

Dutch Institute of Systems and Control



**Course: System Identification
Fall 2020**

Assignment number 1 (concerning Lectures 1, 2 and 3)

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Problem 1

Let \mathcal{S} be a data generating system, defined by

$$G_0(z) = b_0 z^{-1} \quad \text{and} \quad H_0(z) = \frac{1}{1 + d_0 z^{-1}}$$

and let this system generate input/output data according to

$$y(t) = G_0(q)u(t) + H_0(q)e(t)$$

with $\{u(t)\}$ and $\{e(t)\}$ sequences of independent identically distributed zero mean stochastic processes (white noise) that are mutually uncorrelated, having a variance of resp. σ_u^2 and σ_e^2 .

- a. Consider a Box-Jenkins model set \mathcal{M} containing two parameters such that $\mathcal{S} \in \mathcal{M}$. Derive an expression for the asymptotic covariance matrix of the estimated parameters, when applying a standard least squares prediction error identification criterion to data obtained from \mathcal{S} .
- b. Derive a similar expression as under (a) for an Output Error model set that is chosen such that $G_0 \in \mathcal{G}$.
- c. How do the asymptotic variances of the estimates of b_0 in both cases relate to each other?
- d. If d_0 would be known a priori to the user and would not need to be estimated, how would you estimate b_0 to achieve the smallest possible variance, and how does the achieved variance relate to the results in (c.)?

Problem 2

On slides Lecture 2:14-16/61 it is mentioned and motivated that under fairly general conditions (including an open-loop operating data-generating system) it holds that $\bar{E}\varepsilon^2(t, \theta) \geq \sigma_e^2$ with equality for $\hat{\theta}$ if $G(q, \hat{\theta}) = G_0(q)$ and $H(q, \hat{\theta}) = H_0(q)$.

Show that under the condition that

- $C(q)G_0(q)$ and $C(q)G(q, \theta)$ are strictly proper for all θ

the above property also holds if the data has been generated in a closed loop:

$$\begin{aligned} y(t) &= G_0(q)u(t) + H_0(q)e(t) \\ u(t) &= r(t) - C(q)y(t) \end{aligned}$$

with $C(q)$ a (known) controller.

Problem 3

Consider the following measurement problem:

We measure a particular physical variable θ_0 by using 5 different sensors, that each measure θ_0 under the influence of an additive noise disturbance e_i :

$$y_i = \theta_0 + e_i, \quad i = 1, \dots, 5$$

where e_i are independent zero-mean Gaussian random variables with variance σ_i^2 .

- Specify the maximum likelihood estimator for θ_0 on the basis of the five observations.
- Calculate the variance of this estimator and compare it with the Cramér-Rao lower bound.

Problem 4

In Figure 1 a construction method is indicated for generating a sequence of basis functions $\{F_k(z)\}_{k=1,2,\dots}$ that is complete in (spans) the space \mathcal{H}_2 of all stable systems. The sequence of scalar-valued functions $\{F_k(z)\}_{k=1,2,\dots}$ is obtained by considering the transfers from u to the individual scalar components of $x_b^{(i)}(t)$, $i = 1, 2, \dots$, and $x_b^{(i)}$ are balanced state vectors of the systems G_b as indicated in the Figure.

In this construction G_b refers to a stable all-pass system, characterized by the property that $|G_b(e^{i\omega})| = 1$ for all ω .

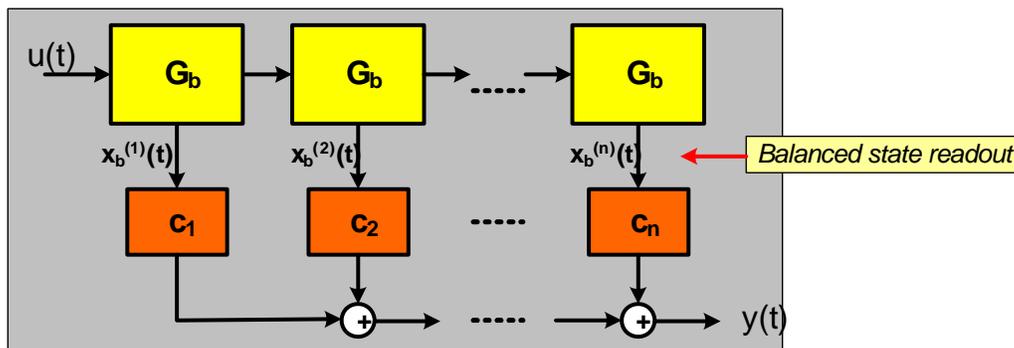


Figure 1: Construction of orthogonal basis functions

A state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with state (x) dimension n_x , has an observability matrix

$$\Gamma := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$$

and controllability matrix:

$$\Delta := [B \ AB \ A^2B \ \dots].$$

A state-space representation is called balanced if it satisfies

$$\Gamma^T \Gamma = \Delta \Delta^T = \Sigma$$

where Σ is diagonal. If the underlying system is an all-pass system it can be shown that $\Sigma = I$ (identity matrix), while for an all-pass system a realization is balanced if and only if it has a realization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

that is unitary¹.

- Show that the construction method for generating an infinite number of basis functions ($n \rightarrow \infty$) through input-to-state transfers as sketched in Figure 1 delivers functions that are all orthogonal in \mathcal{H}_2 .

Problem 5

Consider the system

$$\mathcal{S} : y(t) = b_1 u(t-1) + w_0(t),$$

where $w_0(t) = e_0(t) + k$, k being a constant and $e_0(t)$ a zero-mean white noise. Consider the model set

$$\mathcal{G} = \{G(q, \theta) : G(q, \theta) = \beta_1 q^{-1} + \beta_2 q^{-2}\}$$

and assume that $u(t)$ is a zero-mean white noise input with variance σ_u^2 .

- Does $G_0(q) \in \mathcal{G}$?
- Compute the asymptotic parameter estimates θ^* .
- Does the fact that $E[w_0(t)] \neq 0$ affect the consistency of the identified parameters?

¹A real-valued square matrix X is called unitary if it satisfies $XX^T = X^T X = I$.

Problem 6

We have collected N input-output data on an unknown true system \mathcal{S} : $y(t) = G_0(q)u(t) + H_0(q)e(t)$ and identified the parameter vector $\hat{\theta}_N$ in the following model structure:

$$\mathcal{M} = \left\{ G(q, \theta) = \frac{bq^{-1}}{1 + fq^{-1}} ; H(q, \theta) = 1 \mid \theta = \begin{pmatrix} b \\ f \end{pmatrix} \right\}$$

Using the data and $\hat{\theta}_N$, we have performed the model structure validation based on the residuals $\varepsilon(t, \hat{\theta}_N)$ and we have obtained the result presented in Figure 2, where the figure shows $\hat{R}_\varepsilon(\tau)$ (top) and $\hat{R}_{\varepsilon u}(\tau)$ (bottom) together with their 99% confidence bounds. We have also computed the variance $\text{cov}(G(e^{j\omega}, \hat{\theta}_N))$ of the identified model $G(q, \hat{\theta}_N)$ in the frequency domain using the formula of slide 2-43.

Based on the expression for $\text{cov}(G(e^{j\omega}, \hat{\theta}_N))$ we could derive a probabilistic uncertainty bound for $|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)|$ for each value of ω . Would this approach be valid in the given situation? Motivate your answer.

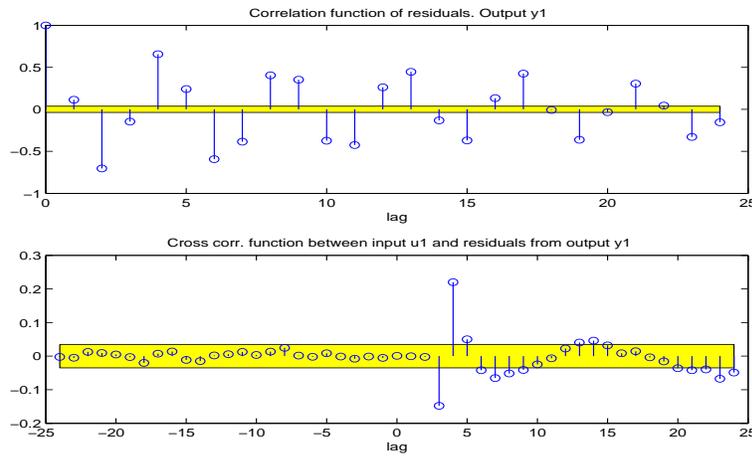


Figure 2: model structure validation