identifiable from (r, w) if in its extended graph: $b_{\mathcal{U}\to\mathcal{W}_j} = |\mathcal{W}_j|$ holds for all $j \in \mathcal{V}$.

[1] van der Woude, J.W., *MCSS,* 1991
[2] Cheng, X. *et al., TAC,* 2021
[3] Hendrickx, J.M., *et al., TAC,* 2019

Path-based conditions: switching networks

Stimulated nodes by switching:

$$\mathcal{S} := \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}_s \wedge b_{\mathcal{U}\to\{w_i\}} \neq 0\}$$



Assumption: independent switching

The matrix $\begin{bmatrix} T_{\mathcal{SU},1} & \cdots & T_{\mathcal{SU},m} \\ I & \cdots & I \end{bmatrix}$ is full row rank.

Theorem: path-based conditions for switching networks

A switching network model set $\mathcal{M}$ is generically identifiable from (r,w) if in its extended graph: $b_{(\mathcal{U}\cup\mathcal{S})\to\mathcal{W}_j} = |\mathcal{W}_j|$ holds for all $j \in \mathcal{V}$.

Example

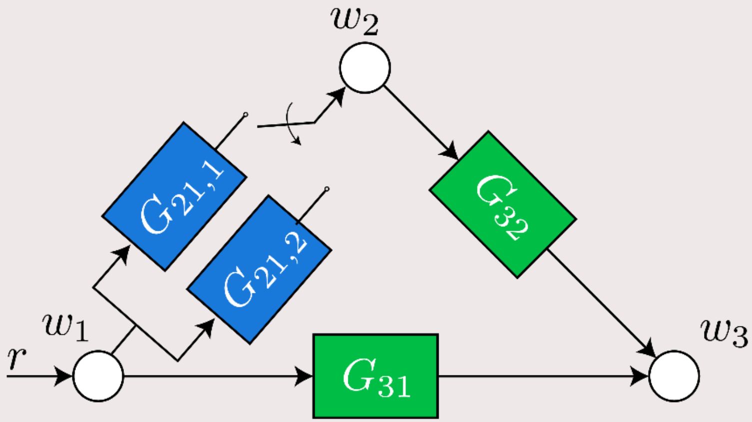
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ G_{21,\ell} & 0 & 0 \\ G_{31}(\theta) & G_{32}(\theta) & 0 \end{bmatrix}}_{G_\ell(\theta)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{U} r$$

Rank condition for node 3: $\breve{T}_{3,\ell} = \begin{bmatrix} 1 \\ G_{21,\ell} \end{bmatrix}$

$$\operatorname{rank} \begin{bmatrix} \breve{T}_{3,1} & \breve{T}_{3,2} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 1 \\ G_{21,1} & G_{21,2} \end{bmatrix} = 2 \checkmark$$

Path-based condition: $\mathcal{S} = \{w_2\}$

$$b_{(\mathcal{U} \cup \mathcal{S}) \to \mathcal{W}_j} = 2 \checkmark$$



Conclusion

Switching modules can be used as alternative for excitation

Rank conditions that generalize:

- Identifiability analysis in dynamics networks [1]
- Switching controllers [2]

Path-based conditions

[1] Weerts, H.H.M. *et al., Automatica*, 2018
[2] Söderström, T. *et al.* TAC, 1976