Identification of Linear Systems

Johan Schoukens

http://homepages.vub.ac.be/~jschouk

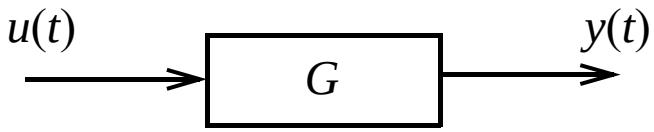
Vrije Universiteit Brussel
Department INDI

Vrije Universiteit Brussel

European Research Council

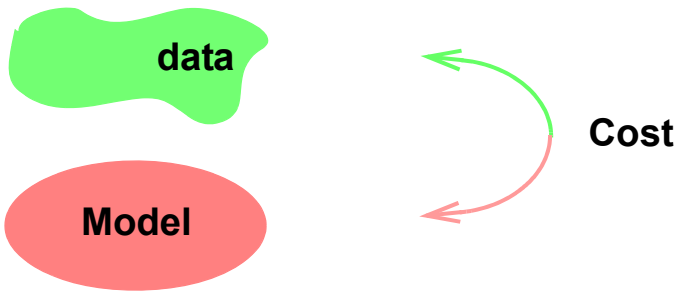
Basic goal

Built a parametric model for a linear dynamic system from sampled data



Initial questions

- sampled data: what's in between the samples?
- plant and noise model?
- cost function?



Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

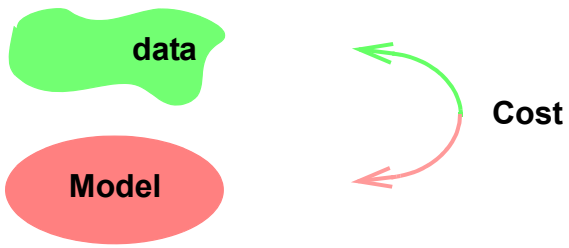
Cost function

Frequency domain formulation
Noise models
Time domain formulation

Validation

Examples

Conclusions

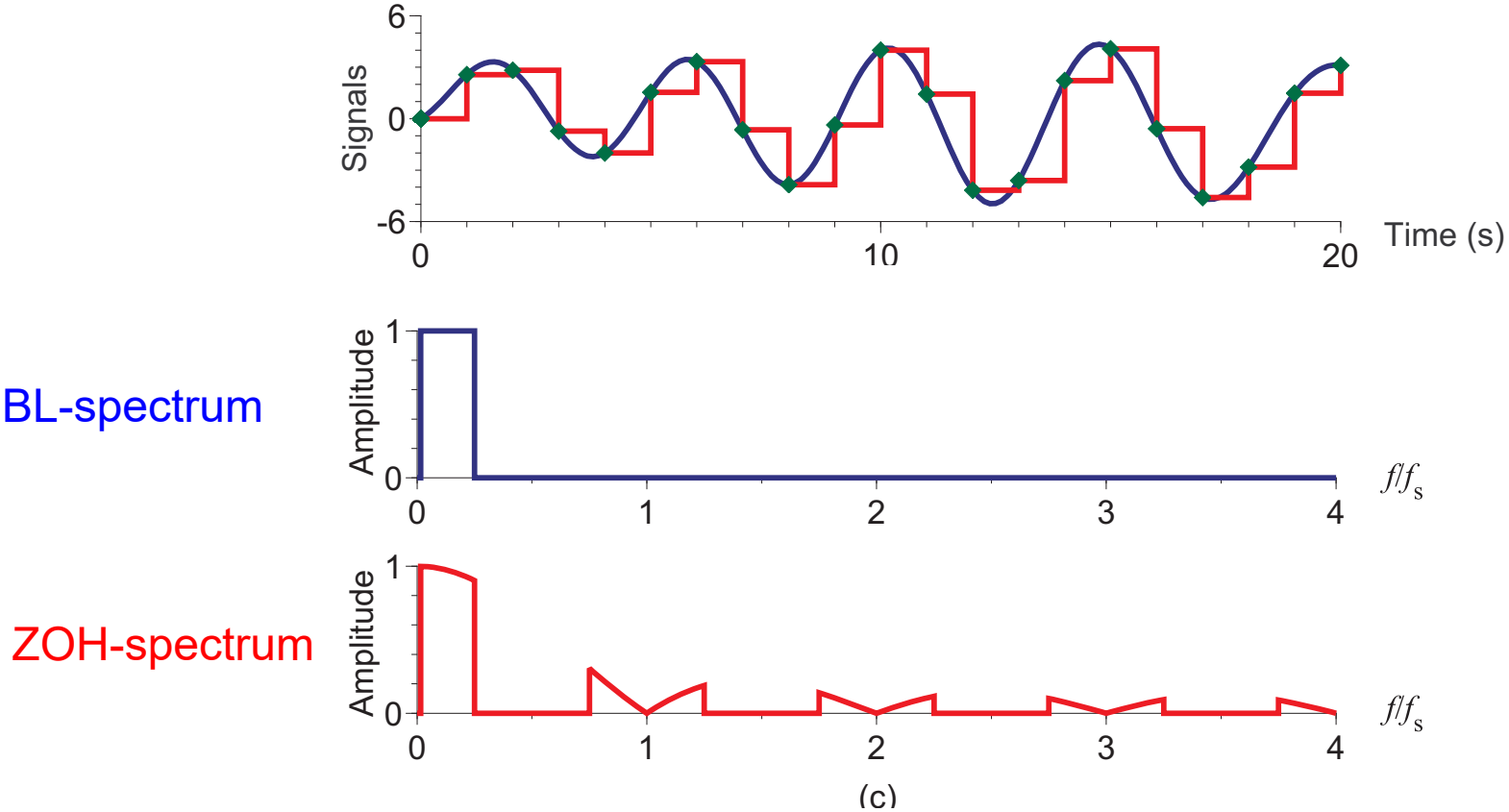


Sampled data
What is going on in between the samples?

Two popular assumptions

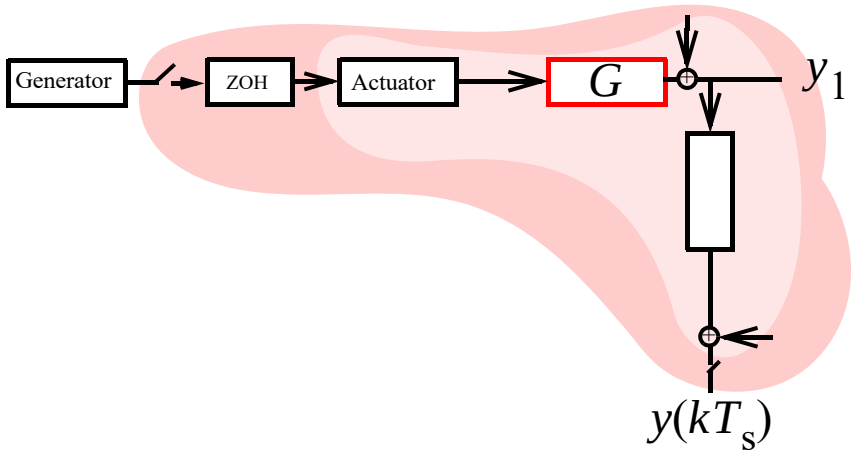
ZOH **zero order hold**: signal piece wise constant

BL **band limited assumption**: no power above $f > f_s/2$

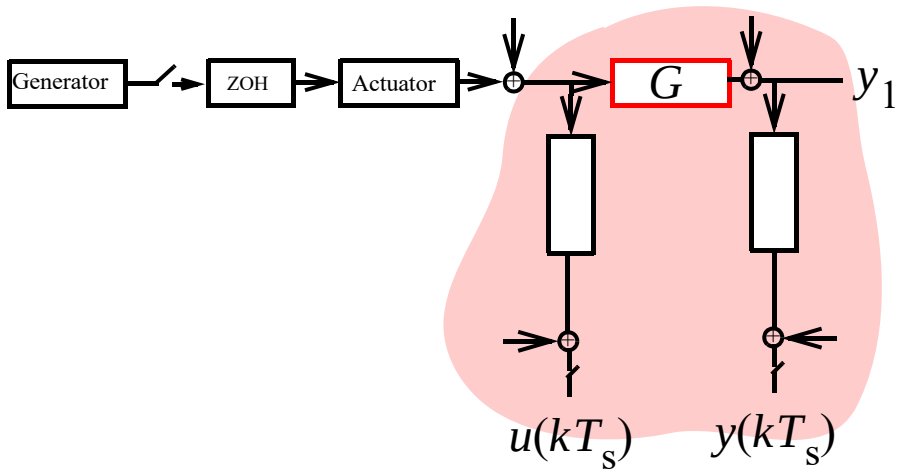


Relation signal assumption / experimental setup

ZOH



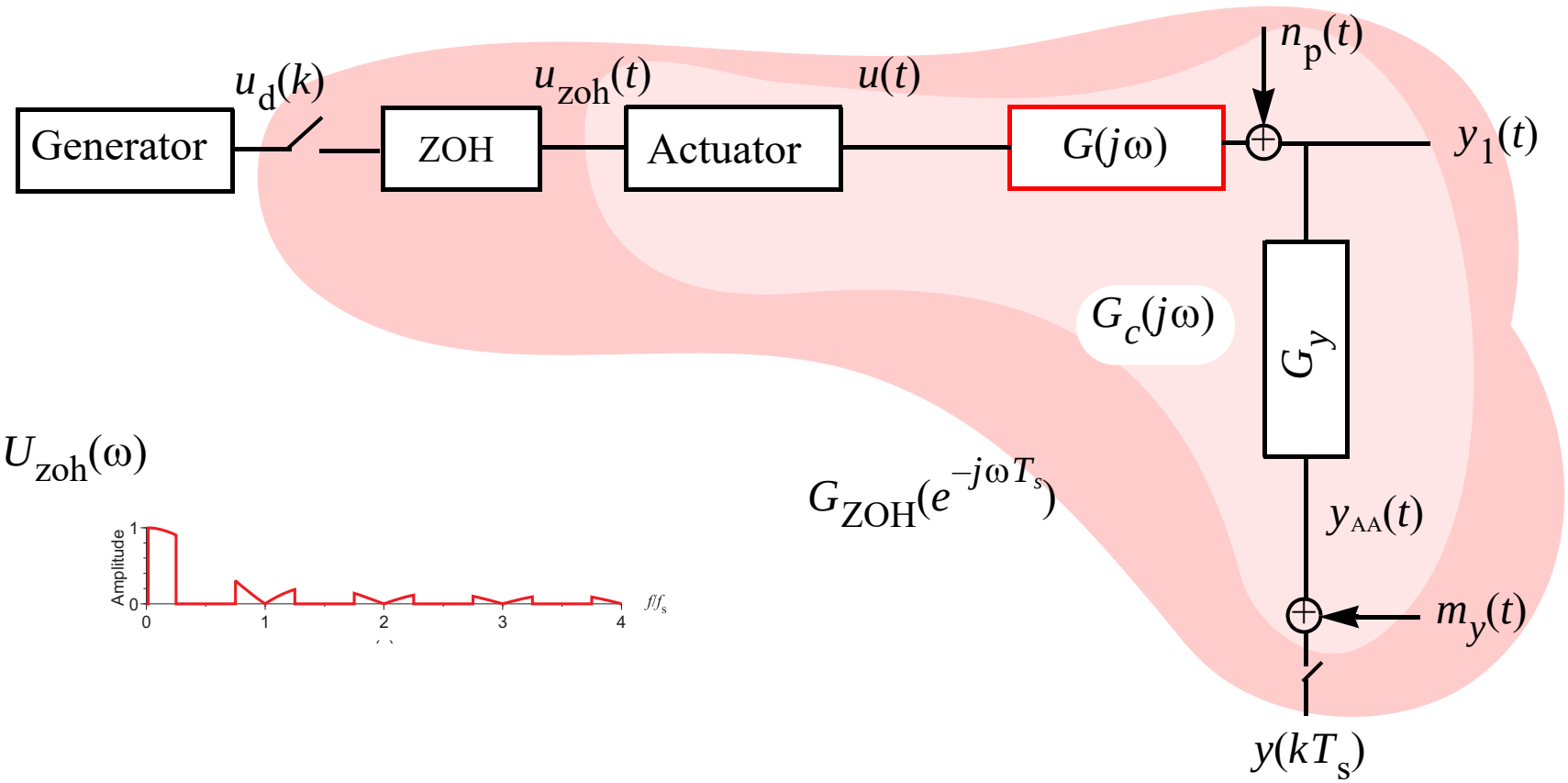
Band limited



Choice driven by the application

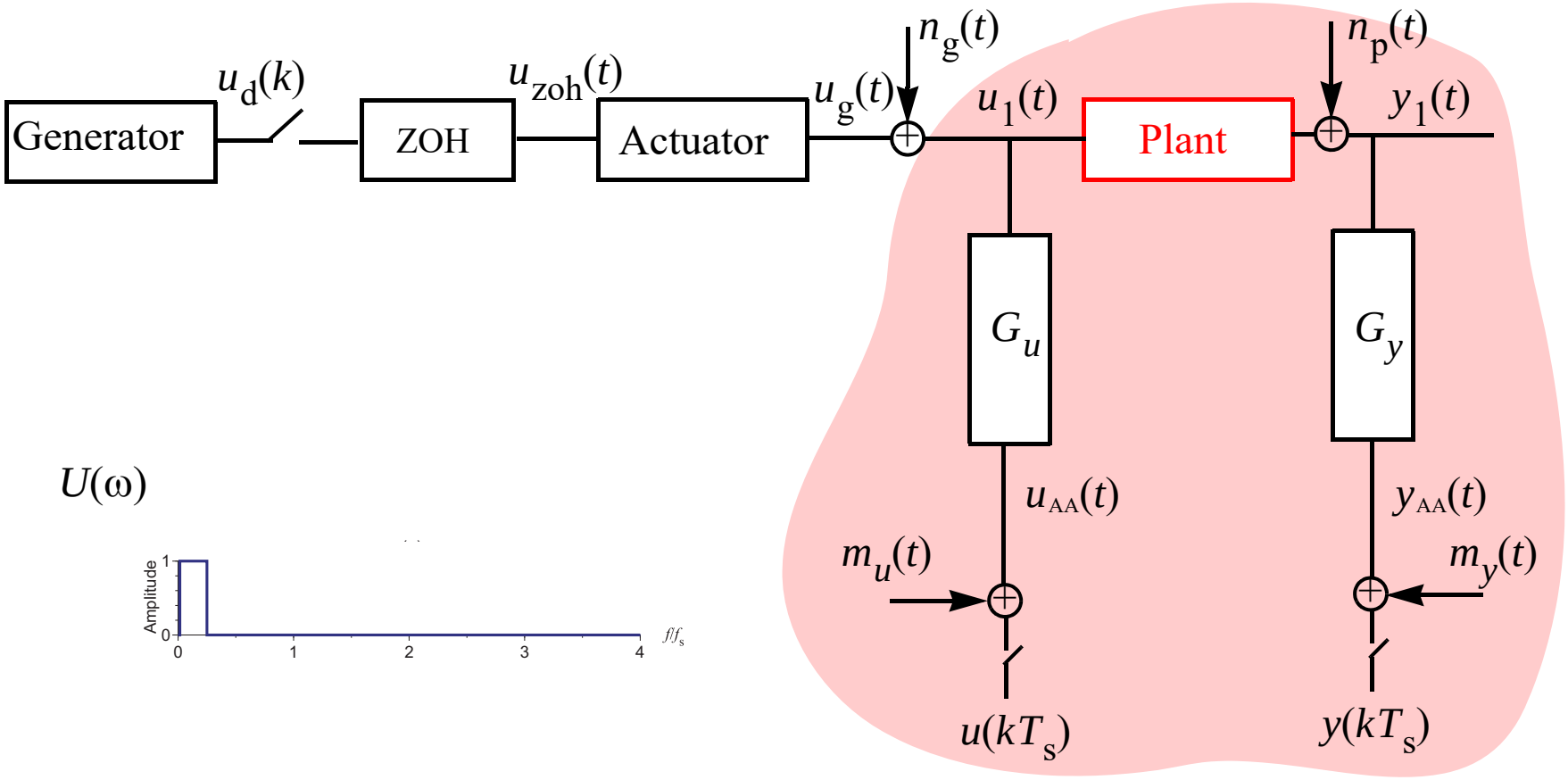
- ZOH: discrete control design
- Band Limited: other applications

ZOH: discrete control design



- input exactly known $u_d(k)$
- high frequency components in input ($f > f_s$)
- absolute calibration $G_y = 1$
- no anti-alias filter allowed
- model: from generator to output (ZOH, actuator, plant, acquisition)

BL: other applications



$U(\omega)$

- input and output measured
- band limited data: no power above $f > f_s/2$ --> anti alias filters G_u, G_y
- relative calibration $G_y/G_u = 1$
- only plant modelled

Conclusions signal assumption

ZOH-assumption

- imposes experimental condition on the excitation signal
- discrete time model from generator to measured plant output
- possible to transform DT --> CT model (perfect ZOH)

BL-assumption

- imposes condition on the observation of the signals
- no constraints on the applied excitation
  (e.g. BL-observations of ZOH-signals can be made).
- continuous-time model of the plant in the observed frequency band

Violating the signal assumption

- still possible to get a behaviour model
- model is no longer independent of the measurement environment
- the inter sample behaviour becomes an intrinsic part of the model

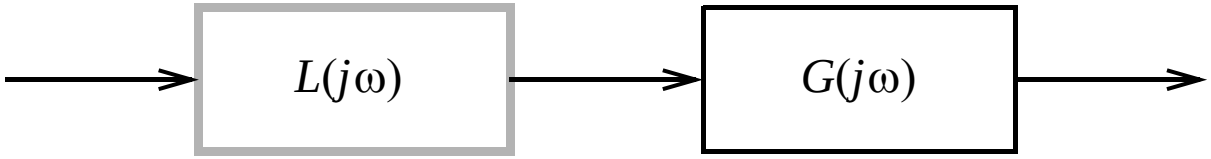
BL-Assumption --> approximate DT-models for simulation
Imperfect ZOH --> model linked to the generator

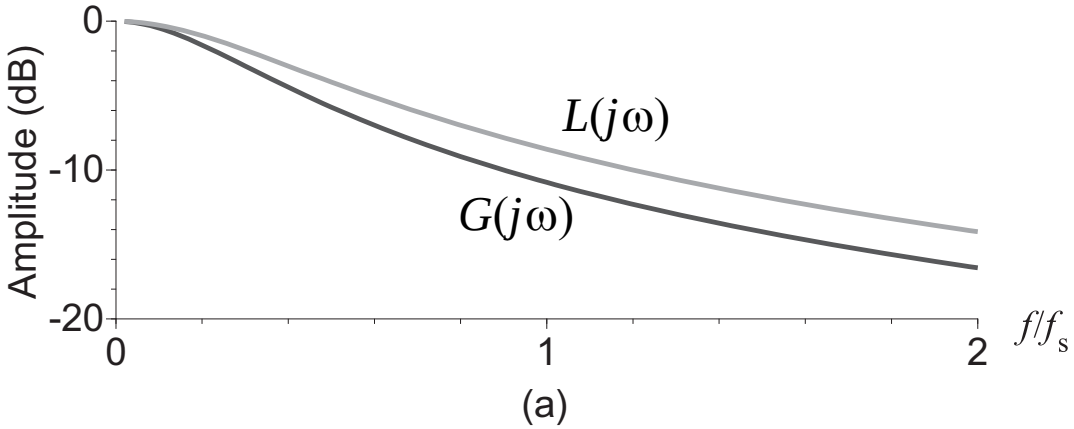
Choice of the model

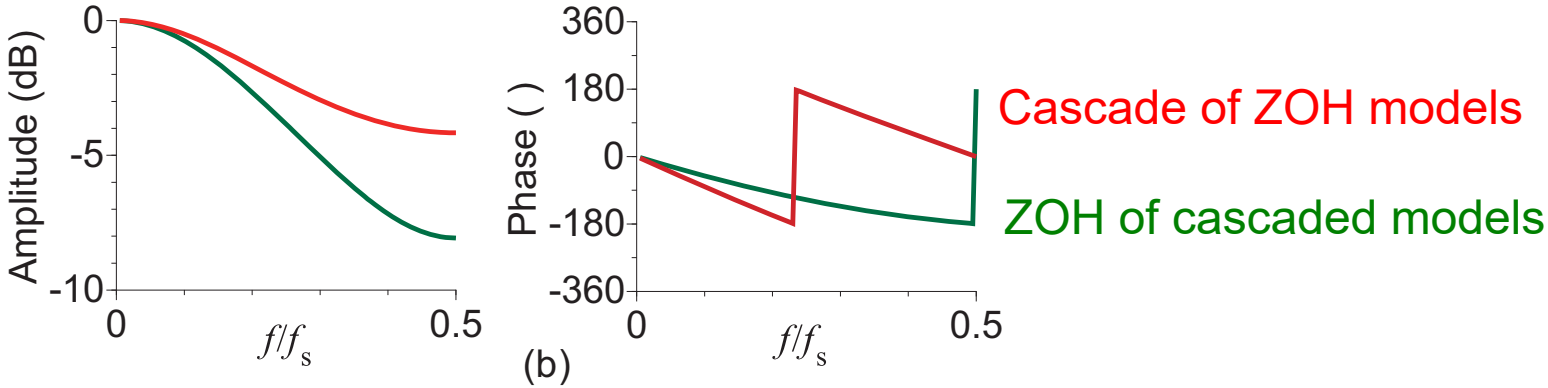
Possible combinations of continuous/discrete-time data and models.

	DT-model (Assuming ZOH-setup)	CT-model (Assuming BL-setup)
ZOH-setup	exact DT-model $G(z) = (1 - z^{-1}) Z\left\{ \frac{G(s)}{s} \right\}$ **'standard conditions DT modelling'**	Not studied
BL-setup	approximate DT model $\tilde{G}(z)$ $\tilde{G}(z = e^{j\omega T_s}) \approx G(s = j\omega),\ \lvert\omega\rvert < \frac{\omega_s}{2}$ **'digital signal processing field'**	exact CT-model $G(s)$ **'standard conditions CT modelling'**

Cascading models in simulations







Conclusion: BL-assumption is needed

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

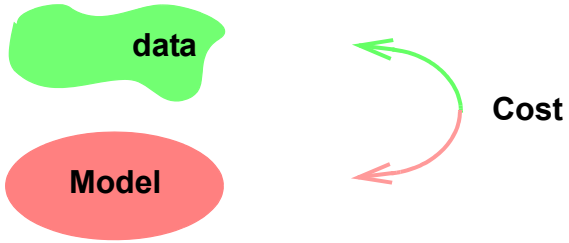
Cost function

Frequency domain formulation
Noise models
Time domain formulation

Validation

Examples

Conclusions



Parametric models of LTI systems

Continuous time

$$G(s, \theta) = \frac{B(s, \theta)}{A(s, \theta)} = \frac{\sum_{r=0}^{n_b} b_r s^r}{\sum_{r=0}^{n_a} a_r s^r}$$

Diffusion

$$G(\sqrt{s}, \theta) = \frac{B(\sqrt{s}, \theta)}{A(\sqrt{s}, \theta)} = \frac{\sum_{m=0}^{n_b} b_m s^{m/2}}{\sum_{n=0}^{n_a} a_n s^{n/2}}$$

Discrete time

$$G(z^{-1}, \theta) = \frac{B(z^{-1}, \theta)}{A(z^{-1}, \theta)} = \frac{\sum_{r=0}^{n_b} b_r z^{-r}}{\sum_{r=0}^{n_a} a_r z^{-r}}$$

General model

$$G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r} \text{ with } \Omega = \begin{cases} s & \text{continuous-time} \\ \sqrt{s} & \text{diffusion} \\ z^{-1} & \text{discrete-time} \end{cases}$$

Parametric models of LTI systems
Relation between input/output DFT spectra

Input/output DFT spectra

$$U(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} u(tT_s) z_k^{-t}, \; Y(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y(tT_s) z_k^{-t}$$

with $z_k = e^{j2\pi k/N}$

Remark

$U(k), Y(k)$ are an $O(N^0)$ for random excitations

Parametric models of LTI systems
Relation between input/output DFT spectra

Periodic signals

$$Y(k) = G(\Omega_k, \theta)U(k)$$

if
- steady state response
- integer number of periods are observed

Parametric models of LTI systems
Relation between input/output DFT spectra

Arbitrary signals

$$Y(k) = G(\Omega_k, \theta)U(k) + T_G(\Omega_k, \theta)$$

with

$$G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r}, \quad T_G(\Omega, \theta) = \frac{I_G(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_{i_g}} i_{g_r} \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r}$$

where

$$\Omega = \begin{cases} z^{-1} & n_{i_g} = \max(n_a, n_b) - 1 \\ s, \sqrt{s} & n_{i_g} > \max(n_a, n_b) - 1 \end{cases}$$

and

$$T_G(\Omega_k, \theta) = \begin{cases} 0 & \text{for periodic and time-limited signals} \\ O(N^{-1/2}) & \text{arbitrary signals} \end{cases}$$

Parametric models of LTI systems
Full equivalence time domain - frequency domain

Frequency domain

leakage $\Leftrightarrow$ begin and end conditions

$$Y(k) = G(\Omega_k, \theta)U(k) + T_G(\Omega_k, \theta)$$

Time domain

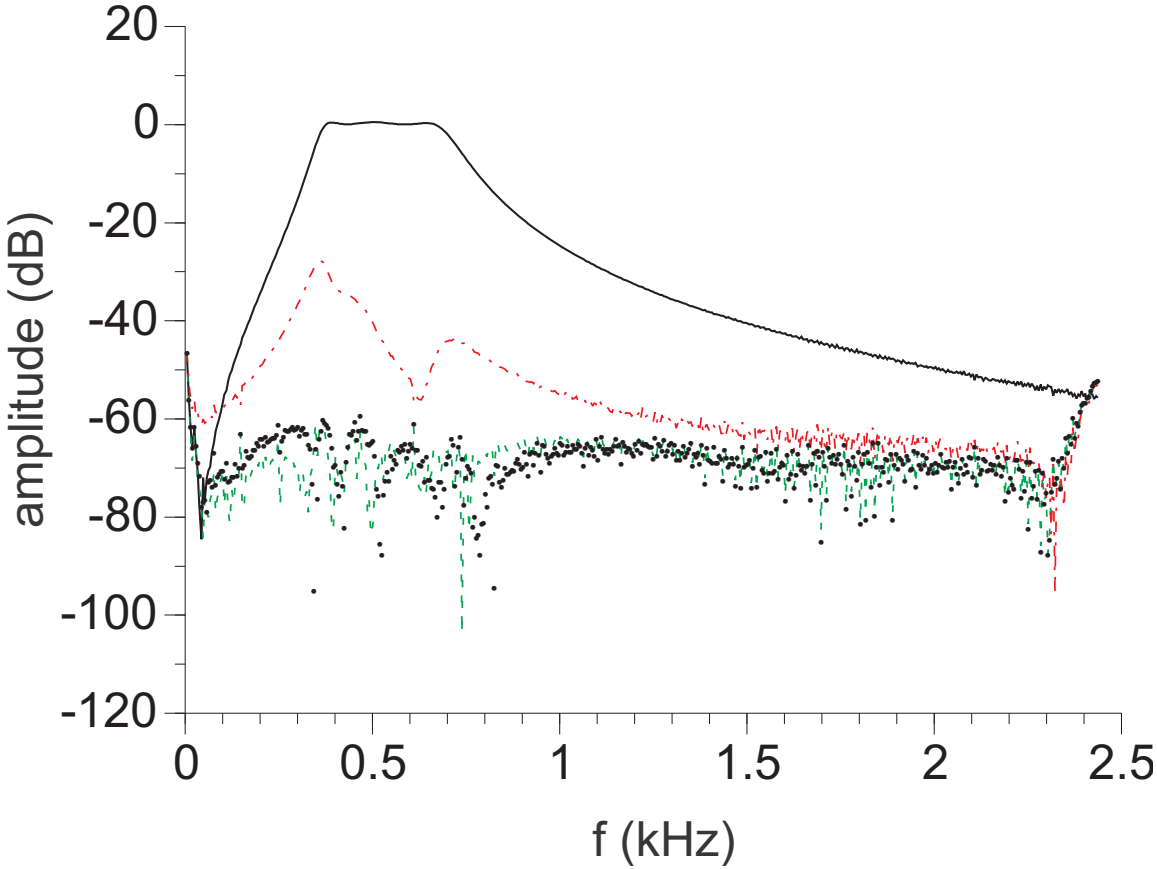
Transient effects due to initial conditions

$$y(t) = G(q, \theta)u(t) + T_G(t, \theta)$$

Experimental illustration: octave band-pass filter

Periodic multisine excitation and random noise excitation

plant model $n_a = 6, n_b = 4$



measured FRF (multisine excit.)

error model without transient (arbitr. excit.)

error model with transient $n_{i_g} = 6$ (arbitr. excit.)

error model (multisine excit.)

Parametric models of LTI systems
Parametrizations

- **Rational form**

$$G(\Omega, \theta) = \frac{B(\Omega, \theta)}{A(\Omega, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega^r}{\sum_{r=0}^{n_a} a_r \Omega^r} \text{ with } \theta^T = [a_0 a_1 \ldots a_{n_a} b_0 b_1 \ldots b_{n_b}]$$

- **Partial fraction expansion**

$$G(\Omega, \theta) = \sum_{\substack{r=-p \\ r \neq 0}}^{p} \frac{L_r}{\Omega - \lambda_r} + \sum_{r=1}^{q} \frac{S_r}{\Omega - \sigma_r} + W(\Omega, w) \text{ for } \Omega = s, \sqrt{s}$$

$$G(z^{-1}, \theta) = \sum_{\substack{r=-p \\ r \neq 0}}^{p} \frac{L_r z^{-1}}{1 - \lambda_r z^{-1}} + \sum_{r=1}^{q} \frac{S_r z^{-1}}{1 - \sigma_r z^{-1}} + W(z^{-1}, w)$$

with

$$\theta^T = [\sigma_1 \ldots \sigma_q \mathrm{Re}(\lambda_1) \mathrm{Im}(\lambda_1) \ldots \mathrm{Re}(\lambda_p) \mathrm{Im}(\lambda_p) \ldots$$
$$S_1 \ldots S_q \mathrm{Re}(L_1) \mathrm{Im}(L_1) \ldots \mathrm{Re}(L_p) \mathrm{Im}(L_p) w_0 \ldots w_{n_w}]$$

Parametric models of LTI systems
Parametrizations (cont'd)

- **State space representation** for proper transfer functions ($n_b \leq n_a$)

$$G(s, \theta) = C(sI_{n_a} - A)^{-1}B + D$$

$$G(z^{-1}, \theta) = z^{-1}C(I_{n_a} - z^{-1}A)^{-1}B + D$$

with

$$\theta^T = [\mathrm{vec}^T(A)\ B^T\ C\ D]$$

- **Pole/zero representation**

$$G(\Omega, \theta) = K\frac{\prod_{r=1}^{n_b}(\Omega - \zeta_r)}{\prod_{r=1}^{n_a}(\Omega - \lambda_r)}$$

disadvantage: ill conditioned for multiple poles/zeroes

- **Systems with time delay**

$$G(\Omega, \theta) = e^{-\tau s}\frac{B(\Omega, \theta)}{A(\Omega, \theta)}$$

$$G(z^{-1}, \theta) = z^{-\tau/T_s}\frac{B(z^{-1}, \theta)}{A(z^{-1}, \theta)}$$

Outline

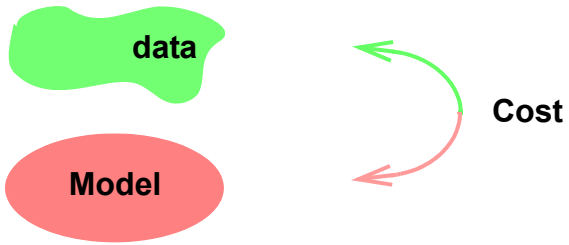
Introduction

Data: what is going on in between the samples

Model: parametric models of LTI-systems

Cost function

Frequency domain formulation
Noise models
Time domain formulation



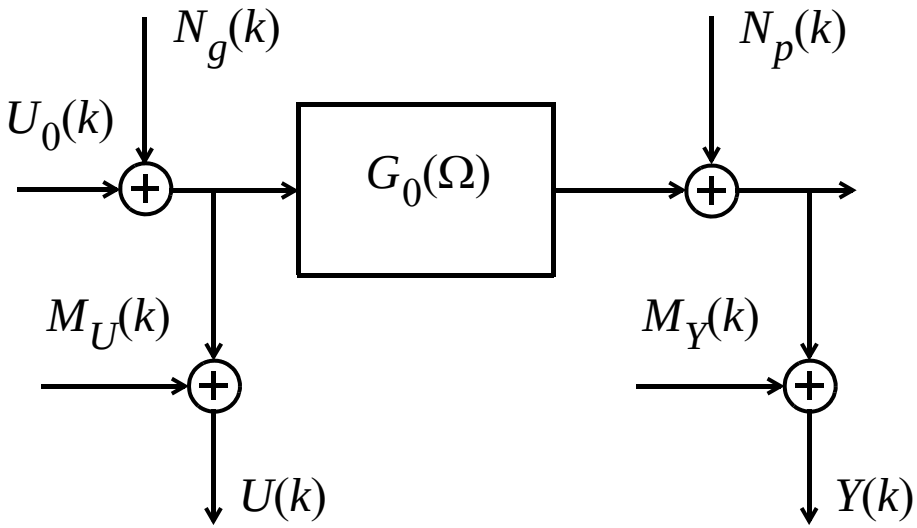
Validation

Examples

Conclusions

Basic problem for BL-setup

Setup



Measurements

$$\begin{aligned} Y(k) &= Y_0(k) + N_Y(k) \\ U(k) &= U_0(k) + N_U(k) \end{aligned}, \; k = 1, \ldots, F$$

Model

$$Y_0(k) = G_0(\Omega_k) U_0(k), \text{ with } G(\Omega_k) = \frac{B(\Omega_k, \theta)}{A(\Omega_k, \theta)} = \frac{\sum_{r=0}^{n_b} b_r \Omega_k^r}{\sum_{r=0}^{n_a} a_r \Omega_k^r}$$

Match model and data: choice of the cost function
Intuitive choice

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|G(\Omega_k) - G(\Omega_k, \theta)\right|^2}{\sigma_G^2(k)}$$

- $G(\Omega_k)$: measured FRF

- $\sigma_G^2(k)$: uncertainty

Works amazingly well in many situations

Problem: good measurements in the presence of input noise

$$\lim_{M \to \infty} G(\Omega_k) = \frac{S_{YU}(\Omega_k)}{S_{UU}(\Omega_k)} = G_0(\Omega_k)\frac{1}{1 + S_{M_U M_U}(\Omega_k)/S_{UU}(\Omega_k)}$$

Alternative

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|U(k) - U_{\mathrm{p}}(k)\right|^2}{\sigma_U^2(k)} + \frac{\left|Y(k) - Y_{\mathrm{p}}(k)\right|^2}{\sigma_Y^2(k)}$$

$$= \frac{1}{F}\sum_{k=1}^{F} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}^H \begin{bmatrix} \sigma_Y^2(k) & 0 \\ 0 & \sigma_U^2(k) \end{bmatrix}^{-1} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}$$

under the constraint

$$Y_{\mathrm{p}}(k) = G(\Omega_k, \theta)U_{\mathrm{p}}(k) \qquad k = 1, 2, \ldots, F$$

Parameters to be estimated:

θ: $n_a + n_b + 1$ real parameters

$U_{\mathrm{p}}, Y_{\mathrm{p}}$: $2F$ complex parameters

Generalized problem

correlated input output noise

$$V_F(\theta, Z) = \frac{1}{F} \sum_{k=1}^{F} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}^H \begin{bmatrix} \sigma_Y^2(k) & \sigma_{YU}^2(k) \\ \sigma_{UY}^2(k) & \sigma_U^2(k) \end{bmatrix}^{-1} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}$$

under the constraint

$$Y_{\mathrm{p}}(k) = G(\Omega_k, \theta) U_{\mathrm{p}}(k) \qquad k = 1, 2, \ldots, F$$

Parameters to be estimated:

θ: $n_a + n_b + 1$ real parameters

$U_{\mathrm{p}}, Y_{\mathrm{p}}$: $2F$ complex parameters

Generalized problem

$$V_F(\theta, Z) = \frac{1}{F} \sum_{k=1}^{F} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}^H \begin{bmatrix} \sigma_Y^2(k) & \sigma_{YU}^2(k) \\ \sigma_{UY}^2(k) & \sigma_U^2(k) \end{bmatrix}^{-1} \begin{pmatrix} Y(k) - Y_{\mathrm{p}}(k) \\ U(k) - U_{\mathrm{p}}(k) \end{pmatrix}$$

under the constraint

$$Y_{\mathrm{p}}(k) = G(\Omega_k, \theta) U_{\mathrm{p}}(k) \qquad k = 1, 2, \ldots, F$$

This is the maximum likelihood estimator

- Gaussian distributed noise

- Known covariance matrix

Elimination of U_p, Y_p

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|Y(k) - G(\Omega_k, \theta)U(k)\right|^2}{\sigma_Y^2(k) + \sigma_U^2(k)\left|G(\Omega_k, \theta)\right|^2 - 2\mathrm{Re}(\sigma_{YU}^2(k)\overline{G}(\Omega_k, \theta))}$$

Symmetric formulation $G = B/A$

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|A(\Omega_k, \theta)Y(k) - B(\Omega_k, \theta)U(k)\right|^2}{\sigma_Y^2(k)\left|A(\Omega_k, \theta)\right|^2 + \sigma_U^2(k)\left|B(\Omega_k, \theta)\right|^2 - 2\mathrm{Re}(\sigma_{YU}^2(k)A(\Omega_k, \theta)\overline{B}(\Omega_k, \theta))}$$

Special case 1: identifying from the measured FRF

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|Y(k) - G(\Omega_k, \theta)U(k)\right|^2}{\sigma_Y^2(k) + \sigma_U^2(k)\left|G(\Omega_k, \theta)\right|^2 - 2\mathrm{Re}(\sigma_{YU}^2(k)\overline{G}(\Omega_k, \theta))}$$

Put

$$Y(k) = G(\Omega_k),\ U(k) = 1,$$

and

$$\sigma_U^2(k) = 0, \text{ and } \sigma_Y^2(k) = \sigma_G^2(k)$$

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|G(\Omega_k) - G(\Omega_k, \theta)\right|^2}{\sigma_G^2(k)}$$

Special case 2: the input is exactly known

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|Y(k) - G(\Omega_k, \theta)U(k)\right|^2}{\sigma_Y^2(k) + \sigma_U^2(k)\left|G(\Omega_k, \theta)\right|^2 - 2\mathrm{Re}(\sigma_{YU}^2(k)\overline{G}(\Omega_k, \theta))}$$

Put

$$\sigma_U^2(k) = 0\,,\ \sigma_{YU}^2(k) = 0$$

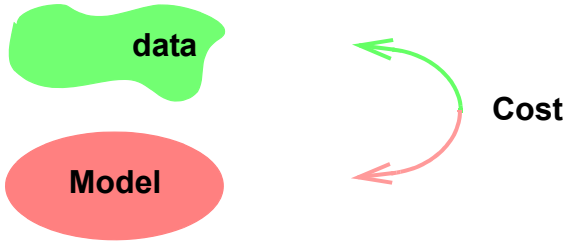
$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=1}^{F} \frac{\left|Y(k) - G(\Omega_k, \theta)U(k)\right|^2}{\sigma_Y^2(k)}$$

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems



Cost function

Frequency domain formulation
Noise models
Time domain formulation

Validation

Examples

Conclusions

Noise models

Time domain

$$\begin{aligned} v(t) &= H(q)e(t) + T_h(t) \\ &\approx H(q)e(t) \end{aligned} \quad \text{and } E\{v(r)v(s)\} = R_{vv}(r-s)$$

Frequency domain

$$\begin{aligned} V(k) &= H(k)E(k) + T_H(k) \\ &\approx H(k)E(k) \end{aligned} \quad \text{and } E\{V(k)^H V(l)\} = \sigma_V^2(k)\delta_{kl} + O(N^{-1})$$

cost frequency domain

$$V^H \begin{bmatrix} \ddots & & O(N^{-1}) \\ & \sigma_V^2(k) & \\ O(N^{-1}) & & \ddots \end{bmatrix}^{-1} V$$

$$\approx$$

$$V^H \begin{bmatrix} \ddots & & 0 \\ & \sigma_V^2(k) & \\ 0 & & \ddots \end{bmatrix}^{-1} V$$

or

cost time domain

$$v^T \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & R_{vv}(k) & \ddots \\ \ddots & \ddots & R_{vv}(0) & \ddots & \ddots \\ \ddots & R_{vv}(k) & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}^{-1} v$$

$$(H^{-1}e)^T \begin{bmatrix} \ddots & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}^{-1} (H^{-1}e)$$

Noise models

cost function frequency domain

$$V^H \begin{bmatrix} \ddots & & 0 \\ & \sigma_V^{-2}(k) & \\ 0 & & \ddots \end{bmatrix} V$$

- nonparametric noise model $\sigma_V^2(k)$
- no interference with plant estimate
- periodic excitation --> very simple extraction
- arbitrary excitation --> more complicated

cost function time domain

$$(H^{-1}e)^T \begin{bmatrix} \ddots & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}^{-1} (H^{-1}e)$$

- simultaneous identification
  parametric plant/noise model
- Errors-in-Variables
  also parametric model excitation
- Only noise on output
  classic prediction error identif.

Noise model frequency domain

Cost function

$$V_F(\theta, Z) = \frac{1}{F} \sum_{k=1}^{F} \begin{pmatrix} Y(k) - Y_p(k) \\ U(k) - U_p(k) \end{pmatrix}^H \begin{bmatrix} \sigma_Y^2(k) & \sigma_{YU}^2(k) \\ \sigma_{UY}^2(k) & \sigma_U^2(k) \end{bmatrix}^{-1} \begin{pmatrix} Y(k) - Y_p(k) \\ U(k) - U_p(k) \end{pmatrix}$$

2nd order moments of the noise needed: to be extracted from the data

Prior analysis

separate signals and noise
extract a nonparametric noise model

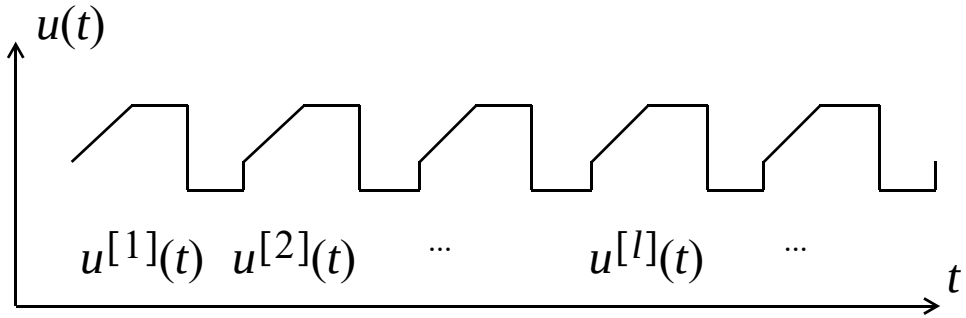
1) periodic excitations

2) arbitrary excitations

Noise model, prior analysis periodic excitation

Identify $\sigma_U^2(k)$, $\sigma_Y^2(k)$ and $\sigma_{YU}^2(k)$

Additional information: the signals are periodic



$$\hat{U}(k) = \frac{1}{M}\sum_{l=1}^{M} U^{[l]}(k), \quad \hat{Y}(k) = \frac{1}{M}\sum_{l=1}^{M} Y^{[l]}(k),$$

$$\hat{\sigma}_U^2(k) = \frac{1}{M-1}\sum_{l=1}^{M} \left|U^{[l]}(k) - \hat{U}(k)\right|^2 \text{ and } \hat{\sigma}_Y^2(k) = \frac{1}{M-1}\sum_{l=1}^{M} \left|Y^{[l]}(k) - \hat{Y}(k)\right|^2$$

$$\hat{\sigma}_{YU}^2(k) = \frac{1}{M-1}\sum_{l=1}^{M} (Y^{[l]}(k) - \hat{Y}(k))\overline{(U^{[l]}(k) - \hat{U}(k))}$$

Noise model, prior analysis
periodic excitation

Properties

consistency: $M = 4$ periods are enough

efficiency:

$M = 6$ periods are enough

'loss' in efficiency $C_{\theta \mathrm{SML}} = \frac{M-2}{M-3} C_{\theta \mathrm{ML}}$

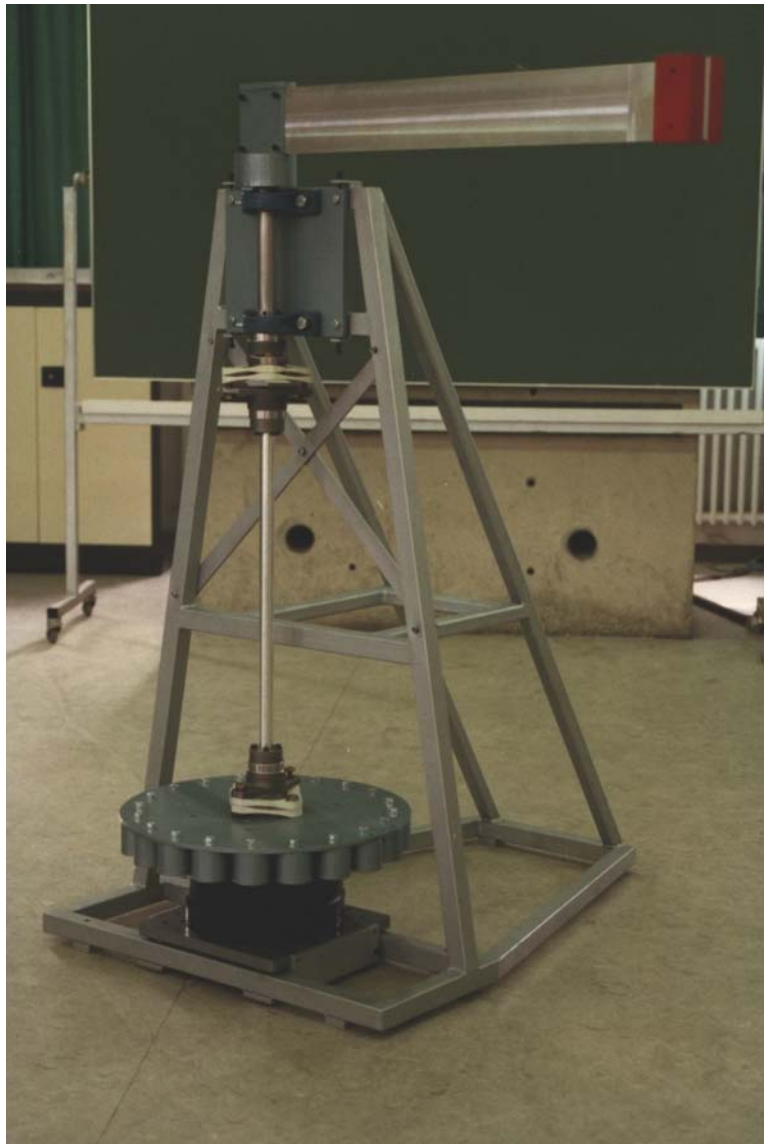
normality: $M = 7$ is enough

Recent results

2 periods + overlapping windows are enough

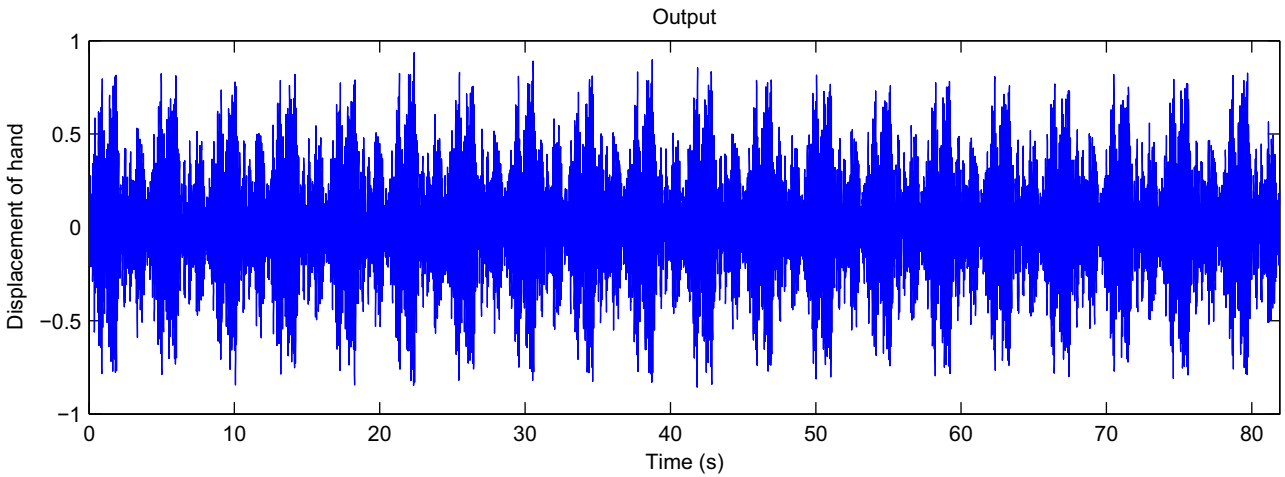
additional loss in efficiency: about 15% (compared to $M = 6$, no overlap)

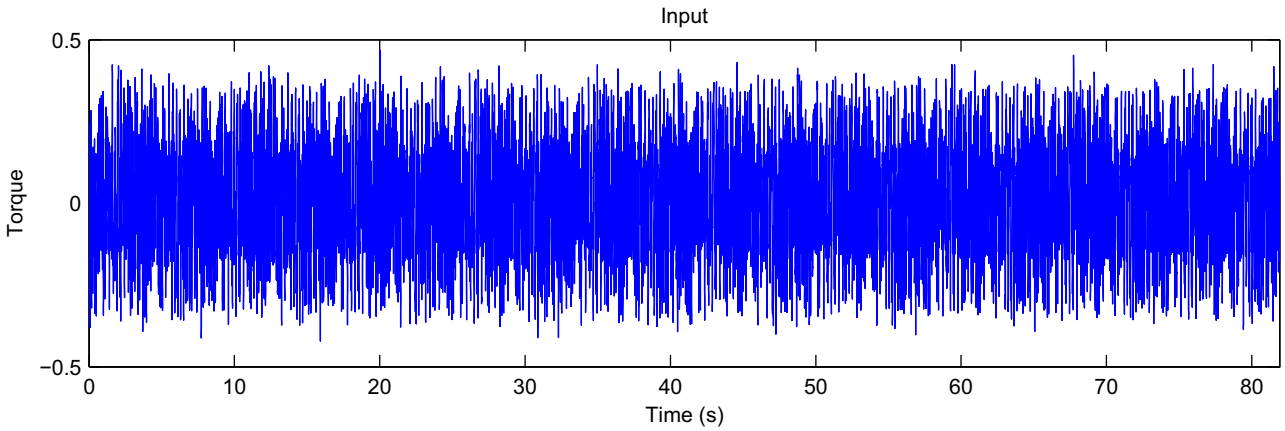
Example of a nonparametric prior noise analysis
The flexible robot arm



Data from Jan Swevers, KULeuven, PMA

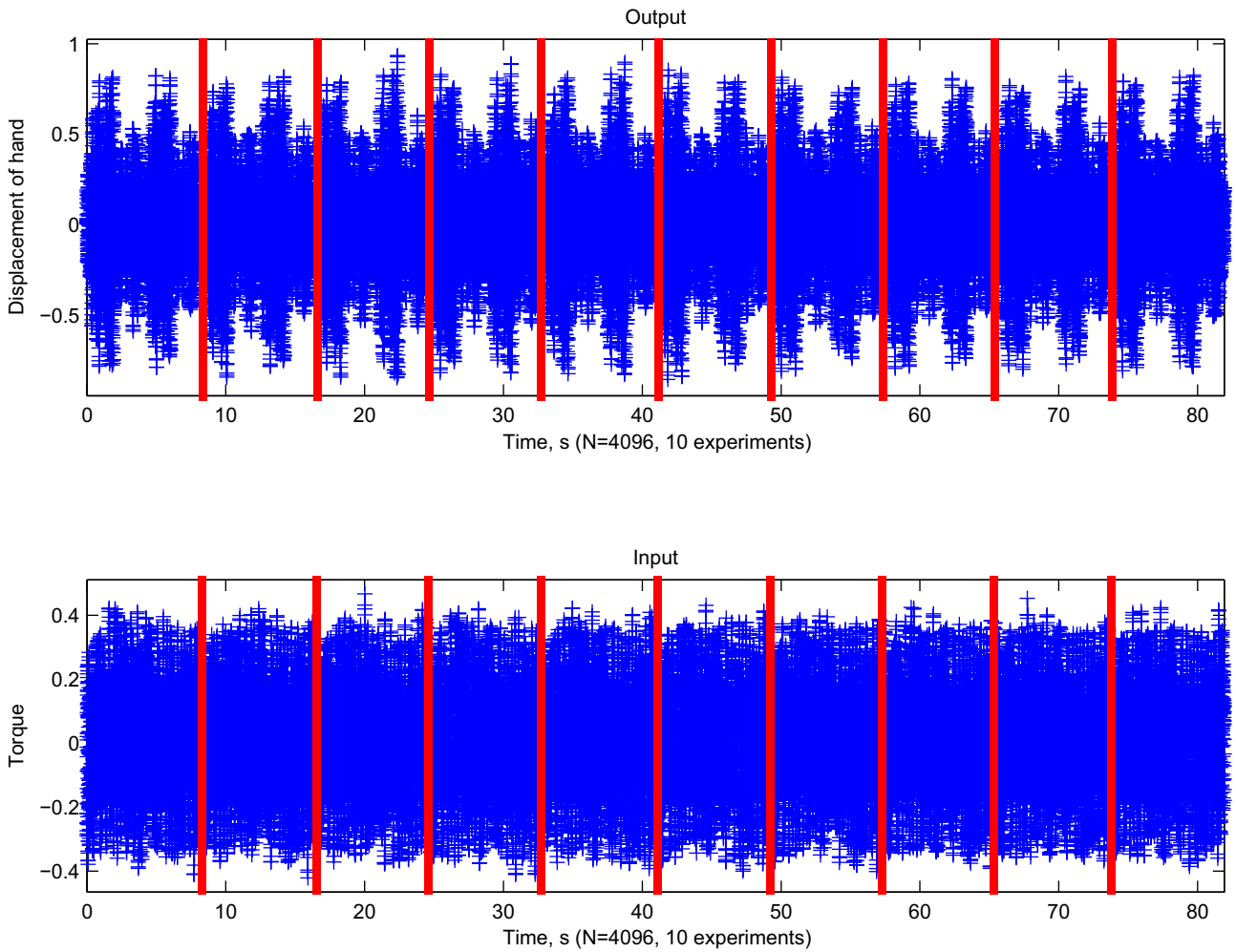
Raw data





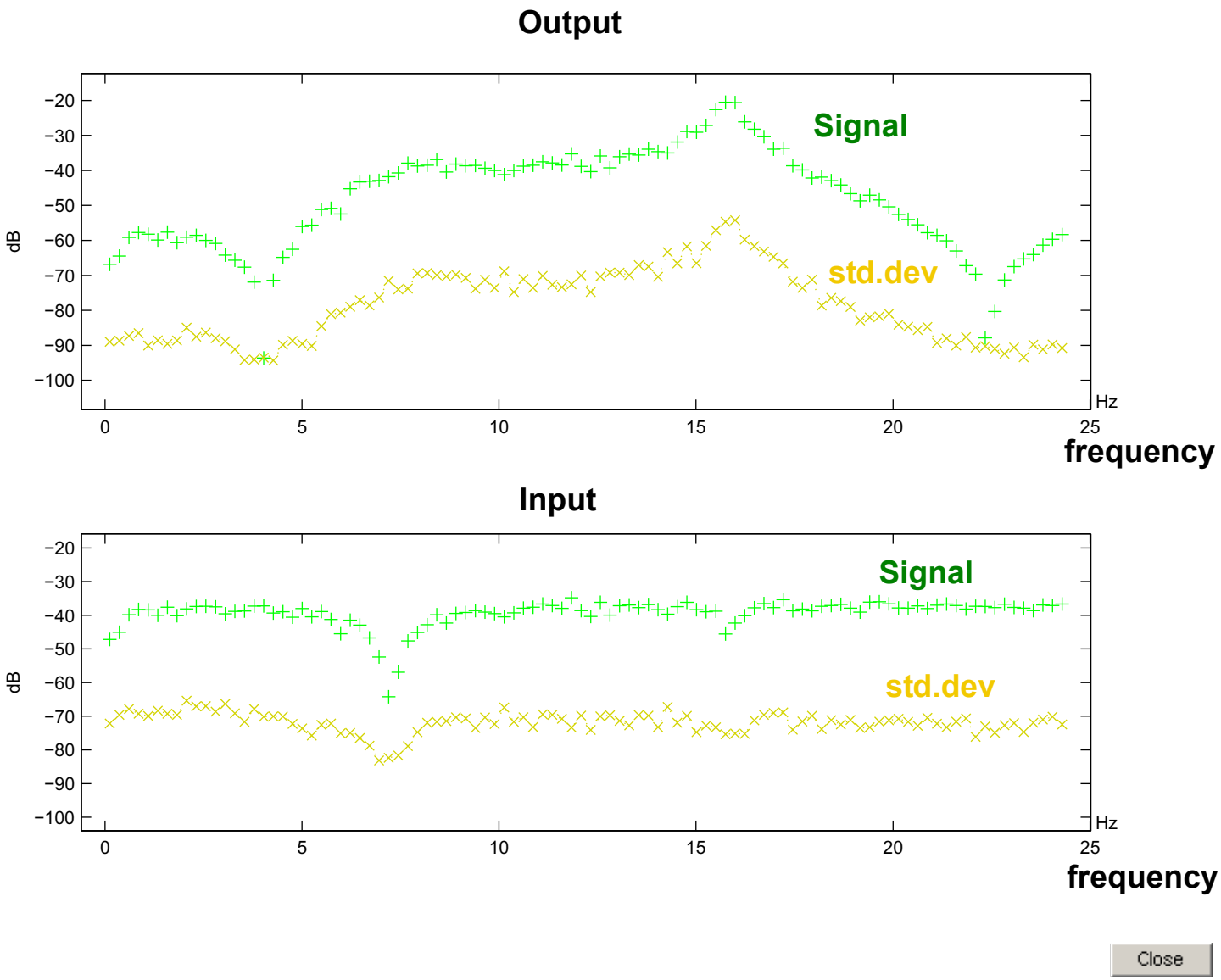
Close

Segment the record
10 periods

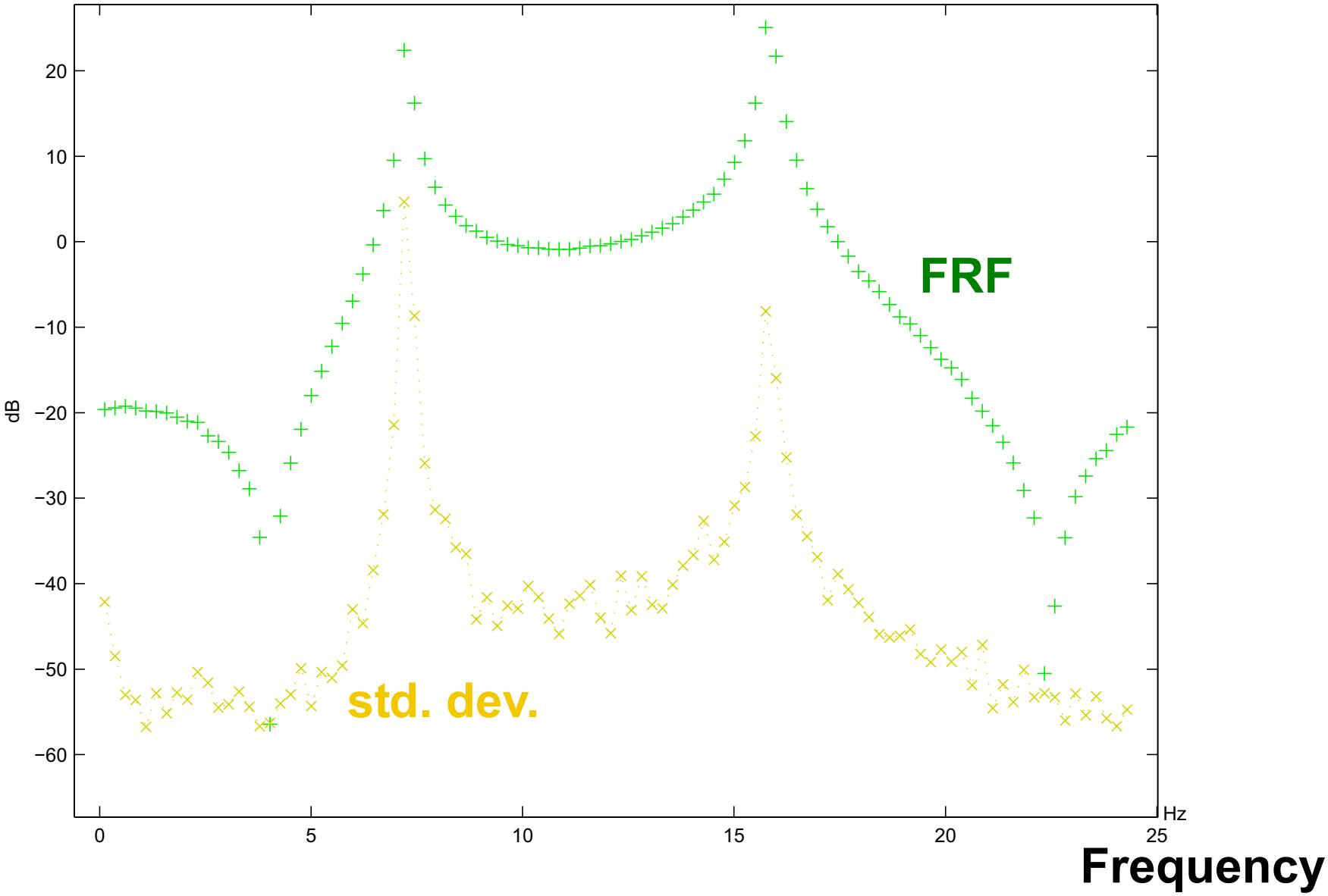


Close

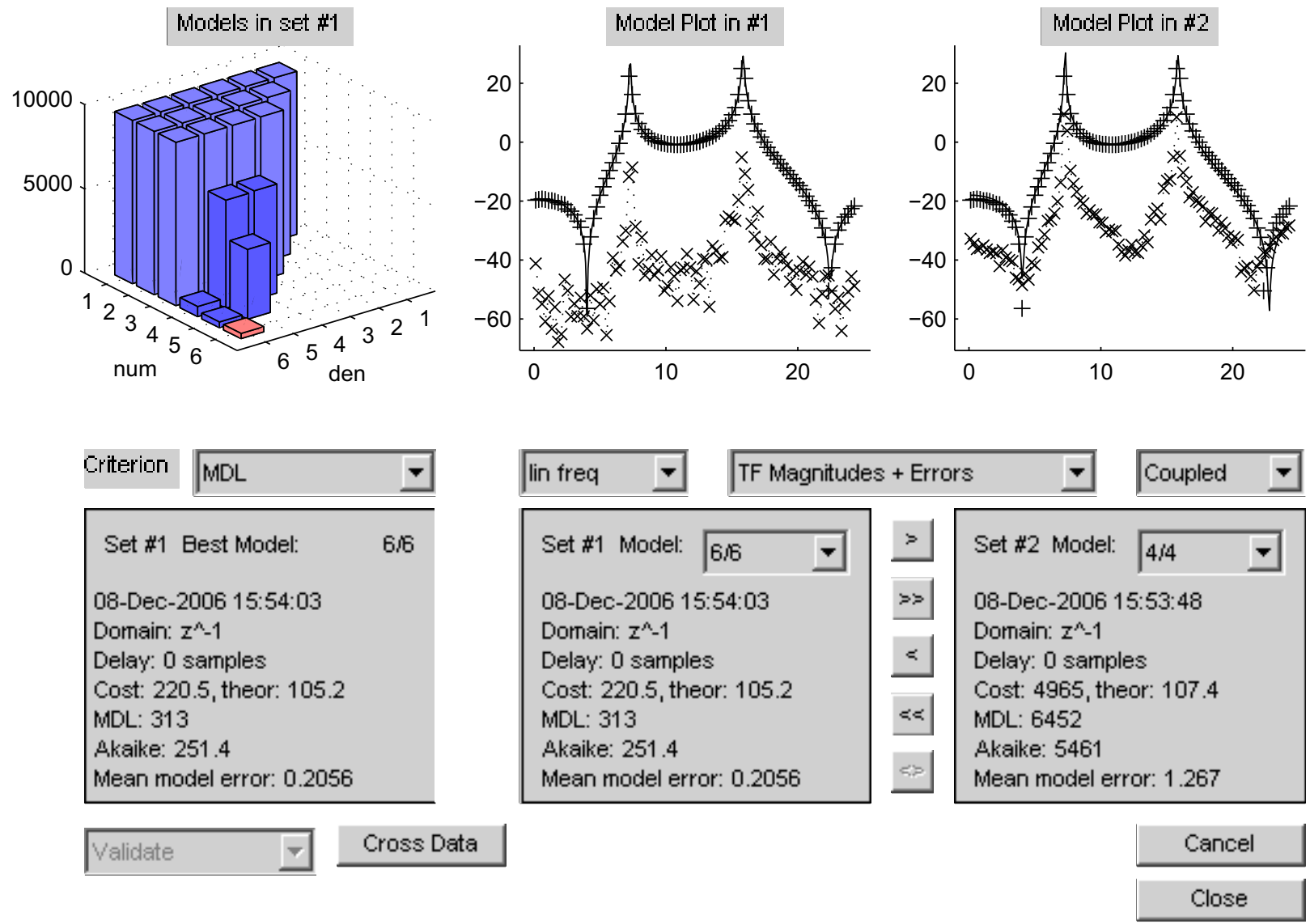
Variance analysis



Variance analysis
FRF



Estimated model



Noise model frequency domain

Cost function

$$V_F(\theta, Z) = \frac{1}{F} \sum_{k=1}^{F} \begin{pmatrix} Y(k) - Y_p(k) \\ U(k) - U_p(k) \end{pmatrix}^H \begin{bmatrix} \sigma_Y^2(k) & \sigma_{YU}^2(k) \\ \sigma_{UY}^2(k) & \sigma_U^2(k) \end{bmatrix}^{-1} \begin{pmatrix} Y(k) - Y_p(k) \\ U(k) - U_p(k) \end{pmatrix}$$

2nd order moments of the noise needed: to be extracted from the data

Prior analysis

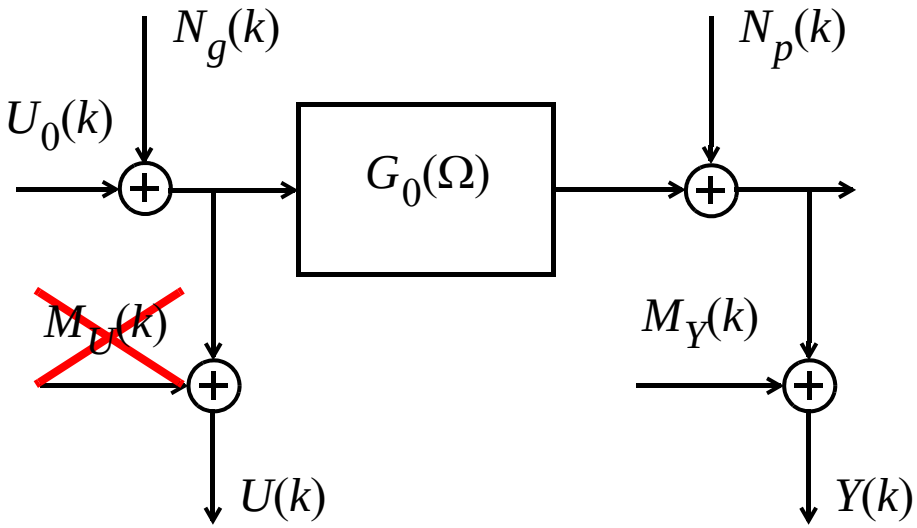
separate signals and noise
extract a nonparametric noise model

1) periodic excitations

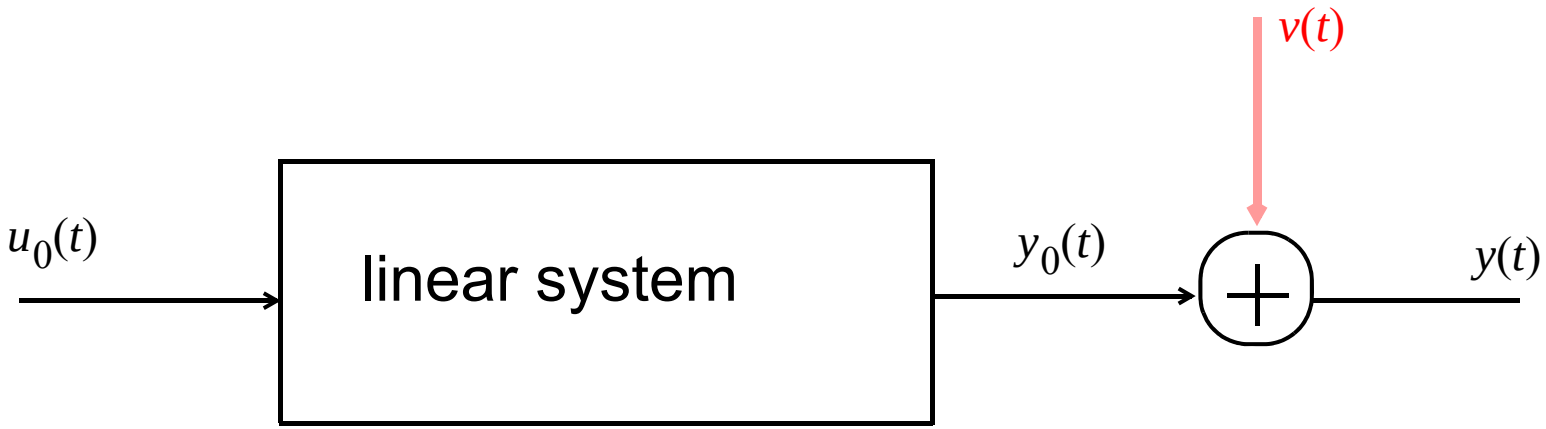
2) arbitrary excitations

Nonparametric noise model, prior analysis arbitrary excitation

Simplification required: only noise on the output



Basic idea



Basic idea: eliminate $G_0(k)U_0(k)$ and $T_0(k)$

coherence $$S_{YY}(f) - \frac{\left|S_{YU_0}(f)\right|^2}{S_{U_0U_0}(f)} = \sigma_v^2(f)$$ (+ leakage errors)

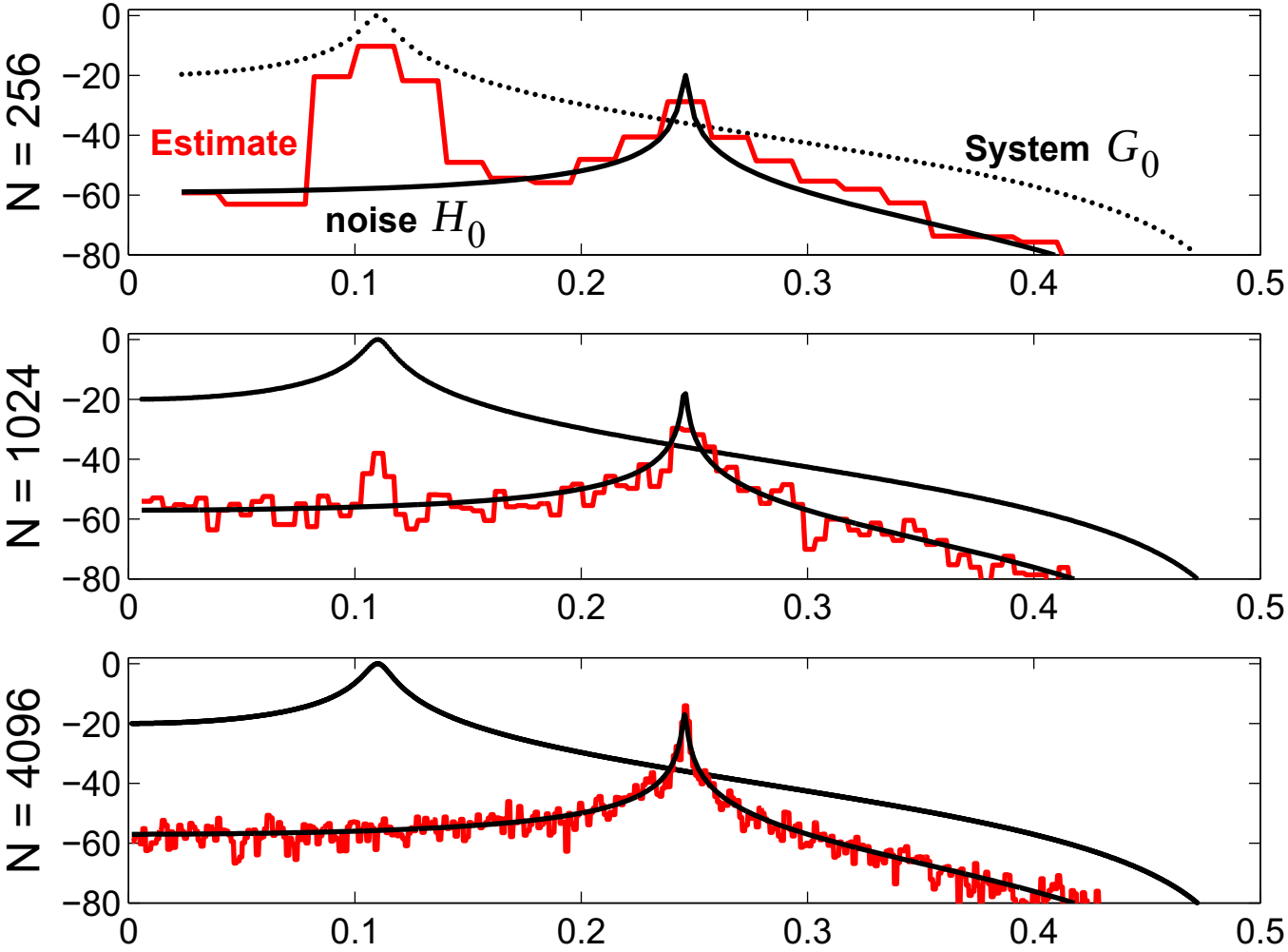
More advanced methods

solve set of equations at multiple frequencies

$G_0(k)$, $T_0(k)$ smooth --> Taylor

$Y(k) = G_0(k)U_0(k) + T_0(k) + V(k)$ with $k-n, \ldots, k, \ldots, k+n$

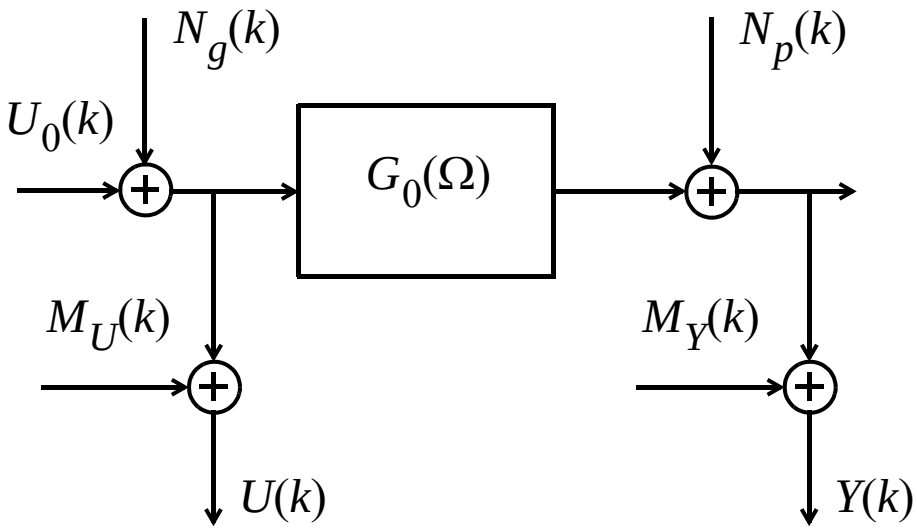
Noise model, prior analysis
arbitrary excitation
Example



Parametric noise model
Simultaneous analysis - General problem

Estimate plant and noise model together

Extract a parametric noise model



Additional constraints needed

- NO cross-correlation between M_U and N_p, M_Y

- input: filtered white noise --> a parametric input model is also estimated

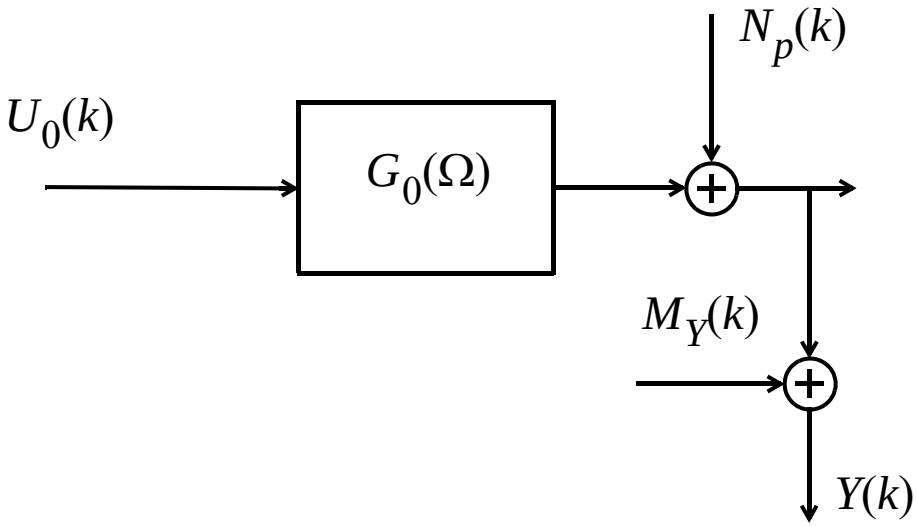
Errors-in-variables problem --> out of the scope of this course

Parametric noise model
Simultaneous analysis - Simplified problem

Input is exactly known

Estimate parametric plant and noise model together

No signal model needed



Classical prediction error method --> time domain identification

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

Cost function

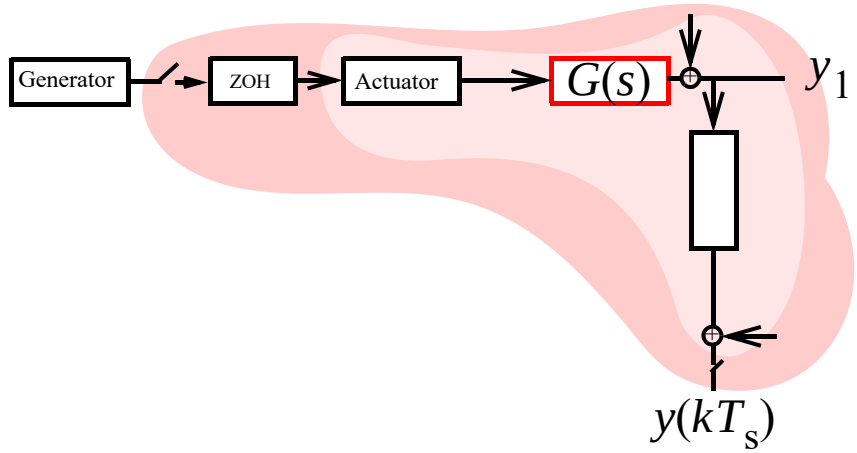
- Frequency domain formulation
- Noise models
- Time domain formulation

Validation

Examples

Conclusions

Time domain identification



- Use discrete time models: $G(z^{-1}, \theta)$

- Assume that the input is exactly known: $\sigma_U^2(k) = 0$, $\sigma_{YU}^2(k) = 0$

- Use a parametric noise model: $\sigma_Y^2(k) = \left|H(z^{-1}, \theta)\right|^2$

- The cost function becomes:

$$V_F(\theta, Z) = \frac{1}{F}\sum_{k=0}^{N-1} \frac{\left|Y(k) - G(z_k^{-1}, \theta)U(k)\right|^2}{\left|H(z_k^{-1}, \theta)\right|^2}, \text{ with } z_k^{-1} = e^{-j\frac{2\pi k}{N}}$$

Time domain identification (Cont'd)

Interpretation in the time domain:

model

$$y(t) = G_0(q)u(t) + H(q)e(t)$$

cost function

$$\hat{y}(t|t-1) = H^{-1}(q,\theta)G(q,\theta)u(t) + (1 - H^{-1}(q,\theta))y(t),$$

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|t-1)$$

$$V_N(\theta, Z) = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon(t,\theta)^2$$

Time domain identification (Cont'd)

$$V = \frac{1}{N}\sum_{t=0}^{N-1} e(t)^2, \text{ or } V = \frac{1}{N}\sum_{k=0}^{N-1} \frac{\left| Y(k) - \frac{B(z_k^{-1}, \theta)}{A(z_k^{-1}, \theta)} U(k) \right|^2}{\left| H(z_k^{-1}, \theta) \right|^2}$$

1) $G(q, \theta) = \frac{B(q, \theta)}{A(q, \theta)}$ and $H(q) = \frac{1}{A(q, \theta)}$ --> $A(q, \theta)y(t) = B(q, \theta)u(t) + e(t)$

problem that is linear-in-the-parameters (ARX)

2) $G(q, \theta) = \frac{B(q, \theta)}{A(q, \theta)}$ and $H(q) = \frac{C(q, \theta)}{A(q, \theta)}$ --> $A(q, \theta)y(t) = B(q, \theta)u(t) + C(q, \theta)e(t)$

problem that is linear-in-the-parameters (ARMAX)

3) $G(q, \theta) = \frac{B(q, \theta)}{A(q, \theta)}$ and $H(q) = 1$ --> $y(t) = \frac{B(q, \theta)}{A(q, \theta)}u(t) + e(t)$

problem that is nonlinear-in-the-parameters (Output Error)

4) $G(q, \theta) = \frac{B(q, \theta)}{A(q, \theta)}$ and $H(q) = \frac{D(q, \theta)}{C(q, \theta)}$ --> $y(t) = \frac{B(q, \theta)}{A(q, \theta)}u(t) + e(t)\frac{D(q, \theta)}{C(q, \theta)}$

problem that is nonlinear-in-the-parameters (Box-Jenkins)

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

Cost function

- Frequency domain formulation
- Noise models
- Time domain formulation

Validation

Examples

Conclusions

Validation

Classic

- Cross-correlation $R_{ue}(\tau) = 0$? --> test

- Auto-correlation residuals $\frac{\varepsilon(\hat{\theta}, k)}{\sigma_e(k)}$ or prediction errors white? --> test

Nonparametric noise models

$$V = \sum_{k=1}^{N} \frac{\varepsilon(\hat{\theta}, k)^2}{\sigma_e^2(k)} \sim \chi^2(N - n_\theta)$$

$E\{V\} = N - n_\theta$ --> check the actual value >< theoretic value

$\sigma_V^2 = 2(N - n_\theta)$

Remark: in classical prediction error framework

$\sigma_e(k)$ estimated from residuals --> includes model errors

Compare FRF modelled transfer function with measured FRF

Linear identification framework

Parametric noise model Classical prediction error frame work	Non-parametric noise model Frequency domain identification
	Preprocessing - non-parametric noise model
Estimates - parametric plant model - parametric noise model (nonlinear and disturbing noise)	Estimates - parametric plant model
Properties - consistent - efficient - normal	Properties - consistent - efficient - normal
Validation - nonlinearity is NOT detected	Validation - nonlinearity is detected - alternative validation scheme
Happy but 'unconscious' user	**Happy but 'conscious' user**

Time domain versus frequency domain identification

- Transforming data from time to frequency domain does not create or delete information!

- There exists a full equivalence between both approaches

- Practical issues are decisive
  - some information easier accessible in one domain than in the other
  - (non causal) prefiltering in frequency domain
  - improved SNR --> simpler generation of starting values
  - combining different sampling frequencies --> wide frequency range

- Use periodic excitations if possible --> access to a nonparametric noise model

Some of these aspects will be illustrated on the examples

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

Cost function

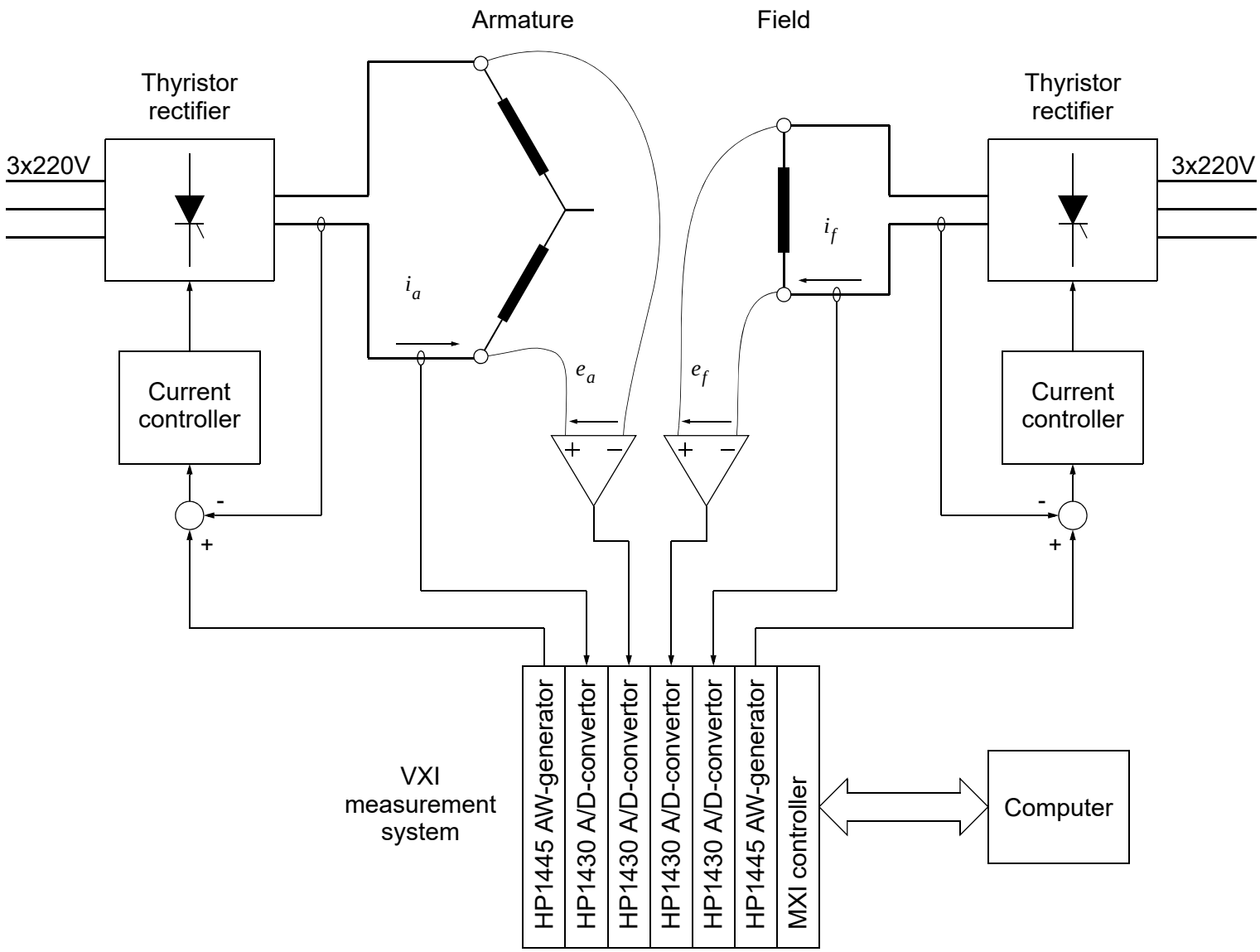
- Frequency domain formulation
- Noise models
- Time domain formulation

Validation

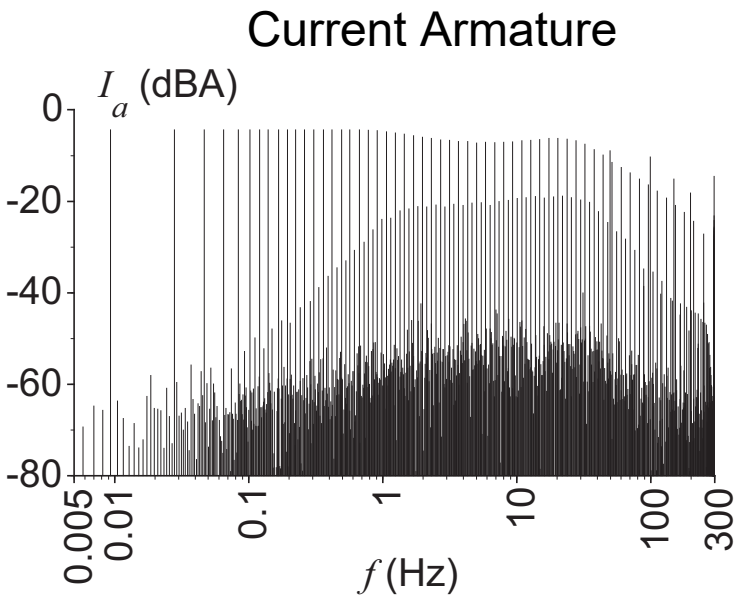
Examples

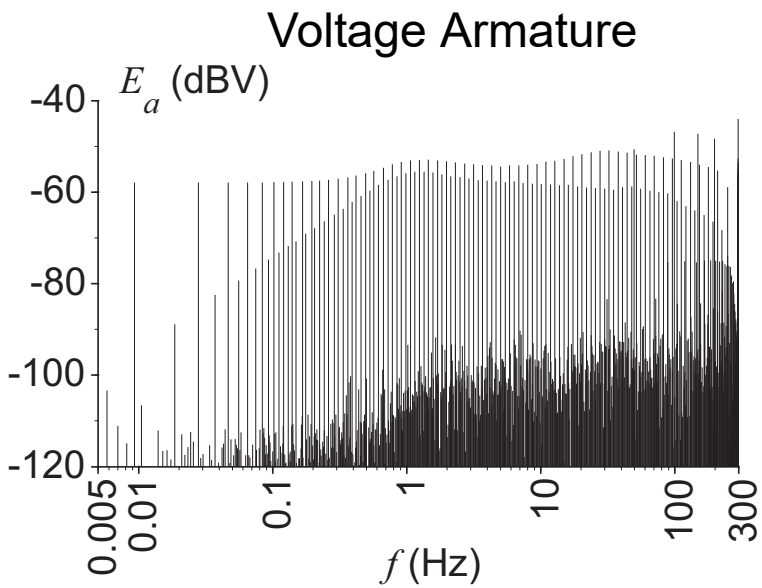
Conclusions

Example 1
Identification d-axis synchronous machine



Identification d-axis synchronous machine

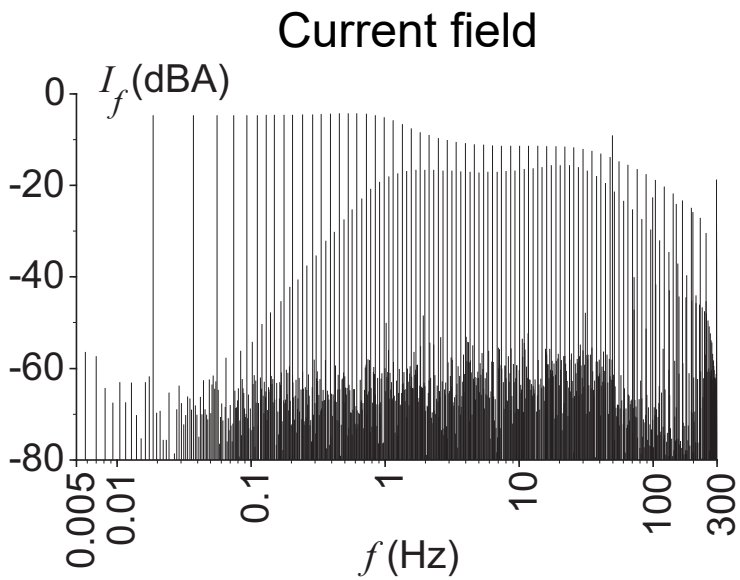


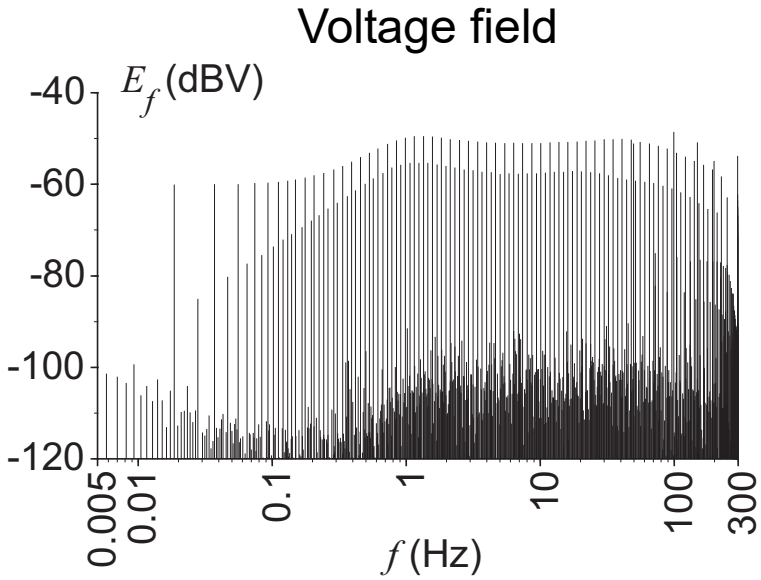


$M = 8$

$N = 65536$

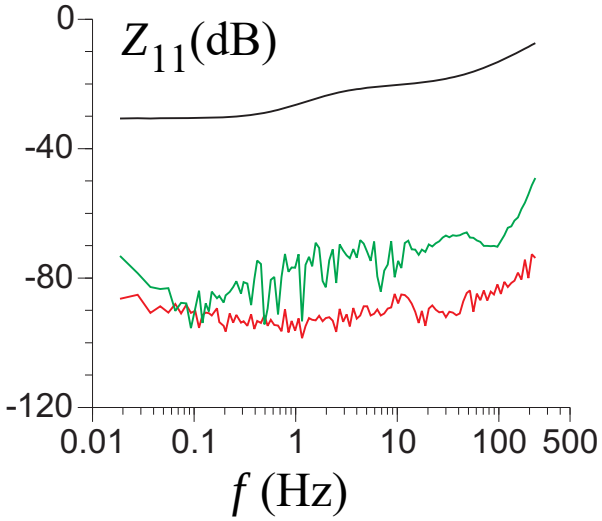
[0,01 Hz, 230 Hz

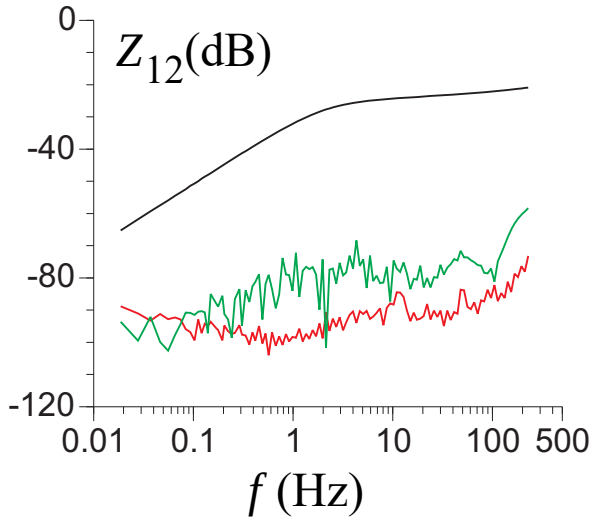


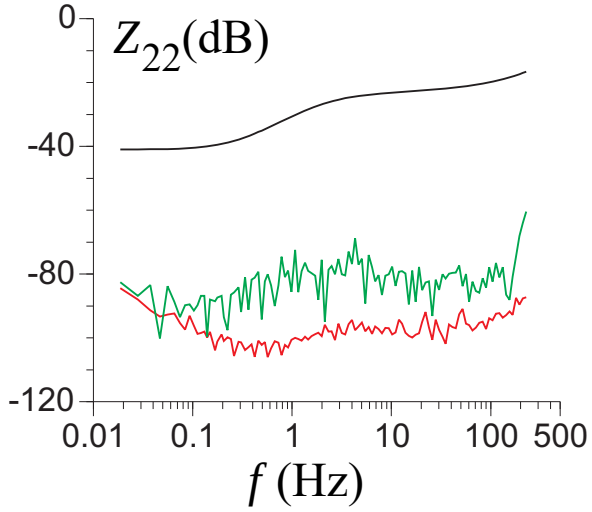


Estimation parametric plant model with estimated nonparametric noise model

Measurement example: identification d-axis synchronous machine (cont'd)







measured FRF

noise variance

difference modelled and measured FRF

$$Z(s, \theta) = \frac{\sum_{r=0}^{n_b} B_r s^r}{\sum_{r=0}^{n_b - 1} a_r s^r}$$

$$B_r^T = B_r$$

$$n_b = 6$$

Estimation parametric plant model with estimated nonparametric noise model

Measurement example: identification d-axis synchronous machine (cont'd)

n_b	$V_{\mathrm{SML}}(\hat{\theta}, Z)$	$V_{\mathrm{Theoretic}}$
2	3.97e5	625.8
3	3.21e4	620.2
4	8.15e3	614.6
5	3.12e3	609.0
6	2.18e3	603.4
7	2.14e3	597.8

Cost function much too large --> model errors

Observations

- a very wide frequency range is covered [0.01 Hz, 230 Hz]

- improper models can be used (more zeros than poles)

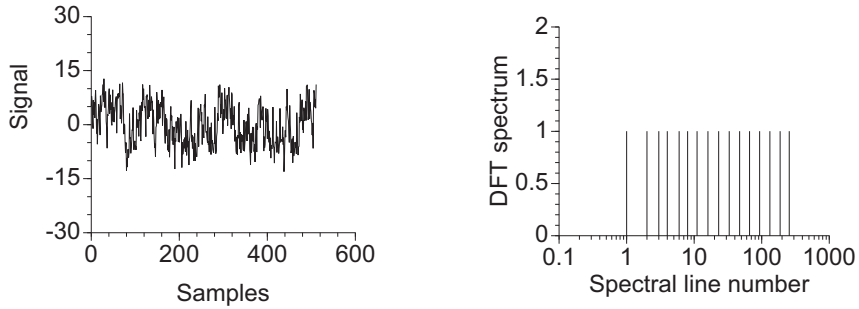
- model errors are easily detected

- only a small number of frequencies is excited

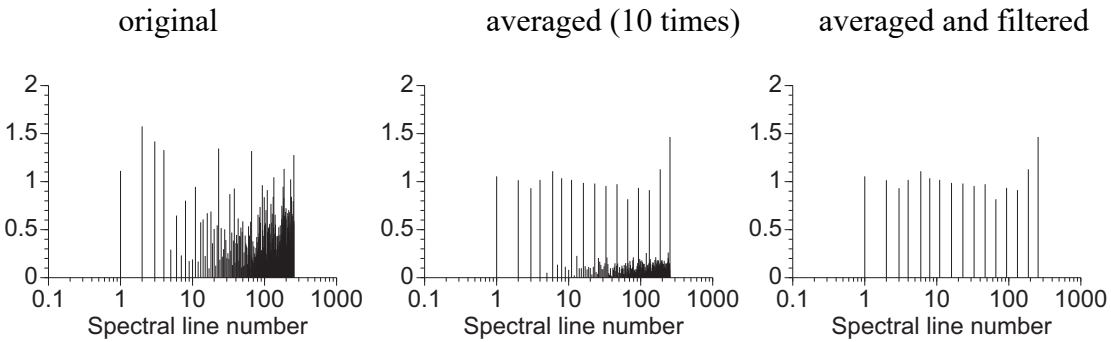
    - a high SNR on these lines --> averaging and filtering effect
    - generation of initial estimates

averaging and filtering: Elimination of non excited frequencies

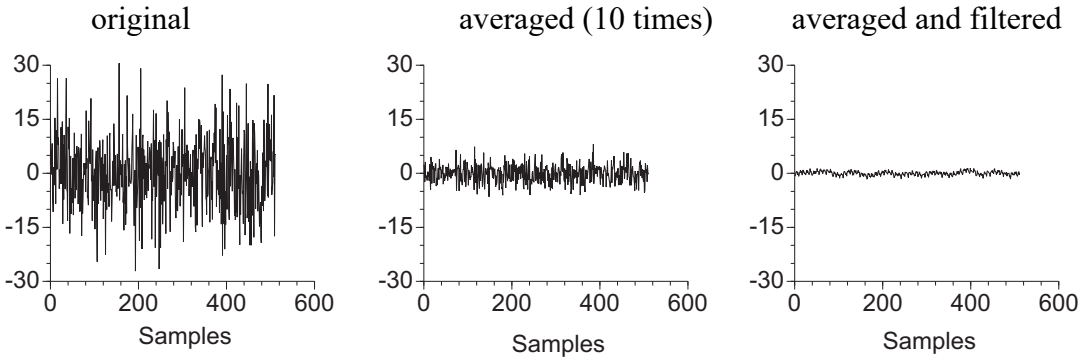
Original signal



Signal + noise (freq. domain)



Additive noise (time domain)

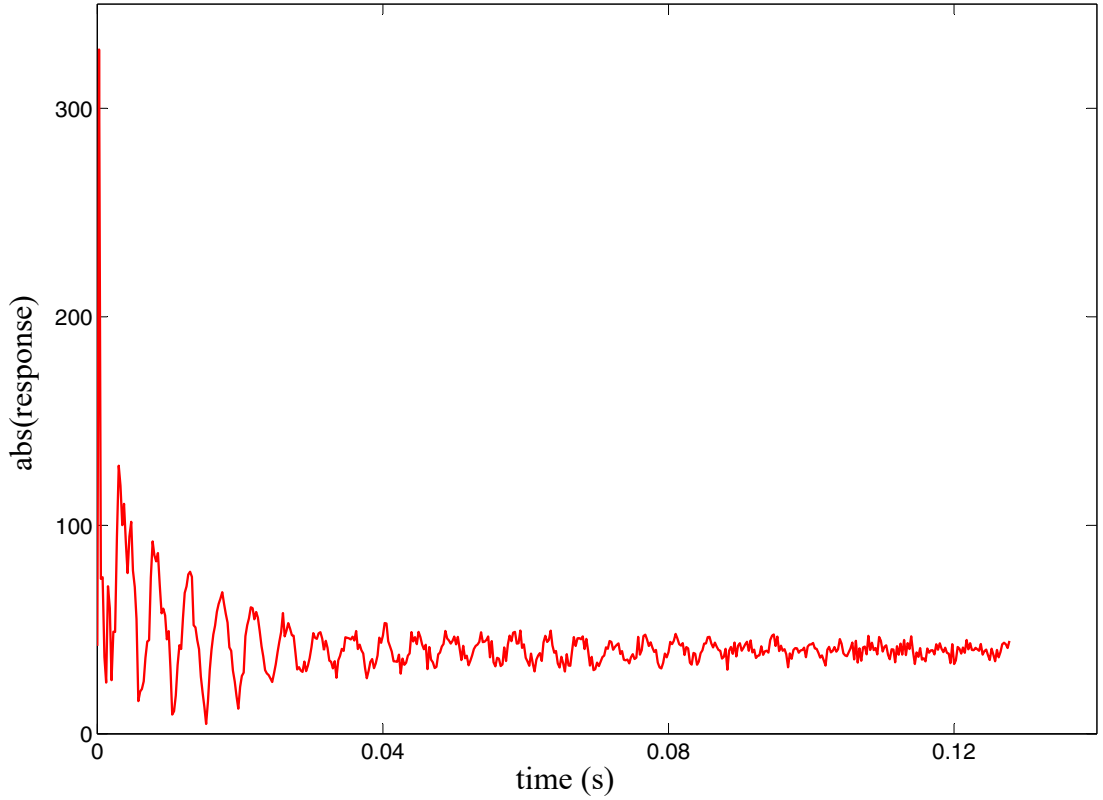


Example 2
Nuclear magnetic resonance (NMR) spectroscopy

Nuclear Magnetic Resonace (NMR) scanner:

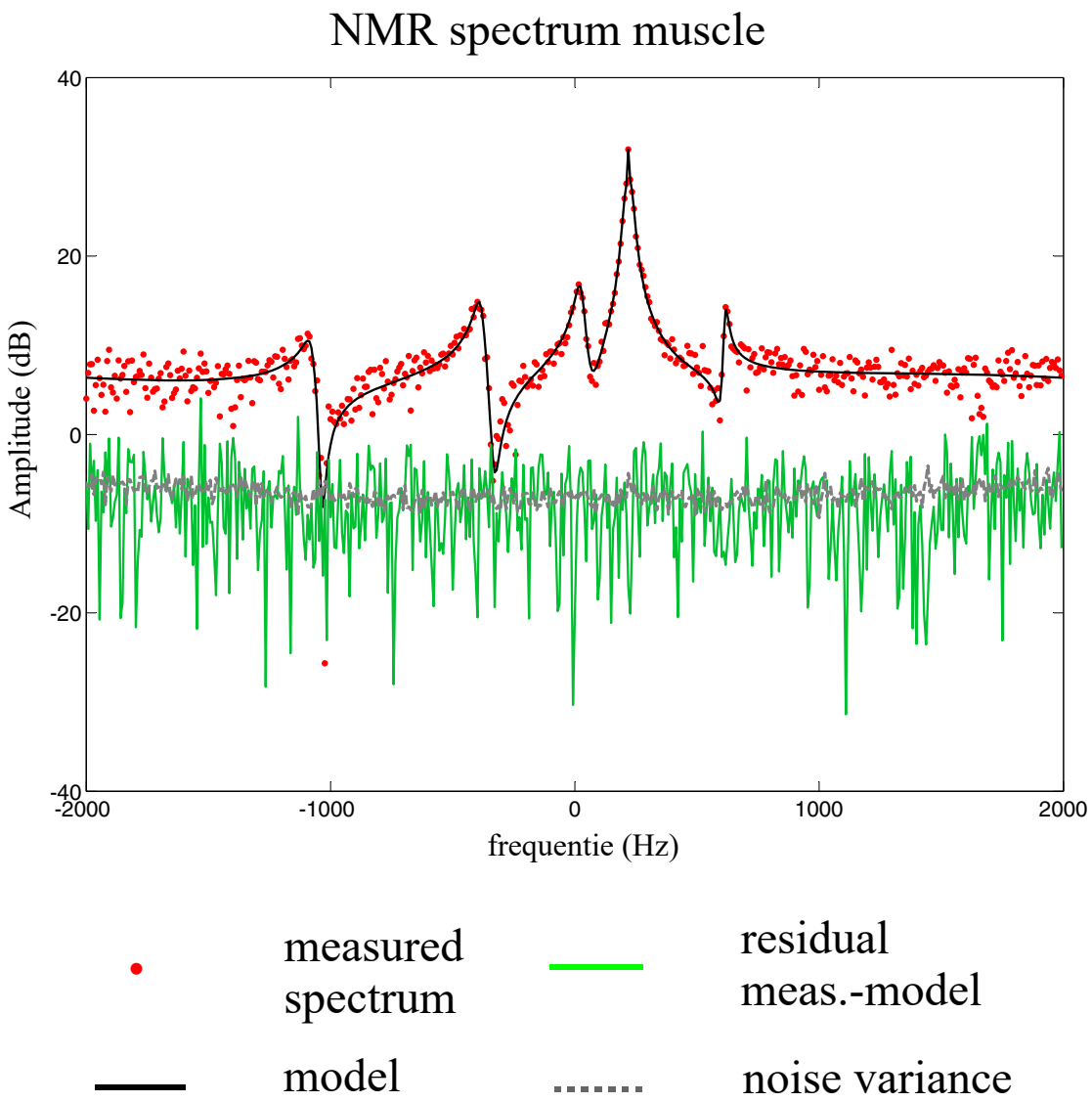
- ~ Tesla static magnetic field,
- ~ MHz oscillating field perpendicular to the static field
- response measured in two orthogonal directions x and y
  $\Rightarrow$ complex signal $x(t) + \mathrm{j}y(t)$

Nuclear magnetic resonance (NMR) spectroscopy (con't)



absolute value demodulated signal $x(t) + jy(t)$
(averaged over 64 measurements)

Nuclear magnetic resonance (NMR) spectroscopy (con't)



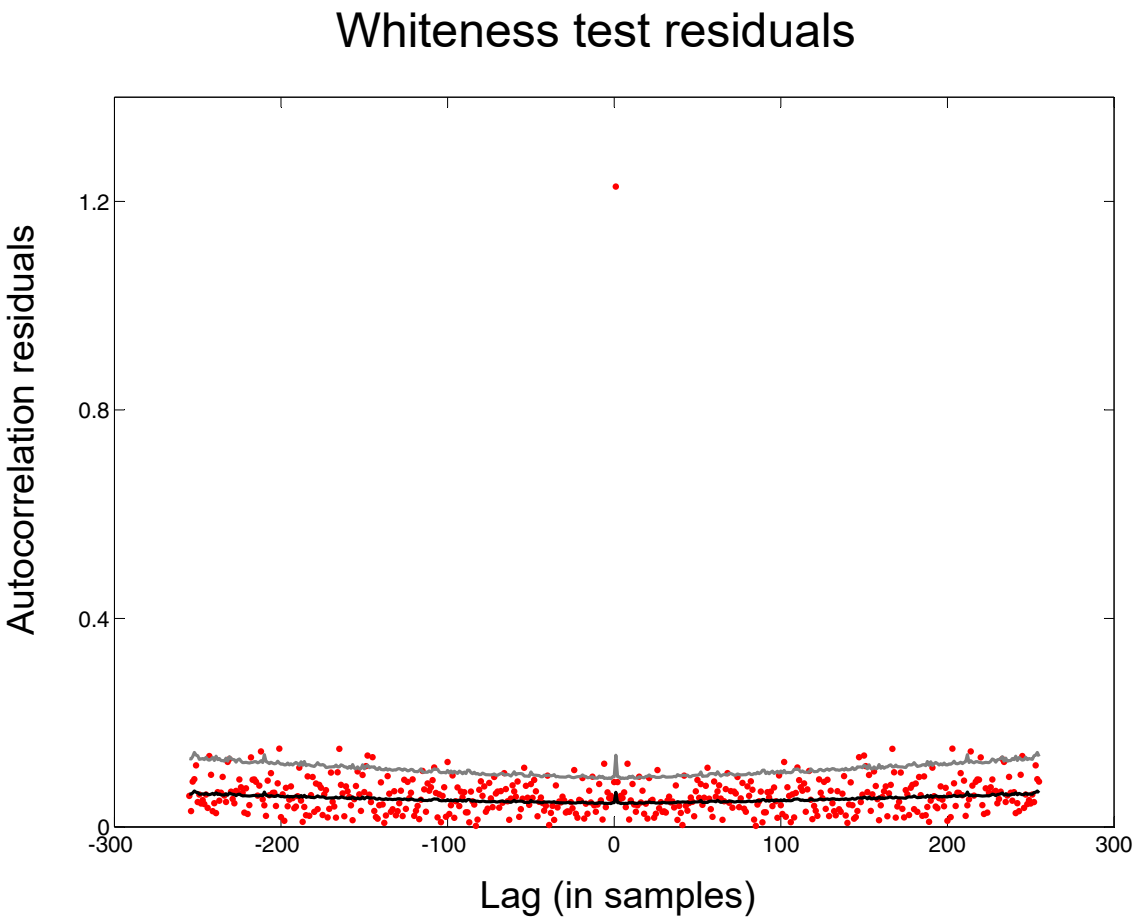
signal model = sum of complex damped exponentials

$$T(z^{-1}, \theta) = \frac{\sum_{r=0}^{n-1} b_r z^{-r}}{\sum_{r=0}^{n} a_r z^{-r}}$$

$$a_r, b_r \in \mathrm{C}$$

$$n = 9$$

Nuclear magnetic resonance (NMR) spectroscopy (con't)



autocorrelation

50% uncertainty bound (fraction outside = 51.6%)

95% uncertainty bound (fraction outside = 5.2%)

$$V_{\mathrm{SML}}(\hat{\theta}, Z) = 584$$

$$V_{\mathrm{noise}} = 502$$

$$V_{\mathrm{noise}}^{1/2} = 22$$

Observations

Transfer functions with complex coefficients

No model errors observed

Outline

Introduction

Data: what is going on between the samples

Model: parametric models of LTI-systems

Cost function

- Frequency domain formulation
- Noise models
- Time domain formulation

Validation

Examples

Conclusions