$\mathcal{H}_\infty$ performance analysis and distributed controller synthesis for interconnected linear systems from noisy input-state data

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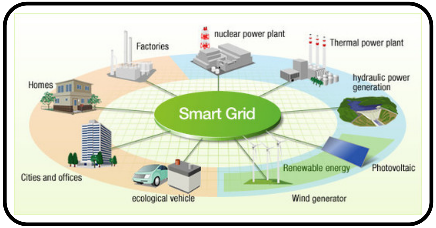
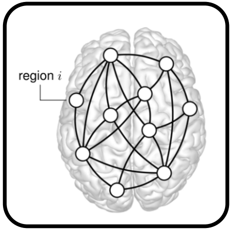
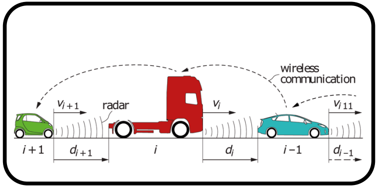
Motivation







From left to right: betterworldsolutions.eu, Ploeg et al., 2013, Baggio et al., 2021

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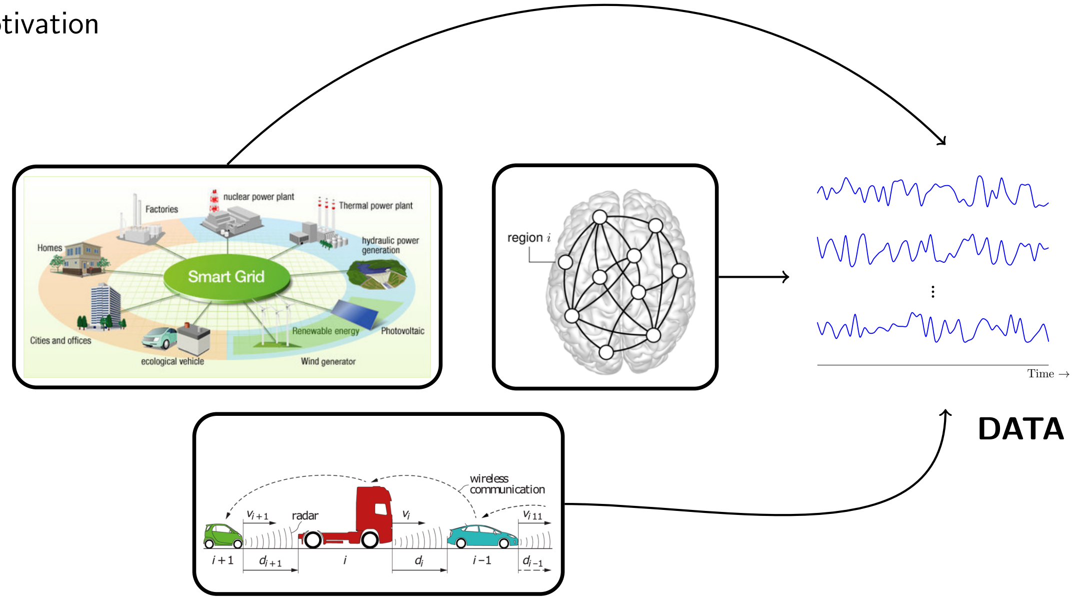

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Challenge

(i) Determine *interconnected* system performance from data with guarantees

(ii) Synthesize *distributed* controller from data with guaranteed performance

Related challenges:

- Data-based dissipativity analysis (Koch et al., 2020), (Van Waarde et al., 2021)
- Data-based controller synthesis (Van Waarde et al., 2020), (De Persis et al., 2020)
- Data-based distributed controller synthesis (Steentjes et al., 2020 & 2021)

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$\Rightarrow$ unstructured/centralized $\vee$ asymptotic guarantees

Underlying interconnected system and data collection

Interconnected system

$$x_i(k+1) = A_i x_i(k) + \sum_{j \in \mathcal{N}_i} A_{ij} x_j(k) + B_i u_i(k) + w_i(k),$$

$$y_i(k) = C_i x_i(k) + D_i u_i(k) \quad \text{for} \quad i \in \mathcal{V} := \{1, 2, \ldots, L\}$$

Known bound on the noise signals: $W_i^- := [w_i(0) \cdots w_i(N-1)] \in \mathcal{W}_i$, with

$$\mathcal{W}_i = \left\{ W_i \mid \begin{bmatrix} W_i^\top \\ I \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q_w^i & S_w^i \\ (S_w^i)^\top & R_w^i \end{bmatrix}}_{=:\Pi_w^i} \begin{bmatrix} W_i^\top \\ I \end{bmatrix} \geq 0 \right\}$$

Transfer $(u_1, \ldots, u_L) \to (y_1, \ldots, y_L)$ is $G_\mathcal{I}$

$\mathcal{H}_\infty$ analysis problem

Given data $\{u_i\}_{t=0,\ldots,N-1}$, $\{x_i\}_{t=0,\ldots,N}$ for each i, verify if $\|G_{\mathcal{I}}\|_{\mathcal{H}_\infty} < \gamma$

$\mathcal{H}_\infty$ analysis problem

Given data $\{u_i\}_{t=0,\dots,N-1}$, $\{x_i\}_{t=0,\dots,N}$ for each i, verify if $\|G_{\mathcal{I}}\|_{\mathcal{H}_\infty} < \gamma$

For each i, collect data in matrices

$$\begin{aligned} X_i^+ &:= \begin{bmatrix} x_i(1) & \cdots & x_i(N) \end{bmatrix} \\ X_i^- &:= \begin{bmatrix} x_i(0) & \cdots & x_i(N-1) \end{bmatrix} \\ U_i^- &:= \begin{bmatrix} u_i(0) & \cdots & u_i(N-1) \end{bmatrix} \end{aligned}$$

$\Rightarrow$ every subsystem compatible with the data is in

$$\Sigma_{\mathcal{D}}^i := \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, \exists W_i \in \mathcal{W}_i : X_i^+ = A_i X_i^- + \sum_{j \in \mathcal{N}_i} A_{ij} X_j^- + B_i U_i^- + W_i\}$$

Data-based $\mathcal{H}_\infty$ analysis of interconnected systems

Unknown data-generating interconnected system has $\mathcal{H}_\infty$ performance γ

$$\Uparrow$$

Every interconnected system with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$, $i \in \mathcal{V}$ has $\mathcal{H}_\infty$ performance γ

Data-based $\mathcal{H}_\infty$ analysis of interconnected systems

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Every interconnected system with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$, $i \in \mathcal{V}$ has $\mathcal{H}_\infty$ performance γ

How to represent all possible interconnected systems?

Parametrization $\Sigma^i_{\mathcal{D}}$

$$\Sigma^i_{\mathcal{D}} = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \underbrace{\begin{bmatrix} \bar{Q}^i_{\mathcal{D}} & \bar{S}^i_{\mathcal{D}} \\ (\bar{S}^i_{\mathcal{D}})^\top & \bar{R}^i_{\mathcal{D}} \end{bmatrix}}_{=:\bar{\Pi}^{\mathcal{P}}_i} \begin{bmatrix} -A_i^\top \\ -A_{\mathcal{N}_i}^\top \\ -B_i^\top \\ I \end{bmatrix} \geq 0\}, \quad \bar{\Pi}^{\mathcal{P}}_i = (\star)\Pi^w_i \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix}^\top$$

Dual parametrization $\Sigma^i_{\mathcal{D}}$

$$\Sigma^i_{\mathcal{D}} = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \underbrace{\begin{bmatrix} Q^i_{\mathcal{D}} & S^i_{\mathcal{D}} \\ (S^i_{\mathcal{D}})^\top & R^i_{\mathcal{D}} \end{bmatrix}}_{=:\Pi^{\mathcal{P}}_i} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} \leq 0\}, \quad \Pi^{\mathcal{P}}_i = (\bar{\Pi}^{\mathcal{P}}_i)^{-1}$$

Parametrization $\Sigma_{\mathcal{D}}^i$

$$\Sigma_{\mathcal{D}}^i = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \underbrace{\begin{bmatrix} \bar{Q}_{\mathcal{D}}^i & \bar{S}_{\mathcal{D}}^i \\ (\bar{S}_{\mathcal{D}}^i)^\top & \bar{R}_{\mathcal{D}}^i \end{bmatrix}}_{=:\bar{\Pi}_i^{\mathcal{P}}} \begin{bmatrix} -A_i^\top \\ -A_{\mathcal{N}_i}^\top \\ -B_i^\top \\ I \end{bmatrix} \geq 0\}, \quad \bar{\Pi}_i^{\mathcal{P}} = (\star)\Pi_i^w \begin{bmatrix} X_i^- & 0 \\ X_{\mathcal{N}_i}^- & 0 \\ U_i^- & 0 \\ X_i^+ & I \end{bmatrix}^\top$$

Dual parametrization $\Sigma_{\mathcal{D}}^i$

$$\Sigma_{\mathcal{D}}^i = \{(A_i, A_{\mathcal{N}_i}, B_i) \,|\, (\star)^\top \underbrace{\begin{bmatrix} Q_{\mathcal{D}}^i & S_{\mathcal{D}}^i \\ (S_{\mathcal{D}}^i)^\top & R_{\mathcal{D}}^i \end{bmatrix}}_{=:\Pi_i^{\mathcal{P}}} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} \leq 0\}, \quad \Pi_i^{\mathcal{P}} = (\bar{\Pi}_i^{\mathcal{P}})^{-1}$$

Data gathering: ensure that $\mathrm{col}(X_i^-, X_{\mathcal{N}_i}^-, U_i^-)$ has full row rank

Data-compatible interconnected system in LFT representation

Write the interconnected systems in LFT representation (cf. (Koch et al., 2020))

$$\begin{bmatrix} x_i(k+1) \\ y_i(k) \\ p_i(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & I \\ C_i & 0 & D_i & 0 \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_i(k) \\ \operatorname{col}_{j \in \mathcal{N}_i} x_j(k) \\ u_i(k) \\ l_i(k) \end{bmatrix} \text{ and } l_i(k) = \begin{bmatrix} A_i & A_{\mathcal{N}_i} & B_i \end{bmatrix} p_i(k)$$

with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_{\mathcal{D}}^i$.

Dual parametrization of $\Sigma_{\mathcal{D}}^i$ $\Rightarrow$ use robust $\mathcal{H}_\infty$ conditions (Van Horssen et al., 2016)

Data-based $\mathcal{H}_\infty$ norm analysis

Proposition (performance from structured data)

If, for each $i \in \mathcal{V}$, there exist P_i, Z_i and α_i so that $P_i > 0$, $\alpha_i > 0$ and

$$J_i(C_i, D_i)^\top \left[\begin{array}{cc|cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_i R_\mathcal{D}^i & -\alpha_i (S_\mathcal{D}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_i S_\mathcal{D}^i & -\alpha_i Q_\mathcal{D}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] J_i(C_i, D_i) < 0$$

then all interconnected systems with $(A_i, A_{\mathcal{N}_i}, B_i) \in \Sigma_\mathcal{D}^i$, $i \in \mathcal{V}$, achieve $\mathcal{H}_\infty$ performance γ

$J_i(C_i, D_i)$ is a known matrix

Distributed control from data

Let u_i be the control input and y_i be the measured output, $i \in \mathcal{V}$:

Distributed controller

$$\begin{bmatrix} \xi_i(k+1) \\ o_i(k) \\ u_i(k) \end{bmatrix} = \Theta_i \begin{bmatrix} \xi_i(k) \\ s_i(k) \\ y_i(k) \end{bmatrix}, \quad i \in \mathcal{V} \quad \text{and} \quad (i,j) \in \mathcal{E} \Rightarrow s_{ij} = o_{ji}$$

$T_{\mathcal{I}}$ transfer $w \to z$ with performance output $z_i = C_i^z x_i + \sum_{j \in \mathcal{N}_i} C_{ij}^z x_j + D_i^z u_i$,

Distributed $\mathcal{H}_\infty$ control problem

Given data $\{u_i\}_{t=0,\ldots,N-1}$, $\{x_i\}_{t=0,\ldots,N}$, find $\Theta_1, \ldots, \Theta_L$ so that $\|T_{\mathcal{I}}\|_{\mathcal{H}_\infty} < \gamma$

Theorem (distributed control from data)

If there exist P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$, α_i such that $P_i \geq \bar{P}_i^{-1} > 0$, $\alpha_i = \beta_i^{-1} > 0$ and

$$\Psi_i^\top T_i^\top \left[\begin{array}{cc|cc|cc|cc} -P_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (Z_i^{12})^\top & Z_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha_i R_{\mathcal{D}}^i & -\alpha_i (S_{\mathcal{D}}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_i S_{\mathcal{D}}^i & -\alpha_i Q_{\mathcal{D}}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] T_i \Psi_i < 0$$

$$\Phi_i^\top S_i^\top \left[\begin{array}{cc|cc|cc|cc} -\bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{P}_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \bar{Z}_i^{11} & \bar{Z}_i^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & (\bar{Z}_i^{12})^\top & \bar{Z}_i^{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\beta_i \bar{R}_{\mathcal{D}}^i & -\beta_i (\bar{S}_{\mathcal{D}}^i)^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_i \bar{S}_{\mathcal{D}}^i & -\beta_i \bar{Q}_{\mathcal{D}}^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^{-2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] S_i \Phi_i > 0$$

then there exist $\Theta_1, \ldots, \Theta_L$, so that $\| T_{\mathcal{I}} \|_{\mathcal{H}_\infty} < \gamma$

Theorem (distributed control from data)

If there exist P_i, $\bar{P}_i$, Z_i, $\bar{Z}_i$, α_i such that $P_i \geq \bar{P}_i^{-1} > 0$, $\alpha_i = \beta_i^{-1} > 0$ and

$$\Psi_i^\top T_i^\top \Lambda_{\mathcal{D}}^i T_i \Psi_i < 0$$

$$\Phi_i^\top S_i^\top \bar{\Lambda}_{\mathcal{D}}^i S_i \Phi_i > 0$$

then there exist $\Theta_1, \ldots, \Theta_L$, so that $\| T_{\mathcal{I}} \|_{\mathcal{H}_\infty} < \gamma$

- Coupled LMIs for fixed scalars α_i

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- Scalable with respect to L (number of subsystems)

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- Coupled LMIs for fixed scalars α_i
- Scalable with respect to L (number of subsystems)
- Data-based distributed controller $(\Theta_1, \ldots, \Theta_L)$ is obtained from LMI solution

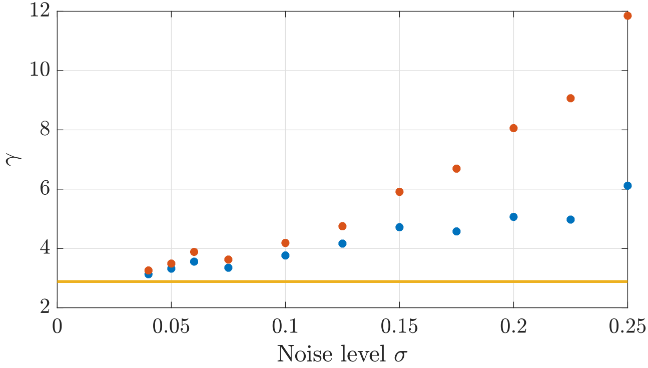
Example 1: $\mathcal{H}_\infty$-norm analysis

Data-generating system ($L = 3$)

$$x(k+1) = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.4 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} x(k) + u(k) + w(k), \quad w(k) \in \{w \mid \|w\|_2 \leq \sigma\}$$

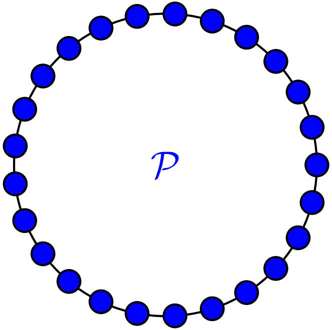
Objective: Find upper-bound on $\mathcal{H}_\infty$ norm of $u \rightarrow x$ from data (U_-, X) with $N = 50$ samples

Example 1: $\mathcal{H}_\infty$-norm analysis



- $\mathcal{H}_\infty$ bound from data (*structured*)
- $\mathcal{H}_\infty$ bound from data (*unstructured*)
- $\mathcal{H}_\infty$ norm true system

Example 2: Distributed $\mathcal{H}_\infty$ controller synthesis



System dynamics ($L = 25$)

$A_i, \sim \mathcal{U}[0, 1]$, $A_{ij} \sim \mathcal{U}[0, 0.1]$,
$B_i = 1$
$w_i(k) \sim \mathcal{U}[-\sigma, \sigma]$

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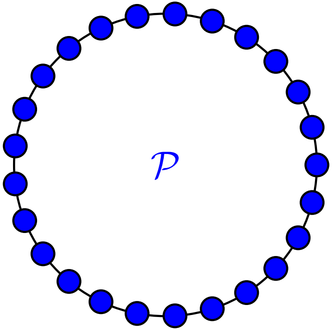

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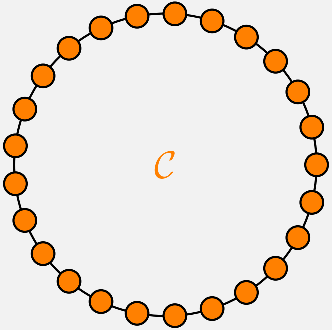
Objective: Synthesize a distributed $\mathcal{H}_\infty$ controller from data (U_i^-, X_i), $N = 50$



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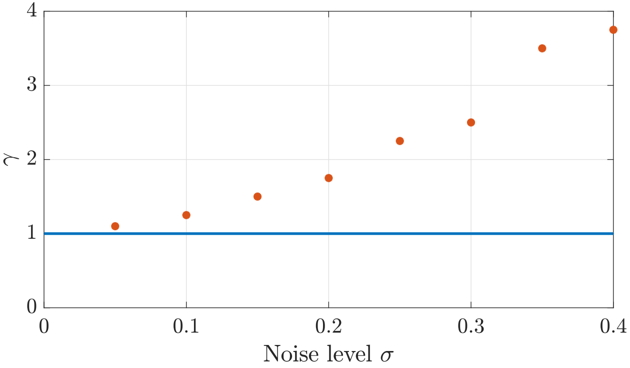

$\mathcal{H}_\infty$ performance bound with

- distributed controller based on *data*
- distributed controller based on *true system*

Concluding remarks

Summary:

- $\mathcal{H}_\infty$ performance analysis and distributed controller synthesis from data
- Dual parametrization of the set of feasible subsystems and interconnected system LFT representation
- Sufficient data-based (existence) conditions in LMI form for guaranteed $\mathcal{H}_\infty$ performance

Remarks & future work:

- Dual parametrization also leads to dual data-based dissipativity conditions
- Enhance practical applicability via weaker noise assumptions (cross-covariance bounds)

Thank you for your attention!

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